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On global smoothness preservation in complex approximation

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Abstract. By using the properties of convergence and global smoothness preservation of multivariate Weierstrass singular integrals, we establish multivariate complex Carleman type approximation results with rates. Here the approximants fulfill the global smoothness preservation property. Furthermore Mergelyan's theorem for the unit disc is strengthened by proving the global smoothness preservation property.

1. Introduction. In the theory of approximation of real-valued functions of real variables, the topic of global smoothness preservation has been intensively studied in recent years (see, e.g., the book [1]).

Combining the classical Weierstrass approximation theorem with the global smoothness preservation property of Bernstein polynomials attached to $f \in C[0,1]$, $B_n(f)(x)$, that is,

$$\omega_1(B_n(f);\delta)_{[0,1]} \le 2\omega_1(f;\delta)_{[0,1]}, \quad \forall\delta > 0,\ \forall n \in \mathbb{N},\ \forall f \in C[0,1],$$

where $\omega_1(f;\delta)_{[0,1]} = \sup\{|f(x_1) - f(x_2)|;\ |x_1 - x_2| \le \delta,\ x_1, x_2 \in [0,1]\}$ (see, e.g., [2] or [1, p. 244], we easily obtain the following

Theorem 1.1. *For any $f \in C[0,1]$ and any $\varepsilon > 0$, there exists an algebraic polynomial P such that $|P(x) - f(x)| < \varepsilon$ for all $x \in [0,1]$ and*

$$\omega_1(P;\delta)_{[0,1]} \le 2\omega_1(f;\delta)_{[0,1]}, \quad \forall\delta > 0.$$

Here the constant 2 is optimal.

On the other hand, natural extensions of Weierstrass' theorem to the complex case are the following well-known results.

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THEOREM 1.2 (Scheinberg [7]). *Let $m \in \mathbb{N}$. For every continuous function $f\colon \mathbb{R}^m \to \mathbb{R}$ and every continuous function $\varepsilon\colon \mathbb{R}^m \to \mathbb{R}_+$, there exists an entire function $g\colon \mathbb{C}^m \to \mathbb{C}$ such that*

$$|f(x) - g(x)| < \varepsilon(x), \quad \forall x \in \mathbb{R}^m.$$

REMARK. For $m = 1$ Theorem 1.2 becomes Carleman's result in [3].

THEOREM 1.3 (Mergelyan [6], or e.g., [5, p. 97]). *Let $K \subset \mathbb{C}$ be compact in $\mathbb{C}$ with $\mathbb{C} \setminus K$ connected and suppose $f\colon K \to \mathbb{C}$ is continuous on K and analytic in K^0. Then for any $\varepsilon > 0$, there exists an algebraic polynomial P such that*

$$|f(z) - P(z)| < \varepsilon, \quad \forall z \in K.$$

It is then natural to ask if there exist analogues of Theorem 1.1 for the cases of Theorems 1.2 and 1.3.

In this paper we give some answers to the above question.

2. Global smoothness preservation. Let f be a function defined on $\mathbb{R}^m$ with values in $\mathbb{R}$. Let $x = (x_1, \ldots, x_m)$, $h = (h_1, \ldots, h_m)$, $\delta = (\delta_1, \ldots, \delta_m) \in \mathbb{R}^m$. Set

$$\Delta_h^r f(x) = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} f(x + rh), \quad r \in \mathbb{N},$$

and define the rth L^s-*modulus of smoothness* over $\mathbb{R}^m$, $1 \le s \le \infty$, by

$$\omega_r(f;\delta)_s := \sup\{\|\Delta_h^r f(\cdot)\|_{L^s(\mathbb{R}^m)};\ |h| \le \delta\},$$

where $|h| = (|h_1|, \ldots, |h_m|)$, $|h| \le \delta$ means $|h_i| \le \delta_i$, $i = \overline{1,m}$, and

$$\|f\|_{L^s(\mathbb{R}^m)} := \begin{cases} \left\{ \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} |f(x_1, \ldots, x_m)|^s \, dx_1 \ldots dx_m \right\}^{1/s} & \text{if } 1 \le s < \infty, \\ \sup\{|f(x_1, \ldots, x_m)|;\ x_i \in \mathbb{R},\ i = \overline{1,m}\} & \text{if } s = \infty. \end{cases}$$

Next we introduce the multivariate Jackson-type generalization of the Weierstrass integral:

$$W_{p,n}(f)(x) = -\left(\prod_{i=1}^{m} \frac{n_i}{\sqrt{\pi}}\right) \sum_{k=1}^{p+1} (-1)^k \binom{p+1}{k} \times \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} f(x_1 + kt_1, \ldots, x_m + kt_m) \left(\prod_{i=1}^{m} e^{-n_i^2 t_i^2}\right) dt_1 \ldots dt_m,$$

where $n = (n_1, \ldots, n_m) \in \mathbb{N}^m$, $p \in \mathbb{N} \cup \{0\}$, $x = (x_1, \ldots, x_m)$ and

$$\int\limits_{-\infty}^{\infty} e^{-n_i^2 t_i^2} \, dt_i = \frac{2}{n_i} \int\limits_{0}^{\infty} e^{-t_i^2} \, dt_i = \frac{\sqrt{\pi}}{n_i}, \quad i = \overline{1,m}.$$

First we present

THEOREM 2.1. *Let $f \in L^1(\mathbb{R}^m)$. For $s = 1$ and $s = \infty$ we have*

$$\|f - W_{p,n}(f)\|_{L^s(\mathbb{R}^m)} \le C_{p,m}\omega_{p+1}(f;1/n)_s,$$

where $1/n := (1/n_1, \ldots, 1/n_m)$, $n = (n_1, \ldots, n_m) \in \mathbb{N}^m$, and

$$\omega_r(W_{p,n}(f);\delta)_s \le (2^{p+1}-1)\omega_r(f;\delta)_s, \quad \forall r \in \mathbb{N},\ \forall \delta > 0,$$

where

$$c_{p,m} = \left[\frac{2}{\sqrt{\pi}}\int_0^\infty (u+1)^{p+1}e^{-u^2}\,du\right]^m < \infty.$$

Proof. First let $s = 1$. We obtain

$$f(x) - W_{p,n}(f)(x) = \left(\prod_{i=1}^m \frac{n_i}{\sqrt{\pi}}\right) \times \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}(-1)^{p+1}\Delta_t^{p+1}f(x)\Big(\prod_{i=1}^m e^{-n_i^2t_i^2}\Big)\,dt_1\ldots dt_m \tag{1}$$

for all $x = (x_1, \ldots, x_m)$, $t = (t_1, \ldots, t_m) \in \mathbb{R}^m$, $n = (n_1, \ldots, n_m) \in \mathbb{N}^m$, $p \in \mathbb{N} \cup \{0\}$.

Taking the absolute value, then integrating with respect to x over $\mathbb{R}^m$, and defining

$$|t|/n = (|t_1|/n_1, \ldots, |t_m|/n_m), \quad n|t| = (n_1|t_1|, \ldots, n_m|t_m|),$$

we get

$$\begin{aligned}
&\|f - W_{p,n}(f)\|_{L^1(\mathbb{R}^m)}\\
&\quad\le \left(\prod_{i=1}^m \frac{n_i}{\sqrt{\pi}}\right)\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\omega_{p+1}(f;n\cdot|t|/n)_1\Big(\prod_{i=1}^m e^{-n_i^2t_i^2}\Big)\,dt_1\ldots dt_m\\
&\quad\le \left(\prod_{i=1}^m \frac{n_i}{\sqrt{\pi}}\right)\omega_{p+1}(f;1/n)_1\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\Big[\prod_{i=1}^m(1+n_i|t_i|)\Big]^{p+1}\\
&\qquad\times\Big(\prod_{i=1}^m e^{-n_i^2t_i^2}\Big)\,dt_1\ldots dt_m\\
&\quad= \left(\prod_{i=1}^m \frac{n_i}{\sqrt{\pi}}\right)\omega_{p+1}(f;1/n)_1\left(\prod_{i=1}^m\frac{2}{n_i}\right)\Big(\int_0^\infty(1+u)^{p+1}e^{-u^2}\,du\Big)^m\\
&\quad= \left(\frac{2}{\sqrt{\pi}}\right)^m\Big(\int_0^\infty(1+u)^{p+1}e^{-u^2}\,du\Big)^m\omega_{p+1}(f;1/n)_1,
\end{aligned}$$

because

$$\int_{-\infty}^{\infty}(1+n_i|t_i|)^{p+1}e^{-n_i^2t_i^2}\,dt_i = \frac{2}{n_i}\int_0^\infty(1+u)^{p+1}e^{-u^2}\,du, \quad \forall i = \overline{1,m}.$$

Now, let $r \in \mathbb{N}$ and fix $\delta = (\delta_1, \ldots, \delta_m) > 0$ (i.e., $\delta_i > 0$, $i = \overline{1,m}$). For any $h = (h_1, \ldots, h_m)$ with $|h| \le \delta$, we have

$$(2) \qquad \Delta_h^r[W_{p,n}(f)](x) = -\left(\prod_{i=1}^m \frac{n_i}{\sqrt{\pi}}\right) \sum_{k=1}^{p+1} (-1)^k \binom{p+1}{k} \times \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \Delta_h^r f(x+kt) \Big(\prod_{i=1}^m e^{-n_i^2 t_i^2}\Big) dt_1 \ldots dt_m.$$

Taking in (2) the absolute value, integrating and taking into account that by $|h| \le \delta$ it follows that

$$\int_{\mathbb{R}^m} |\Delta_h^r f(x+t)|\, dt \le \omega_r(f;\delta)_1,$$

we finally obtain

$$\omega_r(W_{p,n}(f);\delta)_1 \le \sum_{k=1}^{p+1} \binom{p+1}{k} \omega_r(f;\delta)_1 = (2^{p+1}-1)\omega_r(f;\delta)_1.$$

For the case $s = \infty$, by using the relations (1) and (2) above, the reasoning is similar; this establishes the theorem. ∎

Corollary 2.2. *Let $f \in L^1(\mathbb{R}^m)$. For $s = 1$ and $s = \infty$, there exists an entire function depending on f, $W(f)\colon \mathbb{C}^m \to \mathbb{C}$, that satisfies the estimates of Theorem* 2.1.

Proof. By making the substitutions $x_i + kt_i = u_i$, $i = \overline{1,m}$, in $W_{p,n}(f)(x)$, $x \in \mathbb{R}^m$, we easily obtain

$$W_{p,n}(f)(x) = -\left(\prod_{i=1}^m \frac{n_i}{\sqrt{\pi}}\right) \sum_{k=1}^{m} (-1)^k \binom{p+1}{k} \frac{1}{k^m} \times \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} f(u_1, \ldots, u_m) \Big(\prod_{i=1}^m e^{-n_i^2 (u_i - x_i)^2/k^2}\Big)\, du_1 \ldots du_m.$$

If we replace now $x \in \mathbb{R}^m$ by $z \in \mathbb{C}^m$, then obviously $W_{p,n}(f)(z)$ becomes an entire function, which proves the corollary. ∎

Corollary 2.3. *Let $f \in L^1(\mathbb{R}^m)$.*

(i) *For any $\varepsilon > 0$ and any $r \in \mathbb{N}$, there exists an entire function $g\colon \mathbb{C}^m \to \mathbb{C}$ such that*

$$\|f-g\|_{L^1(\mathbb{R}^m)} < \varepsilon \quad \text{and} \quad \omega_1(g;\delta)_1 \le c\,\omega_1(f;\delta)_1$$

for all $\delta > 0$, where $c > 0$ is an absolute constant (*i.e., independent of f, m, ε and δ*).

(ii) *If moreover f is uniformly continuous on $\mathbb{R}^m$, then for any $\varepsilon > 0$ and any $r \in \mathbb{N}$, there exists an entire function $g\colon \mathbb{C}^m \to \mathbb{C}$ such that*

$|f(x) - g(x)| < \varepsilon$ for all $x \in \mathbb{R}^m$ and $\omega_1(g;\delta)_\infty \le c\omega_1(f;\delta)_\infty$ for all $\delta > 0$, where $c > 0$ is an absolute constant.

Proof. (i) This is immediate by Theorem 2.1 and Corollary 2.2, because if $f \in L^1(\mathbb{R}^m)$, then $\omega_r(f;1/n)_1 \to 0$ as $n \to \infty$ (here $n = (n_1, \dots, n_m) \to \infty$ means $n_1 \to \infty, \dots, n_m \to \infty$.

(ii) This is also immediate by Theorem 2.1 and Corollary 2.2, because f being uniformly continuous on $\mathbb{R}^m$ implies that $\omega_r(f;1/n)_\infty \to 0$ as $n \to \infty$. ∎

Open Question. For any uniformly continuous function $f\colon \mathbb{R}^m \to \mathbb{R}$, any continuous error function $\varepsilon\colon \mathbb{R}^m \to \mathbb{R}_+$ and any $r \in \mathbb{N}$, does there exist an entire function $g\colon \mathbb{C}^m \to \mathbb{C}$ such that $|f(x) - g(x)| < \varepsilon(x)$ for all $x \in \mathbb{R}^m$ and

$$\omega_r(g;\delta)_\infty \le c\omega_r(f;\delta)_\infty, \quad \forall \delta > 0,$$

where $c > 0$ is an absolute constant?

Remarks. 1) It is known (see, e.g., [4, p. 285]) that the order λ of an entire function $g\colon \mathbb{C} \to \mathbb{C}$ is given by

$$\lambda = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}, \quad \text{where} \quad M(r) = \max\{|f(z)|;\, |z| = r\}.$$

For $p = 0$ and $m = 1$, $W_{p,n}(f)(z)$ becomes the usual Weierstrass integral. In this case, easy calculations show that the order of $W_{p,n}(f)(z)$ is ≤ 2 if we suppose in addition that f is bounded on $\mathbb{R}$.

2) For $K = \{z \in \mathbb{C};\, |z| \le 1\}$, consider the following operator attached to a function f, continuous on K and analytic on K^0:

$$F_n(f)(z) = \frac{1}{2\pi n} \int_0^{2\pi} f(ze^{iu})\Phi_n(u)\,du, \quad \forall z = re^{ix} \in K,$$

where $\Phi_n(u) = \left(\frac{\sin\frac{nu}{2}}{\sin\frac{u}{2}}\right)^2$ is the Fejér kernel.

First we prove that $F_n(f)(z)$ represents in fact the complex Fejér polynomials of degree $n-1$, given by

$$\frac{1}{n}\sum_{j=0}^{n-1}(n-j)a_j z^j, \quad \text{where} \quad f(z) = \sum_{j=0}^{\infty} a_j z^j$$

(see, e.g., [5, p. 53]).

Indeed, because f is analytic on K^0, we can write

$$f(ze^{iu}) = \sum_{k=0}^{\infty} a_k (ze^{iu})^k = \sum_{k=0}^{\infty} a_k z^k e^{iuk}.$$

On the other hand, if we set $w = e^{iu/2}$, then by the general formula $\sin\alpha = (e^{i\alpha} - e^{-i\alpha})/(2i)$, we get

$$\Phi_n(u) = \left(\frac{w^n - 1/w^n}{w - 1/w}\right)^2 = \frac{1}{(e^{iu})^{n-1}}\,[(e^{iu})^{n-1} + (e^{iu})^{n-2} + \ldots + e^{iu} + 1]^2.$$

Also writing $e^{iu} = t$, we obtain

$$\begin{aligned}
&\frac{1}{t^{n-1}}\,(t^{n-1} + t^{n-2} + \ldots + t + 1)^2 \\
&\quad = \frac{1}{t^{n-1}}\,\{t^{2(n-1)} + t^{2(n-2)} + \ldots + t^2 + 1 + 2[t + t^2 + \ldots + t^{n-1} \\
&\qquad + (t^3 + t^4 + \ldots + t^{n-1} + t^n) + (t^5 + \ldots + t^{n-1} + t^n + t^{n+1}) \\
&\qquad + \ldots + (t^{2n-5} + t^{2n-4}) + t^{2n-3}]\} \\
&\quad = \frac{1}{t^{n-1}}\left\{\sum_{k=n}^{2n-2} c_k t^k + [1 + 2t + 3t^2 + \ldots + nt^{n-1}]\right\} \\
&\quad = \sum_{p=n}^{2n-2} c_p t^{p-(n-1)} + \sum_{j=0}^{n-1}(n-j)t^{-j},
\end{aligned}$$

that is,

$$\Phi_n(u) = \sum_{p=n}^{2n-2} c_p e^{iu[p-(n-1)]} + \sum_{j=0}^{n-1}(n-j)e^{-iju}.$$

In general we have

$$\int_0^{2\pi} e^{i(k+\lambda)u}du = \begin{cases} 0 & \text{if } k+\lambda \neq 0, \\ 2\pi & \text{if } k = -\lambda. \end{cases}$$

Integrating with respect to u the product

$$\begin{aligned}
f(ze^{iu})\Phi_n(u) &= \left(\sum_{k=0}^{\infty} a_k z^k e^{iuk}\right)\left[\sum_{p=n}^{2n-2} c_p e^{iu[p-(n-1)]} + \sum_{j=0}^{n-1}(n-j)e^{-iju}\right] \\
&= \sum_{k=0}^{\infty} b_k z^k e^{iu(k+\lambda_k)} + \sum_{k=0}^{\infty}\sum_{j=0}^{n-1} a_k z^k (n-j) e^{iu(k-j)},
\end{aligned}$$

where $\lambda_k > 0$ for all $k \in \mathbb{N}$, we immediately obtain

$$\frac{1}{2\pi n}\int_0^{2\pi} f(ze^{iu})\Phi_n(u)\,du = \frac{1}{n}\sum_{j=0}^{n-1} a_j z^j (n-j).$$

It is well known (see [5, p. 53]) that complex Fejér polynomials satisfy the assertion of Theorem 1.3 when K is the unit disc.

Now, let $z_1, z_2 \in K$, $\delta > 0$, $|z_1 - z_2| \le \delta$. We have

$$|F_n(f)(z_1) - F_n(f)(z_2)| \le \frac{\sqrt{2}}{2\pi n} \int_0^{2\pi} |f(z_1 e^{iu}) - f(z_2 e^{iu})| \Phi_n(u)\, du$$
$$\le \sqrt{2}\, \omega_1(f; |z_1 - z_2|)_K,$$

because $|z_1 e^{iu} - z_2 e^{iu}| = |z_1 - z_2|$ (here $\omega_1(f;\delta)_K = \sup\{|f(z_1) - f(z_2)|;\ z_1, z_2 \in K, |z_1 - z_2| \le \delta\}$.

This immediately implies the global smoothness preservation property

$$\omega_1(F_n(f);\delta)_K \le \sqrt{2}\, \omega_1(f;\delta)_K, \quad \forall \delta > 0,\ \forall n \in \mathbb{N}.$$

3) It is an open question if for general K and f with the properties as in Theorem 1.3, there exists a polynomial $P(z)$ satisfying $|P(z) - f(z)| < \varepsilon$ for all $z \in K$ and moreover

$$\omega_1(P;\delta)_K \le C\omega_1(f;\delta)_K, \quad \forall \delta > 0,$$

with some constant $C > 0$.

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