ANNALES
POLONICI MATHEMATICI
102.1 (2011)

Uniqueness problem for meromorphic mappings with Fermat moving hypersurfaces

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Abstract. We give unicity theorems for meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ with Fermat moving hypersurfaces.

1. Introduction. Using the Second Main Theorem of value distribution theory and Borel's lemma, Nevanlinna [N] proved that if two nonconstant meromorphic functions f and g on the complex plane $\mathbb{C}$ have the same inverse images for five distinct values, then $f \equiv g$, and if they have the same inverse images, counted with multiplicities, for four distinct values then g is a special type of linear fractional transformation of f.

In 1975, Fujimoto [F1] generalized Nevanlinna's result to the case of meromorphic mappings of $\mathbb{C}$ into $\mathbb{C}P^n$. He showed that if two linearly nondegenerate meromorphic mappings f and g of $\mathbb{C}$ into $\mathbb{C}P^n$ have the same inverse images, counted with multiplicities, for $3n+2$ hyperplanes in $\mathbb{C}P^n$ in general position, then $f \equiv g$, and if they have the same inverse images counted with multiplicities for $3n+1$ hyperplanes in $\mathbb{C}P^n$ in general position, then there exists a projective linear transformation L of $\mathbb{C}P^n$ to itself such that $g = L \cdot f$. Since that time, this problem has been studied intensively for the case of hyperplanes by Fujimoto ([F2], [F3]), Stoll [St], Smiley [Sm], Ji [J], Ru [R], Tu [T], Ye [Y], Dethloff and Tan ([DT1]–[DT3]), and Thai and Quang [TQ]. Motivated by the case of hyperplanes, the uniqueness problem for the case of hypersurfaces arises naturally. However, there are so far only the uniqueness theorem of Dulock–Ru [DR] and the one of Phuong [P] for the case of a large number of (general) fixed hypersurfaces. It seems that the biggest difficulty in studying uniqueness of meromorphic mappings with few hypersurfaces comes from the fact that we do not have good forms of the Second Main Theorem for the case of hypersurfaces. Our purpose in this paper is to give some uniqueness theorems for the case of few Fermat

2010 *Mathematics Subject Classification*: Primary 32H30; Secondary 32H04, 32H25.
Key words and phrases: Nevanlinna theory, Second Main Theorem, uniqueness problem.

DOI: 10.4064/ap102-1-1

moving hypersurfaces. We would like to remark that in [DR] and [P], the Second Main Theorem given by An–Phuong [AP] was used. However, this theorem does not apply to the case of few hypersurfaces. In order to prove our uniqueness theorems, we also establish a Second Main Theorem for a class of Fermat moving hypersurfaces.

Let f be a nonconstant meromorphic map of $\mathbb{C}^m$ into $\mathbb{C}P^n$. We say that a meromorphic function φ on $\mathbb{C}^m$ is *small* with respect to f if $T_\varphi(r) = o(T_f(r))$ as $r \to \infty$ (outside a set of finite Lebesgue measure). Denote by $\mathcal{R}_f$ the field of all small (with respect to f) meromorphic functions on $\mathbb{C}^m$.

Take a reduced representation $(f_0 : \cdots : f_n)$ of f. We say that f is *algebraically nondegenerate over* $\mathcal{R}_f$ if there is no nonzero homogeneous polynomial $Q \in \mathcal{R}_f[x_0, \ldots, x_n]$ such that $Q(f) := Q(f_0, \ldots, f_n) \equiv 0$.

For a homogeneous polynomial $Q \in \mathcal{R}_f[x_0, \ldots, x_n]$, denote by $Q(z)$ the homogeneous polynomial over $\mathbb{C}$ obtained by substituting a specific point $z \in \mathbb{C}^m$ into the coefficients of Q.

We say that a set $\{Q_j\}_{j=0}^n$ of homogeneous polynomials of the same degree in $\mathcal{R}_f[x_0, \ldots, x_n]$ is *admissible* if there exists $z \in \mathbb{C}^m$ such that the system of equations

$$Q_j(z)(w_0, \ldots, w_n) = 0, \quad 0 \le j \le n,$$

has only the trivial solution $w = (0, \ldots, 0)$ in $\mathbb{C}^{n+1}$. Denote by $\mathcal{S}(\{Q_j\}_{j=0}^n)$ the set of all homogeneous polynomials $P = \sum_{j=0}^n b_j Q_j$, where $b_j \in \mathcal{R}_f$.

Let $\{P_i\}_{i=1}^q$ $(q \ge n+1)$ be homogeneous polynomials in $\mathcal{S}(\{Q_j\}_{j=0}^n)$, $P_i = \sum_{j=0}^n b_{ij} Q_j$. We say that $\{P_i\}_{i=1}^q$ are *in general position* if for any $1 \le i_0 < \cdots < i_n \le q$, $\det(b_{i_k j}, 0 \le k, j \le n) \not\equiv 0$.

Theorem 1.1. *Let f, g be nonconstant meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and $\{Q_j\}_{j=0}^n$ be an admissible set of homogeneous polynomials of degree d in $\mathcal{R}_f[x_0, \ldots, x_n]$. Let $\gamma_0, \ldots, \gamma_n$ be nonzero meromorphic functions in $\mathcal{R}_f$. Put $P = \gamma_0 Q_0^p + \cdots + \gamma_n Q_n^p$, where p is a positive integer, $p > n(d(n+1)+2)/d$. Assume that f, g are algebraically nondegenerate over $\mathcal{R}_f$ and $\mathcal{R}_g$ respectively, and*

(i) $\mathrm{Zero}(P(f)) = \mathrm{Zero}(P(g))$,
(ii) $f = g$ *on* $\mathrm{Zero}(P(f))$.

Then $f = g$.

Theorem 1.2. *Let f, g be nonconstant meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and $\{Q_j\}_{j=0}^n$ be an admissible set of homogeneous polynomials of degree $d \ge n+2$ in $\mathcal{R}_f[x_0, \ldots, x_n]$. Let $\{P_i\}_{i=0}^{2n+1}$ be homogeneous polynomials in $\mathcal{S}(\{Q_j\}_{j=0}^n)$ in general position. Assume that f, g are algebraically nondegenerate over $\mathcal{R}_f$ and $\mathcal{R}_g$ respectively, and*

(i) $\mathrm{Zero}(P_i(f)) = \mathrm{Zero}(P_i(g))$, $i \in \{1, \ldots, 2n+1\}$,

(ii) $\dim(\mathrm{Zero}(P_i(f)) \cap \mathrm{Zero}(P_j(f))) \le m-2$ *for all* $1 \le i < j \le 2n+1$,
(iii) $f = g$ *on* $\bigcup_{i=1}^{2n+1} \mathrm{Zero}(P_i(f))$.

Then $f = g$.

2. Preliminaries. For $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$, we set $\|z\| = (\sum_{j=1}^m |z_j|^2)^{1/2}$ and define

$$B(r) = \{z \in \mathbb{C}^m : \|z\| < r\}, \quad S(r) = \{z \in \mathbb{C}^m : \|z\| = r\},$$

$$d^c = \frac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial), \quad \mathcal{V} = (dd^c\|z\|^2)^{m-1}, \quad \sigma = d^c \log\|z\|^2 \wedge (dd^c \log\|z\|)^{m-1}.$$

Let F be a nonzero holomorphic function on $\mathbb{C}^m$. For each $a \in \mathbb{C}^m$, expanding F as $F = \sum H_i(z-a)$ with homogeneous polynomials H_i of degree i around a, we define $v_F(a) = \min\{i : H_i \not\equiv 0\}$.

Let φ be a nonzero meromorphic function on $\mathbb{C}^m$. For each $a \in \mathbb{C}^m$, we choose nonzero holomorphic functions F and G on a neighborhood U of a such that $\varphi = F/G$ on U and $\dim(F^{-1}(0) \cap G^{-1}(0)) \le m-2$ and we define the map $v_\varphi : \mathbb{C}^m \to \mathbb{N}_0$ by $v_\varphi(a) = v_F(a)$. Set

$$|v_\varphi| = \overline{\{z : v_\varphi(z) \neq 0\}}.$$

Let k be a positive integer or $+\infty$. Set $v_\varphi^{[k]}(z) = \min\{v_\varphi(z), k\}$, and

$$N_\varphi^{[k]}(r) := \int_1^r \frac{n^{[k]}(t)}{t^{2m-1}}\, dt \qquad (1 < r < +\infty)$$

where

$$n^{[k]}(t) = \begin{cases} \displaystyle\int_{|v_\varphi| \cap B(t)} v_\varphi^{[k]} \cdot \mathcal{V} & \text{for } m \ge 2, \\ \displaystyle\sum_{|z| \le t} v_\varphi^{[k]}(z) & \text{for } m = 1. \end{cases}$$

Set $N_\varphi(r) := N_\varphi^{[+\infty]}(r)$. We have the following *Jensen's formula*:

$$N_\varphi(r) - N_{1/\varphi}(r) = \int_{S(r)} \log|\varphi|\, \sigma - \int_{S(1)} \log|\varphi|\, \sigma.$$

Let f be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$. For fixed homogeneous coordinates $(w_0 : \cdots : w_n)$ of $\mathbb{C}P^n$, we take a reduced representation $f = (f_0 : \cdots : f_n)$, which means that each f_i is a holomorphic function on $\mathbb{C}^m$ and $f(z) = (f_0(z) : \cdots : f_n(z))$ outside the analytic set $\{z : f_0(z) = \cdots = f_n(z) = 0\}$ of codimension ≥ 2. Set $\|f\| = \max\{|f_0|, \ldots, |f_n|\}$.

The characteristic function of f is defined by

$$T_f(r) := \int_{S(r)} \log\|f\|\, \sigma - \int_{S(1)} \log\|f\|\, \sigma, \qquad 1 < r < +\infty.$$

For a meromorphic function φ on $\mathbb{C}^m$, the characteristic function $T_\varphi(r)$ of φ is defined, as φ is a meromorphic map of $\mathbb{C}^m$ into $\mathbb{C}P^1$.

The *proximity function* $m(r,\varphi)$ is defined by

$$m(r,\varphi) = \int\limits_{S(r)} \log^+ |\varphi|\, \sigma,$$

where $\log^+ x = \max\{\log x, 0\}$ for $x \geq 0$. Then

$$T_\varphi(r) = N_{1/\varphi}(r) + m(r,\varphi) + O(1).$$

For a homogeneous polynomial $Q := \sum_I a_I x^I \in \mathcal{R}_f[x_0,\dots,x_n]$ with degree $d \geq 1$, we define

$$N_f^{[k]}(r,Q) := N^{[k]}_{Q(f_0,\dots,f_n)}(r).$$

For brevity we will omit the superscript $[k]$ in the counting function if $k = +\infty$. It is clear that

$$\log |Q(f)| \leq \log \sum_I |a_I| + \log \|f\|^d \leq \sum_I \log^+ |a_I| + d \log \|f\| + O(1).$$

From this fact and Jensen's formula, we easily get the following First Main Theorem of value distribution theory.

Theorem 2.1 (First Main Theorem). *Let f be a nonconstant meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and Q be a homogeneous polynomial of degree d in $\mathcal{R}_f[x_0,\dots,x_n]$ such that $Q(f) \not\equiv 0$. Then*

$$N_f(r,Q) \leq dT_f(r) + o(T_f(r))$$

for all r except for a subset E of $(1,+\infty)$ of finite Lebesgue measure.

For a hyperplane $H: a_0w_0 + \cdots + a_nw_n = 0$ in $\mathbb{C}P^n$ with $\mathrm{im} f \not\subseteq H$, we denote

$$(f,H) := a_0f_0 + \cdots + a_nf_n,$$

where $(f_0 : \cdots : f_n)$ again is a reduced representation of f. Now we formulate the Second Main Theorem.

Theorem 2.2 ([F2, Theorem 2.13]; Second Main Theorem). *Let f be a linearly nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and $H_1,\dots,H_q$ $(q \geq n+1)$ be hyperplanes in $\mathbb{C}P^n$ in general position. Then*

$$(q-n-1)T_f(r) \leq \sum_{j=1}^q N^{[n]}_{(f,H_j)}(r) + o(T_f(r))$$

for all r except for a subset E of $(1,+\infty)$ of finite Lebesgue measure.

3. Proofs. First of all we give the following lemma:

LEMMA 3.1. *Let f be a nonconstant meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and $\{Q_j\}_{j=0}^n$ be an admissible set of homogeneous polynomials of degree d in $\mathcal{R}_f[x_0,\dots,x_n]$. Let $\{P_i\}_{i=0}^q$ $(q \geq n+2)$ be homogeneouus polynomials in $\mathcal{S}(\{Q_j\}_{j=0}^n)$ in general position. Assume that f is algebraically nondegenerate over $\mathcal{R}_f$. Then*

$$\frac{qd}{n+2}T_f(r) \leq \sum_{i=1}^{q} N_f^{[n]}(r,P_i) + o(T_f(r))$$

for all r except for a subset E of $(1,+\infty)$ of finite Lebesgue measure.

Proof. Set $\mathcal{T}_d := \{I := (i_0,\dots,i_n) \in \mathbb{N}_0^{n+1} : |I| := i_0 + \cdots + i_n = d\}$.
Assume that

$$Q_j = \sum_{I\in\mathcal{T}_d} a_{jI}x^I \qquad (j = 0,\dots,n),$$

$$P_i = \sum_{j=0}^{n} b_{ij}Q_j \qquad (i = 1,\dots,q),$$

where $a_{jI}, b_{ij} \in \mathcal{R}_f$, $x^I = x_0^{i_0}\cdots x_n^{i_n}$.

In order to prove Lemma 3.1, we only have to show that for any subset $\{k_1,\dots,k_{n+2}\} \subset \{1,\dots,q\}$,

$$dT_f(r) \leq \sum_{i=1}^{n+2} N_f^{[n]}(r,P_{k_i}) + o(T_f(r)). \tag{3.1}$$

Without loss generality, we may assume that $\{k_1,\dots,k_{n+2}\} = \{1,\dots,n+2\}$. Set

$$N_{n+2} := \begin{pmatrix} b_{10} & \dots & b_{n+1,0} \\ b_{11} & \dots & b_{n+1,1} \\ \vdots & \ddots & \vdots \\ b_{1n} & \dots & b_{n+1,n} \end{pmatrix}$$

and define N_i $(i \in \{1,\dots,n+1\})$ to be N_{n+2} with the ith column changed to $\begin{pmatrix} b_{n+2,0} \\ \vdots \\ b_{n+2,n} \end{pmatrix}$. Set

$$c_i = \det(N_i), \qquad i \in \{1,\dots,n+2\}.$$

It is easy to see that $c_i \in \mathcal{R}_f$, $c_i \not\equiv 0$ and

$$\sum_{i=1}^{n+1} c_iP_i(f) = c_{n+2}P_{n+2}(f). \tag{3.2}$$

Set

$$F = (c_1P_1(f) : \cdots : c_{n+1}P_{n+1}(f)) : \mathbb{C}^m \to \mathbb{C}P^n.$$

It is easy to see that F is linearly nondegenerate (over $\mathbb{C}$).

Assume that $(c_1P_1(f)/h : \cdots : c_{n+1}P_{n+1}(f)/h)$ is a reduced representation of F, where h is a meromorphic function on $\mathbb{C}^m$. Put $F_i = c_iP_i(f)/h$, $i \in \{1, \ldots, n+1\}$. We have

$$hF_i = \sum_{j=0}^{n} c_i b_{ij} Q_j(f), \quad 1 \le i \le n+1.$$

This implies that

$$Q_j(f) = \sum_{i=1}^{n+1} \gamma_{ij} hF_i(f), \quad 0 \le j \le n,$$

where $\gamma_{ij} \in \mathcal{R}_f$. We have

$$\begin{aligned} \max_{0\le j\le n} |Q_j(f)| &= \max_{0\le j\le n} \Big| \sum_{i=1}^{n+1} h\gamma_{ij}F_i \Big| \\ &\le |h| \Big(\sum_{\substack{0\le j\le n\\ 1\le i\le n+1}} |\gamma_{ij}| \Big) \max_{1\le i\le n+1} |F_i|. \end{aligned} \tag{3.3}$$

Let $t = (\ldots, t_{kI}, \ldots)$ be a family of variables ($k \in \{0, \ldots, n\}, I \in \mathcal{T}_d$). Set

$$\widetilde{Q}_j = \sum_{I \in \mathcal{T}_d} t_{jI} x^I \in \mathbb{Z}[t, x], \quad j = 0, \ldots, n.$$

Let $\widetilde{R} \in \mathbb{Z}[t]$ be the resultant of $\widetilde{Q}_0, \ldots, \widetilde{Q}_n$.

Since $\{Q_j\}_{j=0}^n$ is an admissible set, $R := \widetilde{R}(\ldots, a_{kI}, \ldots) \not\equiv 0$. It is clear that $R \in \mathcal{R}_f$ since $a_{kI} \in \mathcal{R}_f$.

By Proposition 2.1 in [DT4], there exists a positive integer s such that

$$x_i^s \widetilde{R} = \sum_{j=0}^{n} \widetilde{R}_{ij} \widetilde{Q}_j \quad \text{for all } i \in \{0, \ldots, n\}, \tag{3.4}$$

where $\{\widetilde{R}_{ij}\}_{0\le i,j\le n}$ are polynomials in $\mathbb{Z}[t, x]$. Without loss of generality, after multiplying both sides of (3.4) by x_i^d, we may assume that $s \ge d$.

For each polynomial $H \in \mathbb{Z}[t, x]$,

$$H = \sum_{I\in A,\, J\in B} a_{IJ} t^J x^I, \quad \text{where } a_{IJ} \in \mathbb{Z},\ A \subset \mathbb{N}_0^{n+1},\ B \subset \mathbb{N}_0^{(n+1)\cdot(\sharp\mathcal{T}_d)},$$

we denote

$$H^{(1)} = \sum_{I\in A,\,|I|>s-d,\,J\in B} a_{IJ}t^J x^I, \qquad H^{(2)} = \sum_{I\in A,\,|I|<s-d,\,J\in B} a_{IJ}t^J x^I,$$
$$H^{(3)} = \sum_{I\in A,\,|I|=s-d,\,J\in B} a_{IJ}t^J x^I.$$

By (3.4), we have

$$x_i^s\widetilde{R} = \sum_{j=0}^{n}\widetilde{R}_{ij}^{(1)}\widetilde{Q}_j + \sum_{j=0}^{n}\widetilde{R}_{ij}^{(2)}\widetilde{Q}_j + \sum_{j=0}^{n}\widetilde{R}_{ij}^{(3)}\widetilde{Q}_j \quad \text{for all } i\in\{0,\dots,n\}.$$

Hence, since $\widetilde{Q}_j$ ($j\in\{0,\dots,n\}$) are homogeneous polynomials of degree d in variables $(x_0,\dots,x_n)$ and $\widetilde{R}\in\mathbb{Z}[t]$, we have

$$\sum_{j=0}^{n}\widetilde{R}_{ij}^{(1)}\widetilde{Q}_j = 0 \quad\text{and}\quad \sum_{j=0}^{n}\widetilde{R}_{ij}^{(2)}\widetilde{Q}_j = 0 \quad \text{for all } i\in\{0,\dots,n\}.$$

Hence, without loss of generality after replacing $\widetilde{R}_{ij}$ by $\widetilde{R}_{ij}^{(3)}$, we may assume that $\widetilde{R}_{ij}$ are homogeneous polynomials of degree $s-d$ in $(x_0,\dots,x_n)$. Set

$$R_{ij} = \widetilde{R}_{ij}((\dots,a_{kI},\dots),(f_0,\dots,f_n)), \quad 0\le i,j\le n.$$

Then

$$f_i^sR = \sum_{j=0}^{n} R_{ij}\cdot Q_j(f_0,\dots,f_n) \quad \text{for all } i\in\{0,\dots,n\}. \tag{3.5}$$

So,

$$\begin{aligned}|f_i^sR| &= \Big|\sum_{j=0}^{n} R_{ij}\cdot Q_j(f_0,\dots,f_n)\Big| \\ &\le \sum_{j=0}^{n}|R_{ij}|\cdot\max_{k\in\{0,\dots,n\}}|Q_k(f_0,\dots,f_n)|\end{aligned} \tag{3.6}$$

for all $i\in\{0,\dots,n\}$.

We write

$$R_{ij} = \sum_{I\in\mathcal{T}_{s-d}}\beta_I^{ij}f^I, \qquad \beta_I^{ij}\in\mathcal{R}_f.$$

By (3.6), we have

$$|f_i^sR| \le \Big(\sum_{\substack{0\le j\le n\\ I\in\mathcal{T}_{s-d}}}|\beta_I^{ij}|\,\|f\|^{s-d}\Big)\max_{k\in\{0,\dots,n\}}|Q_k(f_0,\dots,f_n)|, \quad i\in\{0,\dots,n\}.$$

So,

$$\frac{|f_i|^s}{\|f\|^{s-d}} \le \Big(\sum_{\substack{0\le j\le n\\ I\in\mathcal{T}_{s-d}}} |\beta_I^{ij}/R|\Big) \max_{k\in\{0,\dots,n\}} |Q_k(f_0,\dots,f_n)|$$

for all $i\in\{0,\dots,n\}$. Thus

$$\|f\|^d \le \Big(\sum_{\substack{0\le j\le n\\ I\in\mathcal{T}_{s-d}}} |\beta_I^{ij}/R|\Big) \max_{k\in\{0,\dots,n\}} |Q_k(f_0,\dots,f_n)|. \tag{3.7}$$

By (3.3) and (3.7) we have

$$\|f\|^d \le \Big(\sum_{\substack{0\le j\le n\\ I\in\mathcal{T}_{s-d}}} |\beta_I^{ij}/R|\Big)\cdot |h| \cdot \Big(\sum_{\substack{0\le j\le n\\ 1\le i\le n+1}} |\gamma_{ij}|\Big)\cdot \|F\|. \tag{3.8}$$

Take a meromorphic function u on $\mathbb{C}^m$ such that $(Q_0(f)/u : \cdots : Q_n(f)/u)$ is a reduced representation of the meromorphic mapping $(Q_0(f):\cdots:Q_n(f))$. By (3.5) we have

$$N_u(r) \le N_R(r) + \sum_{\substack{0\le j\le n\\ I\in\mathcal{T}_d}} N_{1/a_{jI}}(r) + \sum_{\substack{0\le i,j\le n\\ I\in\mathcal{T}_{s-d}}} N_{1/\beta_I^{ij}}(r) = o(T_f(r)).$$

Since $(c_1P_1(f)/h : \cdots : c_{n+1}P_{n+1}(f)/h)$ is a reduced representation of the meromorphic mapping F, we have

$$N_h(r) \le N_{\det(c_iub_{ij},1\le i\le n+1,0\le j\le n)}(r) = o(T_f(r)),$$

$$N_{1/h}(r) \le \sum_{i=1}^{n+1} N_{1/c_i}(r) + \sum_{\substack{0\le j\le n\\ 1\le i\le n+1}} N_{1/b_{ij}}(r) + \sum_{\substack{0\le j\le n\\ I\in\mathcal{T}_{s-d}}} N_{1/a_{jI}}(r) = o(T_f(r)).$$

By (3.8), we have

$$\begin{aligned} dT_f(r) &= d\int\limits_{S(r)} \log\|f\|\,\sigma + O(1) \\ &\le \int\limits_{S(r)} \log\Big(\sum_{\substack{0\le j\le n\\ I\in\mathcal{T}_{s-d}}} |\beta_I^{ij}/R|\Big)\cdot|h|\cdot\Big(\sum_{\substack{0\le j\le n\\ 1\le i\le n+1}} |\gamma_{ij}|\Big)\sigma \\ &\quad + T_F(r) + O(1) \\ &\le \int\limits_{S(r)} \log^+\Big(\sum_{\substack{0\le j\le n\\ I\in\mathcal{T}_{s-d}}} |\beta_I^{ij}/R|\Big)\sigma + \int\limits_{S(r)} \log^+\Big(\sum_{\substack{0\le j\le n\\ 1\le i\le n+1}} |\gamma_{ij}|\Big)\sigma \\ &\quad + \int\limits_{S(r)} \log|h|\,\sigma + T_F(r) + O(1) \end{aligned} \tag{3.9}$$

$$
\begin{aligned}
&\leq \sum_{\substack{0\leq j\leq n\\ I\in\mathcal{T}_{s-d}}} m(r,\beta_I^{ij}/R) + \sum_{\substack{0\leq j\leq n\\ 1\leq i\leq n+1}} m(r,\gamma_{ij})\\
&\quad + N_h(r) - N_{1/h}(r) + T_F(r) + O(1)\\
&= T_F(r) + o(T_f(r)).
\end{aligned}
$$

(note that $\beta_I^{ij}/R, \gamma_{ij} \in \mathcal{R}_f$). By (3.2), (3.9) and the Second Main Theorem, we have

$$
\begin{aligned}
dT_f(r) &\leq T_F(r) + o(T_f(r))\\
&\leq \sum_{i=1}^{n+1} N^{[n]}_{c_iP_i(f)/h}(r) + N^{[n]}_{\sum_{i=1}^{n+1} c_iP_i(f)/h}(r) + o(T_f(r))\\
&\leq \sum_{i=1}^{n+2} N^{[n]}_{c_iP_i(f)/h}(r) + o(T_f(r))\\
&\leq \sum_{i=1}^{n+2} N^{[n]}_{P_i(f)}(r) + \sum_{i=1}^{n+2} N^{[n]}_{c_i}(r) + (n+2)N_{1/h}(r) + o(T_f(r))\\
&\leq \sum_{i=1}^{n+2} N^{[n]}_f(r,P_i)(r) + o(T_f(r)).
\end{aligned}
$$

We get (3.1), completing the proof of Lemma 3.1. ∎

Lemma 3.2 ([J, Lemma 5.1]). *Let $A_1,\dots,A_k$ be pure $(m-1)$-dimensional analytic subsets of $\mathbb{C}^m$ with $\operatorname{codim}(A_i\cap A_j)\geq 2$ whenever $i\neq j$. Let f_1, f_2 be linearly nondegenerate mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$. Then there exists a dense subset $\mathcal{P}\subset\mathbb{C}_*^{m+1}$ such that for any $p:=(p_0,\dots,p_n)\in\mathcal{P}$ the hyperplane H_p defined by $p_0w_0+\dots+p_nw_n=0$ satisfies*

$$\operatorname{codim}\Big(\Big(\bigcup_{j=1}^k A_j\Big)\cap f_i^{-1}(H_p)\Big)\geq 2,\qquad i\in\{1,2\}.$$

Proof of Theorem 1.1. Assume that $f\not\equiv g$. Then there exist hyperplanes H_1, H_2 in $\mathbb{C}P^n$ such that

$$\dim\{P(f)=0=(f,H_i)\}\leq m-2,\quad \dim\{P(g)=0=(g,H_i)\}\leq m-2,$$

for all $i\in\{1,2\}$ and

$$\frac{(f,H_1)}{(f,H_2)}\not\equiv\frac{(g,H_1)}{(g,H_2)}.$$

Indeed, suppose that this does not hold. Then by Lemma 3.2,

$$\frac{(f,H_1)}{(f,H_2)}\equiv\frac{(g,H_1)}{(g,H_2)}$$

for all hyperplanes H_1, H_2 in $\mathbb{C}P^n$. In particular, $f_0/f_i \equiv g_0/g_i$ for all $i \in \{0, \ldots, n\}$. Then $f \equiv g$, which is a contradiction.

By the assumption of Theorem 1.1 and by the First Main Theorem,

$$\begin{aligned} N_f^{[1]}(r,P) &\le N_{\frac{(f,H_1)}{(f,H_2)} - \frac{(g,H_1)}{(g,H_2)}}(r) \le T_{\frac{(f,H_1)}{(f,H_2)} - \frac{(g,H_1)}{(g,H_2)}}(r) + O(1) \\ &\le T_{\frac{(f,H_1)}{(f,H_2)}}(r) + T_{\frac{(g,H_1)}{(g,H_2)}}(r) + O(1) \le T_f(r) + T_g(r) + O(1). \end{aligned}$$

Similarly,

$$N_g^{[1]}(r,P) \le T_f(r) + T_g(r) + O(1).$$

Thus,

$$N_f^{[1]}(r,P) + N_g^{[1]}(r,P) \le 2(T_f(r) + T_g(r)) + O(1). \tag{3.10}$$

Since the $n+2$ homogeneous polynomials $Q_0^p, \ldots, Q_n^p, P$ are in general position in $\mathcal{S}(\{Q_j^p\}_{j=0}^n)$, by Lemma 3.1 and the First Main Theorem we have

$$\begin{aligned} pdT_f(r) &\le \sum_{j=0}^{n} N_f^{[n]}(r, Q_j^p) + N_f^{[n]}(r,P) + o(T_f(r)) \\ &\le \frac{n}{p} \sum_{j=0}^{n} N_f(r, Q_j^p) + nN_f^{[1]}(r,P) + o(T_f(r)) \\ &\le dn(n+1)T_f(r) + nN_f^{[1]}(r,P) + o(T_f(r)). \end{aligned}$$

This implies that

$$\frac{d(p - n(n+1))}{n} \cdot T_f(r) \le N_f^{[1]}(r,P) + o(T_f(r)). \tag{3.11}$$

Since $\mathrm{Zero}(P(f)) = \mathrm{Zero}(P(g))$, we have

$$N_f^{[1]}(r,P) = N_g^{[1]}(r,P).$$

Thus, by (3.11) and the First Main Theorem,

$$\begin{aligned} \frac{d(p - n(n+1))}{n} \cdot T_f(r) &\le N_g^{[1]}(r,P) + o(T_f(r)) \\ &\le N_g(r,P) + o(T_f(r)) \le dp \cdot T_g(r) + o(T_f(r)). \end{aligned}$$

This implies that $\mathcal{R}_f \subset \mathcal{R}_g$. So, by Lemma 3.1, similarly to (3.11) we have

$$\frac{d(p - n(n+1)}{n} \cdot T_g(r) \le N_g^{[1]}(r,P) + o(T_g(r)). \tag{3.12}$$

By (3.11) and (3.12),

$$\begin{aligned} &\frac{d(p - n(n+1))}{n}(T_f(r) + T_g(r)) \\ &\qquad \le N_f^{[1]}(r,P) + N_g^{[1]}(r,P) + o(T_f(r) + T_g(r)). \end{aligned}$$

Combining this with (3.10) we obtain

$$\frac{d(p-n(n+1))}{n}\cdot(T_f(r)+T_g(r))\le 2(T_f(r)+T_g(r))+o(T_f(r)+T_g(r)).$$

This contradicts $p>n(d(n+1)+2)/d$. Thus, $f\equiv g$, which completes the proof of Theorem 1.1. ∎

Proof of Theorem 1.2. Assume that $f\not\equiv g$. By an argument similar to the proof of Theorem 1.1, there exist hyperplanes H_1,H_2 in $\mathbb{C}P^n$ such that

$$\dim\{P_j(f)=0=(f,H_i)\}\le m-2,\quad \dim\{P_j(g)=0=(g,H_i)\}\le m-2,$$

for all $i\in\{1,2\}$, $j\in\{1,\dots,2n+1\}$ and

$$\frac{(f,H_1)}{(f,H_2)}\not\equiv\frac{(g,H_1)}{(g,H_2)}.$$

By the assumption of Theorem 1.2 and by the First Main Theorem,

$$\begin{aligned}\sum_{i=1}^{2n+1}N_f^{[1]}(r,P_i)&\le N_{\frac{(f,H_1)}{(f,H_2)}-\frac{(g,H_1)}{(g,H_2)}}(r)\le T_{\frac{(f,H_1)}{(f,H_2)}-\frac{(g,H_1)}{(g,H_2)}}(r)+O(1)\\&\le T_{\frac{(f,H_1)}{(f,H_2)}}(r)+T_{\frac{(g,H_1)}{(g,H_2)}}(r)+O(1)\le T_f(r)+T_g(r)+O(1).\end{aligned}$$

Similarly,

$$\sum_{i=1}^{2n+1}N_g^{[1]}(r,P_i)\le T_f(r)+T_g(r)+O(1).$$

Thus,

$$\sum_{i=1}^{2n+1}N_f^{[1]}(r,P_i)+\sum_{i=1}^{2n+1}N_g^{[1]}(r,P_i)\le 2(T_f(r)+T_g(r))+O(1).\tag{3.13}$$

By Lemma 3.1, we have

$$(2n+1)T_f(r)\le\sum_{i=1}^{2n+1}N_f^{[n]}(r,P_i)+o(T_f(r))\le n\sum_{i=1}^{2n+1}N_f^{[1]}(r,P_i)+o(T_f(r))$$

(note that $d\ge n+2$). So

$$\frac{2n+1}{n}T_f(r)\le\sum_{i=1}^{2n+1}N_f^{[1]}(r,P_i)+o(T_f(r)).\tag{3.14}$$

Since $\mathrm{Zero}(P_i(f))=\mathrm{Zero}(P_i(g))$ for all $i\in\{1,\dots,2n+1\}$, we have

$$\sum_{i=1}^{2n+1}N_f^{[n]}(r,P_i)=\sum_{i=1}^{2n+1}N_g^{[n]}(r,P_i)\le(2n+1)dT_g(r)+O(1).$$

Combining this with (3.14) we get

$$T_f(r)\le ndT_g(r)+o(T_f(r)).$$

This implies that $\mathcal{R}_f \subset \mathcal{R}_g$. Thus, by Lemma 3.1, similarly to (3.14) we have

$$\frac{2n+1}{n} T_g(r) \le \sum_{i=1}^{2n+1} N_g^{[1]}(r, P_i) + o(T_g(r)).$$

Combining this with (3.13) and (3.14) we obtain

$$\frac{2n+1}{n}(T_f(r) + T_g(r)) \le 2(T_f(r) + T_g(r)) + o(T_f(r) + T_g(r)).$$

This is a contradiction.

Thus, $f \equiv g$, completing the proof of Theorem 1.2. ■

Acknowledgements. The research of the authors is supported by an NAFOSTED grant of Vietnam.

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Received 25.5.2010
and in final form 10.1.2011 (2225)