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Quasi-homogeneous linear systems on $\mathbb{P}^2$ with base points of multiplicity 7, 8, 9, 10

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Abstract. We prove that the Segre–Gimigliano–Harbourne–Hirschowitz conjecture holds for quasi-homogeneous linear systems on $\mathbb{P}^2$ for $m = 7, 8, 9, 10$, i.e. systems of curves of a given degree passing through points in general position with multiplicities at least $m, \dots, m, m_0$, where $m = 7, 8, 9, 10$, m_0 is arbitrary.

1. Introduction. In what follows we assume that the ground field $\mathbb{K}$ is of characteristic zero. Let $d \in \mathbb{Z}$ and $m_1, \dots, m_r \in \mathbb{N}$. Pick r *base points* $p_1, \dots, p_r \in \mathbb{P}^2 := \mathbb{P}^2(\mathbb{K})$. We denote by $\mathcal{L}(d; m_1, \dots, m_r)$ the linear system of curves of degree d passing through points $p_1, \dots, p_r$ with multiplicities at least $m_1, \dots, m_r$, respectively:

$$\mathcal{L}(d; m_1, \dots, m_r) := \{C \subset \mathbb{P}^2 : \deg(C) = d, \\ \operatorname{mult}_{p_j}(C) \geq m_j \text{ for } j = 1, \dots, r\}$$

(by a curve $C \subset \mathbb{P}^2$ of degree $\deg(C) = d$ we understand its defining homogeneous polynomial). The dimension of such a system (as a projective linear space) is denoted by $\dim \mathcal{L}(d; m_1, \dots, m_r)$. Define the *virtual dimension* of $L = \mathcal{L}(d; m_1, \dots, m_r)$ to be

$$\operatorname{vdim} L := \binom{d+2}{2} - \sum_{j=1}^{r} \binom{m_j + 1}{2} - 1$$

and the *expected dimension* of L to be

$$\operatorname{edim} L := \max\{\operatorname{vdim} L, -1\}.$$

Observe that $\dim L \geq \operatorname{edim} L$. If this inequality is strict then L is called *special* (*non-special* otherwise). The system L is called *non-empty* if $\dim L \geq 0$ (*empty* otherwise). We will use the notation $m^{\times p}$ for repeated multiplici-

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ties,

$$m^{\times p} := \underbrace{m, \ldots, m}_{p}.$$

Let $\pi : X \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at $p_1, \ldots, p_r$ with exceptional divisors $E_1, \ldots, E_r$, respectively. The Picard group $\mathrm{Pic}(X)$ of X is generated by $H, E_1, \ldots, E_r$, where H is the pullback of the class of a line in $\mathbb{P}^2$. The system $\mathcal{L}(d; m_1, \ldots, m_r)$ is isomorphic to the complete linear system L (on X) associated to the divisor $dH - m_1E_1 - \cdots - m_rE_r$. Observe that in this way we can define $\mathcal{L}(d; m_1, \ldots, m_r)$ for $d, m_1, \ldots, m_r \in \mathbb{Z}$. In what follows we always allow negative multiplicities unless stated otherwise.

Consider the standard intersection form on X given by $H^2 = 1$, $E_j^2 = -1$, $H.E_j = 0$, $E_j.E_\ell = 0$ for $j \neq \ell$. Now (by Riemann–Roch)

$$\text{(1)} \qquad \operatorname{vdim} L = \frac{L^2 - L.K_X}{2},$$

where K_X is the canonical divisor on X.

Definition 1. We say that a curve $C \subset X$ is a -1-*curve* on X if C is irreducible and $C^2 = C.K_X = -1$.

We recall the following definition of a -1-special system (see e.g. [Cil-Mir 98]):

Definition 2. A linear system $L = \mathcal{L}(d; m_1, \ldots, m_r)$ (considered as a system on X) is -1-*special* if there exist -1-curves $C_1, \ldots, C_s \subset X$ such that

- $L.C_j = -k_j$, $k_j \geq 1$ for $j = 1, \ldots, s$,
- $k_j \geq 2$ for some j,
- the system $M = L - (k_1C_1 + \cdots + k_sC_s)$ has non-negative virtual dimension and non-negative intersection with every -1-curve.

From the above definition it is clear that if L is -1-special then it is non-empty and its dimension is at least $\dim M$. To compare the virtual dimensions of L and M, assume that $L.C_j = -k_j$ for $j = 1, \ldots, s$. By formula (1) and Definition 1, we obtain

$$\operatorname{vdim} L = \operatorname{vdim} M + \sum_{j=1}^{s} \frac{k_j - k_j^2}{2},$$

hence every -1-special system is special.

The converse is only conjectured to hold (see e.g. [Hir 89]), and only for base points in general position:

Conjecture 3 (Segre–Gimigliano–Harbourne–Hirschowitz). *A linear system $\mathcal{L}(d; m_1, \ldots, m_r)$ with base points in general position is special if and only if it is -1-special.*

The above formulation, involving -1-curves, makes sense only for base points in general position. In the non-general case we would also have to consider curves with self-intersection less than -1 and the problem seems to be far more complicated. In this paper we deal with the general case since the main methods (the reduction algorithm and Cremona transformations of systems) work only for base points in general position. So from now on we always assume that the base points are in general position.

The Segre–Gimigliano–Harbourne–Hirschowitz conjecture is known to hold in some cases. The case $r \leq 9$ has been solved in [Nag 60]. For low multiplicities (i.e. bounded by some constant) it has begun with [Hir 85], where the case $m_1 = \cdots = m_r \leq 3$ has been solved. The case when all multiplicities are bounded by 4 has been settled by [Mig 00]; the bound has been extended to 7 in [Yan 07] and 11 in [Dum-Jar 07].

The homogeneous case ($m_1 = \cdots = m_r$) with multiplicities up to 42 has been solved in [Dum 07a]. The quasi-homogeneous case ($m_1 = \cdots = m_{r-1}$, m_r arbitrary) has been handled for $m_1 = 3$ in [Cil-Mir 98], for $m_1 = 4$ in [Sei 01, Laf 99], for $m_1 = 5$ in [Laf-Uga 03], and for $m_1 = 6$ in [Kun 05]. Our result is the following:

THEOREM 4. *The Segre–Gimigliano–Harbourne–Hirschowitz conjecture holds for quasi-homogeneous systems with almost all multiplicities equal to* 7, 8, 9 *or* 10, *i.e. for systems* $\mathcal{L}(d; m^{\times r}, m_0)$ *with* $r+1$ *base points in general position, with multiplicities at least* $m, \dots, m, m_0$ *respectively,* $m \in \{7, 8, 9, 10\}$.

The methods of proving the conjecture for $m = 3, 4, 5, 6$ used by the authors mentioned above are of the same type. Namely, using the degeneration method introduced in [Cil-Mir 98], we can show the non-speciality of a large family of systems with many base points. In fact, with the help of this method, we can show that if the family of systems

$$\{\mathcal{L}(d; m^{\times r}, m_0) : d, m_0 \in \mathbb{N},\ r_1 \leq r \leq r_2\}$$

(for carefully chosen r_1 and r_2) contains only non-special ones then all systems of the form $\mathcal{L}(d; m^{\times r}, m_0)$ for $r \geq r_2$ are non-special. Another task is to show that if the difference between d and m_0 in the system $\mathcal{L}(d; m^{\times r}, m_0)$ is large enough then the system is non-special.

Having the above, we are left with a family of cases that can be solved using the degeneration method, Cremona transformations, ad hoc arguments and computations of the rank of interpolation matrices.

The authors of [Cil-Mir 98, Sei 01, Laf 99, Laf-Uga 03, Kun 05] used computer programs to deal with a large number of cases. These programs are of two kinds. The first one is an implementation of the degeneration technique—for a large number of cases we must check whether the degeneration exists or not. The result (for each single case) can be easily checked by hand, the reason for using software is the number of cases. The second kind uses computer programs to evaluate the dimension of a given system by direct computation of the rank of the interpolation matrix, which, in interesting cases, is of large size (e.g. 105×105). This cannot be done by hand, for obvious reasons.

In this paper we use the same approach, but different methods. Instead of the degeneration method we use the reduction algorithm introduced in [Dum 07b] and [Dum-Jar 07] together with direct computation of dimensions of systems. To deal with the remaining cases we use the Cremona transformation (see Definition 7), "glueing" of points (see Theorem 6) and known results.

We note here that both approaches, by the degeneration and the reduction algorithm, promise to be usable for larger values of m (quasi-homogeneous multiplicity). We prefer the second one—observe that this paper is not much longer than [Sei 01, Laf-Uga 03, Kun 05], although we deal with four higher multiplicities at once.

The paper is organized as follows: The next section is devoted to presenting some methods of showing -1-speciality or non-speciality of systems. Section 3 contains a brief introduction to the reduction method together with the results obtained with the help of this method and computer programs. In Section 4 we deal with the remaining cases, i.e. systems with few base points and low difference between the degree and the quasi-homogeneous multiplicity. The last section contains a note on Seibert's work.

2. Tools. We begin by recalling [Dum 09b, Theorem 1]. Let us recall that the base points are always in general position.

THEOREM 5 (splitting). *Let* $d, k, m_1, \ldots, m_r, m_{r+1}, \ldots, m_s \in \mathbb{N}$. *If*

- $L_1 = \mathcal{L}(k; m_1, \ldots, m_s)$ *is non-special,*
- $L_2 = \mathcal{L}(d; m_{s+1}, \ldots, m_r, k+1)$ *is non-special,*
- $(\operatorname{vdim} L_1 + 1)(\operatorname{vdim} L_2 + 1) \geq 0$,

then the system $L_3 = \mathcal{L}(d; m_1, \ldots, m_r)$ *is non-special.*

However, it would be more convenient to "glue" equal multiplicities, so we will use the following weaker version of the above theorem.

THEOREM 6 (glueing). *Let* $\mathcal{L}(k; m^{\times s})$ *be non-special, and set*

$$L_3 = \mathcal{L}(d; m_1, \ldots, m_r, m^{\times s}), \qquad L_2 = \mathcal{L}(d; m_1, \ldots, m_r, k+1).$$

If either $-1 \leq \operatorname{vdim} L_2 \leq \operatorname{vdim} L_3$ *or* $\operatorname{vdim} L_3 \leq \operatorname{vdim} L_2 \leq -1$ *then in order to show non-speciality of* L_3 *it is enough to show non-speciality of* L_2.

Proof. Follows immediately from Theorem 5. ∎

Definition 7. Let $d, m_1, \ldots, m_r \in \mathbb{Z}$ and $k = d - (m_1 + m_2 + m_3)$. Define the *Cremona transformation of the system* $\mathcal{L}(d; m_1, \ldots, m_r)$ to be

$$\mathrm{Cr}(\mathcal{L}(d; m_1, \ldots, m_r)) := \mathcal{L}(d + k; m_1 + k, m_2 + k, m_3 + k, m_4, \ldots, m_r).$$

Theorem 8. *Let* $L = \mathcal{L}(d; m_1, \ldots, m_r)$ *be a linear system. Then:*

(1) $\dim \mathrm{Cr}(L) = \dim L$,
(2) L *is special if and only if* $\mathrm{Cr}(L)$ *is special,*
(3) L *is* -1*-special if and only if* $\mathrm{Cr}(L)$ *is* -1*-special.*

Proof. The proof can be found, for example, in [Gim 89]. The idea is to show that the Cremona transformation induces an action on $\mathrm{Pic}(X)$ such that $H \mapsto 2H - E_1 - E_2 - E_3$, $E_j \mapsto H - (E_1 + E_2 + E_3) + E_j$ for $j = 1, \ldots, 3$ and $E_j \mapsto E_j$ for $j \geq 4$. Observe that -1-curves are transformed into -1-curves. ■

Definition 9. We say that $\mathcal{L}(d; m_1, \ldots, m_r)$ *is in standard form* if either $d < 0$, or:

- $m_1, \ldots, m_r$ are non-increasing,
- $d - (m_1 + m_2 + m_3) \geq 0$.

Every system can be transformed (by a finite number of Cremona transformations and sorting of multiplicities) into a standard form. For a system L choose one of its standard forms and denote it by $\mathrm{Cr}^\circ(L)$. The standard form is not unique, since e.g. $\mathrm{Cr}(\mathcal{L}(-1; 1, 0, 0)) = \mathcal{L}(-3; -1, -2, -2)$ and both systems are in standard form. However, if a system cannot be transformed (by a sequence of Cremona transformations) into another system with negative degree then its standard form is unique. To see this, consider the following anti-symmetric relation:

$$(d; m_1, \ldots, m_r) \sim (d'; m'_1, \ldots, m'_r)$$

if and only if $d > d'$ and we can map $\mathcal{L}(d; m_1, \ldots, m_r)$ to $\mathcal{L}(d'; m'_1, \ldots, m'_r)$ by a single Cremona transformation. Now the relation is obviously noetherian, and by easy computation it can be shown that it is locally confluent (by definition, $\sim$ is *locally confluent* if whenever $a \sim b_1$, $a \sim b_2$ there exists c such that $b_1 \sim^* c$ and $b_2 \sim^* c$, where $\sim^*$ denotes a sequence of $\sim$'s). Such relations enjoy the property of having unique normal forms, which, in our case, are exactly standard forms.

Let $L = \mathcal{L}(d; m_1, \ldots, m_r)$ be a linear system in standard form. From [Gim 89] we may understand what happens if some of $d, m_1, \ldots, m_r$ are negative:

(1) If $d < 0$ then L is empty.
(2) If $m_j = -1$ then E_j is a fixed component for L; let

$$L' = \mathcal{L}(d; m_1, \ldots, m_{j-1}, 0, m_{j+1}, \ldots, m_r).$$

Since $\mathrm{vdim}\, L = \mathrm{vdim}\, L'$ and $\dim L = \dim L'$, it is enough to study L'.

(3) If $m_j \leq -2$ then E_j is a multiple fixed component; since E_j is a -1-curve, the system L is special if and only if

$$L' = \mathcal{L}(d; m_1, \ldots, m_{j-1}, 0, m_{j+1}, \ldots, m_r)$$

is non-empty. Moreover, if L' is non-empty and non-special, or if it is -1-special, then L is -1-special.

Observe also that the intersection number $L.C$, where C is a -1-curve, is non-negative for any system in standard form with non-negative multiplicities (see [Gim 89]), hence such a system cannot be -1-special. If, additionally, it is a system for which the Segre–Gimigliano–Harbourne–Hirschowitz conjecture has been proved (e.g. multiplicities bounded by 11 or based on at most 9 points) then it is non-special.

We recall the following result (mentioned in the introduction).

THEOREM 10 ([Dum-Jar 07]). *The Segre–Gimigliano–Harbourne–Hirschowitz conjecture holds for systems with multiplicities bounded by* 11.

3. Results using the reduction method. The first step is to show that the systems with the number of base points large enough and the difference between the degree and the free multiplicity greater than $m-2$ are non-special. To do this we will use the reduction method introduced in [Dum 07b] and then exploited in [Dum-Jar 07]. The results are gathered in Table 1.

For a finite $D \subset \mathbb{T}^2 := \{x^{\alpha_1} y^{\alpha_2} \subset \mathbb{K}[x,y] : \alpha_1, \alpha_2 \in \mathbb{N}\}$ and multiplicities $m_1, \ldots, m_r$ define

$$V(D; m_1, \ldots, m_r) := \Big\{ f = \sum_{t \in D} c_t t \in \mathbb{K}[x,y] : \operatorname{mult}_{p_j}(f) \geq m_j \Big\}$$

for $p_1, \ldots, p_r \in \mathbb{K}^2$ in general position. We say that $V(D; m_1, \ldots, m_r)$ is *non-special* if its dimension (as a vector space over $\mathbb{K}$) is equal to its expected dimension

$$\operatorname{edim} V(D; m_1, \ldots, m_r) := \max\{\operatorname{vdim} V(D; m_1, \ldots, m_r), 0\},$$
$$\operatorname{vdim} V(D; m_1, \ldots, m_r) := \#D - \sum_{j=1}^{r} \binom{m_j + 1}{2}.$$

Observe that $\mathcal{L}(d; m_1, \ldots, m_r)$ is non-special if and only if $V(D; m_1, \ldots, m_r)$ is non-special for $D = \{t \in \mathbb{T}^2 : \deg t \leq d\}$.

DEFINITION 11. Let $a_1, \ldots, a_k \in \mathbb{N}$, $a_j \leq j$, $j = 1, \ldots, k$. Define the *diagram*

$$(a_1, \ldots, a_k) = \bigcup_{j=1}^{k} \{x^{\alpha_1} y^{\alpha_2} \in \mathbb{T}^2 : \alpha_1 + \alpha_2 = j - 1,\ \alpha_2 < a_j\}.$$

A single set $\{x^{\alpha_1}y^{\alpha_2} \in \mathbb{T}^2 : \alpha_1 + \alpha_2 = j-1,\ \alpha_2 < a_j\}$ will be called the *jth layer*, or simply a *layer*. For $a, a_1, \dots, a_k \in \mathbb{N}$ with $a_j \le a + j$ define

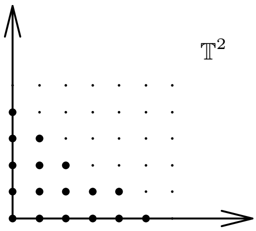
$$(\overline{a}, a_1, \dots, a_k) := (1, 2, \dots, a-1, a, a_1, \dots, a_k).$$

We will also use the notation

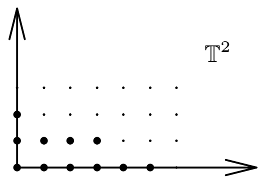
$$(\overline{a}, \{b\}^{\times p}, a_1, a_2, \dots) := (\overline{a}, \underbrace{b, \dots, b}_{p \text{ times}}, a_1, a_2, \dots).$$

Observe that for $d \ge 1$ we have $\{t \in \mathbb{T}^2 : \deg t \le d\} = (\overline{d+1})$.

EXAMPLE 12.



the diagram $(\overline{5}, 2)$



the diagram $(\overline{3}, 2, 2, 1)$

DEFINITION 13. Let $m \in \mathbb{N}^*$, and let $D = (b_1, \dots, b_\ell, a_1, \dots, a_m)$ be a diagram with $a_m > 0$. Define $v_j \in \mathbb{N}$, $j = 1, \dots, m$, and sets V_j, $j = 0, \dots, m$ by downward induction (beginning with m, going down to 0):

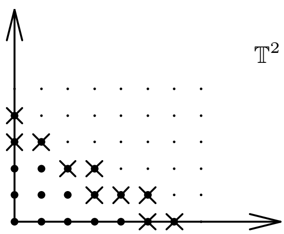
$$V_m := \{1, \dots, m\},$$
$$V_{j-1} := V_j \setminus \{v_j\}, \qquad v_j := \begin{cases} a_j, & a_j < m \text{ and } \max V_j \ge a_j, \\ \max V_j, & \text{otherwise.} \end{cases}$$

If $V_0 = \emptyset$ then we say that *D is m-reducible*. The diagram

$$\mathrm{red}_m(D) := (b_1, \dots, b_\ell, a_1 - v_1, \dots, a_m - v_m)$$

will be called the *m-reduction of D*.

EXAMPLE 14.



the 4-reduction of $(\overline{5}, 3, 2)$ is equal to $(\overline{3}, 3, 1)$

As another example take the diagram $(\overline{32})$ and perform one 12-reduction and four 9-reductions to obtain the diagram $(\overline{19}, 18, 17, 16, 14, 10, 5)$:

$(\overline{19},$	20,	21,	22,	23,	24,	25,	26,	27,	28,	29,	30,	31,	32)
		−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11	−12
$(\overline{19},$	20,	20,	20,	20,	20,	20,	20,	20,	20,	20,	20,	20,	20)
					−1	−2	−3	−4	−5	−6	−7	−8	−9
$(\overline{19},$	20,	20,	20,	20,	19,	18,	17,	16,	15,	14,	13,	12,	11)
					−1	−2	−3	−4	−5	−6	−7	−8	−9
$(\overline{19},$	20,	20,	20,	20,	18,	16,	14,	12,	10,	8,	6,	4,	2)
					−1	−3	−5	−7	−9	−8	−6	−4	−2
$(\overline{19},$	20,	20,	20,	20,	17,	13,	9,	5,	1)				
	−2	−3	−4	−6	−7	−8	−9	−5	−1				
$(\overline{19},$	18,	17,	16,	14,	10,	5)							

We can perform another five 9-reductions to obtain $(\overline{6}, 6, 6, 5, 5, 2)$. The sequence of reductions presented above will be used later to show non-speciality of $\mathcal{L}(31; 12, 9^{\times 9})$.

Definition 15. For an m_r-reducible diagram D we will say that the space

$$V(\mathrm{red}_{m_r}(D); m_1, \ldots, m_{r-1})$$

is the *m_r-reduction* of $V(D; m_1, \ldots, m_r)$.

The reduction method is based on the following fact (see [Dum 07b] for the detailed proof; also a sketch of proof can be found in [Dum-Jar 07]):

Theorem 16. *Let $m_1, \ldots, m_r \in \mathbb{N}$. Let $V = V(D; m_1, \ldots, m_r)$. If D is m_r-reducible and the m_r-reduction of V is non-special then V is non-special.*

Let $V = V(D; m_1, \ldots, m_r)$. We can reduce V until all the multiplicities disappear or the resulting diagram is no longer m_j-reducible for all the remaining m_j's. Observe that an m-reduction is performed on the last m layers of a diagram. Let $D = (\ldots, b_1, \ldots, b_m)$. We will try to reduce D to $(\ldots, b_1', \ldots, b_m')$. Observe that D is m-reducible if and only if, at each step of the reduction procedure described in Definition 13, the set V_{j-1} has exactly one element less than V_j. If $\#V_{j-1} = \#V_j - 1$ then we say that *the m-reduction is possible on the b_j-layer*. We have the following:

- if $b_j + 1 \geq b_{j+1} \geq 2m$ then the m-reduction on the b_j-layer is possible; after the reduction we will have $b_j' \geq b_{j+1}' \geq m$;
- if $b_j \geq b_{j+1} \geq m$ then the m-reduction on the b_j-layer is possible; after the reduction we will have $b_j' > b_{j+1}'$;
- finally, if $b_j > b_{j+1}$ then the m-reduction on the b_j-layer is possible.

In [Dum 07b] one can find additional information on how long one can reduce. We deduce that a diagram $D = (\overline{b}, \{b\}^{\times p})$ for $b \geq m$ and a given $\ell \leq p$ can be reduced (using m-reductions; if $p - \ell \leq m - 1$ then we can use no reduction) to a diagram $(\overline{b}, \{b\}^{\times \ell}, b_1, \ldots, b_{m-1})$. If, moreover, $2m \leq c \leq b$ then D can be reduced to a diagram $(\overline{c}, b_1, \ldots, b_{m-1})$.

We will use reductions to show non-speciality of large families of systems. First define (for a diagram D and $m > 0$)

$$p(D) := \left\lfloor \frac{\#D}{\binom{m+1}{2}} \right\rfloor.$$

PROPOSITION 17. *Let $m > 0$, $k \geq m-1$, and $s \geq 0$. Set*

$$\mathcal{D} = \{(\overline{k+1}, \{k+1\}^{\times s}, a_1, \ldots, a_{m-1}) : k+1 \geq a_1 \geq \cdots \geq a_{m-1}\}.$$

If for all $D \in \mathcal{D}$ the spaces $V(D; m^{\times p(D)})$ and $V(D; m^{\times (p(D)+1)})$ are non-special then for every $r \geq \max\{p(D) : D \in \mathcal{D}\} + 2$ and $m_0 \in \mathbb{N}$ the system $\mathcal{L}(m_0 + k; m_0, m^{\times r})$ is non-special.

Proof. We show that $V(D; m_0, m^{\times r})$ is non-special for $D = (\overline{m_0 + k + 1})$. We can m_0-reduce our space to $V' = V((\overline{k+1}, \{k+1\}^{\times \ell}); m^{\times r})$ for some $\ell \geq 0$. If $\ell < s$ then $\operatorname{vdim} V' < 0$ and, since r is large enough, the same holds for $V((\overline{k+1}, \{k+1\}^{\times s}); m^{\times r})$. So, without loss of generality, we may assume $\ell \geq s$. Performing m-reductions on the diagram $(\overline{k+1}, \{k+1\}^{\times \ell})$ (which is possible due to the preceding discussion) leads to some diagram $D \in \mathcal{D}$, or we obtain a system without conditions. In any case, using Theorem 16, we complete the proof. ∎

PROPOSITION 18. *Let $m \in \mathbb{N}$ and suppose $k + 1 \geq 2m$. Set*

$$\mathcal{D} = \{(\overline{k+1}, a_1, \ldots, a_{m-1}) : k+1 \geq a_1 - 1 \geq a_2 - 2 \geq \cdots \geq a_{m-1} - (m-1)\}.$$

If for all $D \in \mathcal{D}$ the spaces $V(D; m^{\times p(D)})$ and $V(D; m^{\times (p(D)+1)})$ are non-special then for every $m_0 \in \mathbb{N}$, $r \geq \max\{p(D) : D \in \mathcal{D}\} + 2$ and $d \geq m_0 + k$ the system $\mathcal{L}(d; m_0, m^{\times r})$ is non-special.

Proof. The proof is analogous to the previous one. We begin with an m_0-reduction to obtain $(\overline{b}, \{b\}^{\times \ell})$ for $b = d + 1 - m_0 \geq k + 1$ and some $\ell \in \mathbb{N}$. The last diagram can be reduced to some $D \in \mathcal{D}$, or we end up with a system without conditions. ∎

For given m, k and s the set $\mathcal{D}$ defined in Proposition 17 or 18 can be very large. On the other hand we do not need to consider diagrams which cannot be obtained as reductions of diagrams of type $(\overline{k+1}, \{k+1\}^{\times \ell})$.

PROPOSITION 19. *Let $D = (\ldots, a, b, \ldots)$ be a diagram obtained by a sequence of m-reductions from a diagram $D = (\ldots, a', b', \ldots)$ (a' and b' stand at the same position as a and b). If $b > 0$ then*

$$(*) \qquad a + (a - b + b' - a')m \geq a'.$$

Proof. Assume the contrary. Each reduction working on the a-layer works also on the b-layer, moreover, the b-layer is reduced more strongly. Therefore at most $a-b+b'-a'$ such m-reductions are possible, each lowers the a-layer by at most m, which gives the (initial) size of this layer at most $a+(a-b+b'-a')m$, a contradiction. ∎

EXAMPLE 20. Let $\mathcal{D} = \{(\overline{16}, a_1, \dots, a_5) : 16 \geq a_1 - 1 \geq \cdots \geq a_5 - 5\}$. We have $\#\mathcal{D} = 27896$, but if we remove all the diagrams not satisfying $(*)$ for $m = 6$ we have 12799 diagrams.

Observe that the dimension of $V(D; m_1, \dots, m_r)$ can be computed by solving some (large) system of linear equations. Usually this involves the computation of the rank of a $\#D \times \sum_{j=1}^{r} \binom{m_j+1}{2}$ matrix (see e.g. [Dum 07b]).

To prove non-speciality of a large class of systems we will use the following algorithms.

Algorithm InitialCasesA

Input: $m, k, s \in \mathbb{N}$.
Output: r_0 such that $\mathcal{L}(m_0 + k; m_0, m^{\times r})$ is non-special for each $m_0 \geq 0$, $r \geq r_0$, or NOT OK.

if $k < m - 1$ **then return** NOT OK;
set $\mathcal{D} = \{(\overline{k+1}, \{k+1\}^{\times s}, a_1, \dots, a_{m-1}) : k+1 \geq a_1 \geq \cdots \geq a_{m-1}\}$;
$\mathcal{D} \leftarrow \{D \in \mathcal{D} : D \text{ satisfies } (*)\}$;
for each $D \in \mathcal{D}$ **do**
  compute $p = p(D)$;
  check non-speciality of $V(D; m^{\times p})$ and $V(D; m^{(\times p+1)})$ by direct computation;
  if one of these systems is special **then return** NOT OK;
end for each
return $\max\{p(D) : D \in \mathcal{D}\} + 2$;

Algorithm InitialCasesB

Input: $m, k \in \mathbb{N}$.
Output: r_0 such that $\mathcal{L}(d; m_0, m^{\times r})$ is non-special for each $m_0 \geq 0$, $d \geq m_0 + k$, $r \geq r_0$, or NOT OK.

if $a < 2m - 1$ **then return** NOT OK;
set $\mathcal{D} = \{(\overline{k+1}, a_1, \dots, a_{m-1}) : k+1 \geq a_1 - 1 \geq a_2 - 2 \geq \cdots \geq a_{m-1} - (m-1)\}$;
$\mathcal{D} \leftarrow \{D \in \mathcal{D} : D \text{ satisfies } (*)\}$;
for each $D \in \mathcal{D}$ **do**
  compute $p = p(D)$;
  check non-speciality of $V(D; m^{\times p})$ and $V(D; m^{(\times p+1)})$ by direct computation;
  if one of these systems is special **then return** NOT OK;
end for each
return $\max\{p(D) : D \in \mathcal{D}\} + 2$;

These algorithms have been implemented by the author (in Free Pascal; the source code can be downloaded from [Dum 09a]). In Table 1 we present the results of InitialCases for $m = 7, 8, 9, 10$ and various values of k and s. During the implementation the following trick has been added. Let $\mathcal{D}$

be a set of diagrams whose non-speciality is to be decided, and choose $p \geq 1$. First, we compute

$$\mathcal{D}^p_{\mathrm{red}} = \{\mathrm{red}^p_m(D) : D \in \mathcal{D}\},$$

where

$$\mathrm{red}^p_m(D) := \underbrace{\mathrm{red}_m(\mathrm{red}_m(\ldots \mathrm{red}_m(D)\ldots))}_{p \text{ times}}.$$

Then, by matrix computation, create

$\mathcal{D}^p_{\mathrm{red,ok}} = \{D \in \mathcal{D}^p_{\mathrm{red}} : V(D; m^{\times p(D)})$ and $V(D; m^{\times (p(D)+1)})$ are non-special$\}$.

Let

$$\mathcal{D}^p_{\mathrm{done}} = \{D \in \mathcal{D} : \mathrm{red}^p_m(D) \in \mathcal{D}^p_{\mathrm{red,ok}}\}.$$

By the reduction algorithm, $V(D; m^{\times p(D)})$ and $V(D; m^{\times (p(D)+1)})$ are non-special for all $D \in \mathcal{D}^p_{\mathrm{done}}$. We must check the remaining cases belonging to $\mathcal{D} \setminus \mathcal{D}^p_{\mathrm{done}}$. To do this, proceed with new $\mathcal{D} := \mathcal{D} \setminus \mathcal{D}^p_{\mathrm{done}}$ and new $p := p - 1$. For $p = 0$ (final step) no reduction is performed, we only check non-speciality.

4. Remaining cases. According to Propositions 17, 18 and Table 1 we are left with the cases presented in Table 2. In what follows we will solve all these cases. This may be boring; for every system we must show that it is either non-special or -1-special. We will use Cremona transformations, glueing (Theorem 6; in most cases we glue four points) and the known facts about the Segre–Gimigliano–Harbourne–Hirschowitz conjecture (e.g. that it holds for multiplicities bounded by 11). Observe that if a system with non-negative multiplicities is in standard form and is based on at most nine points then it is non-special.

In what follows we write (for simplicity) $\mathcal{L}(d; \ldots, m^{\times a,b,c,\ldots})$ for a family of systems

$$\{\mathcal{L}(d; \ldots, m^{\times \ell}) : \ell = a, b, c, \ldots\}.$$

The remaining cases can be divided according to methods of showing non-speciality or -1-speciality. Therefore we present all the methods used (and an example for each of them); then, for each method, we give a list of cases that can be handled by this method. For all systems considered we assume that $m_0 \geq 12$.

4.1. Glueing. Glue four points $\mathcal{L}(m_0 + k; m_0, m^{\times r}) \to \mathcal{L}(m_0 + k; m_0, 2m + 1, m^{\times (r-4)})$. The resulting system should be in standard form, based on at most nine points, hence non-special, and with non-negative dimension. As an example take $\mathcal{L}(m_0 + k; m_0, 7^{\times 9,10,11})$, $k \geq 22$, $m_0 \geq 12$. After glueing we have $\mathcal{L}(m_0 + k; m_0, 15, 7^{\times 5,6,7})$. This system is in standard form since $m_0 + k - m_0 - 15 - 7 = k - 22 \geq 0$ and $m_0 + k - 15 - 7 - 7 \geq 5$. We also have

$\operatorname{vdim}\mathcal{L}(m_0+k;m_0,15,7^{\times 7}) = (k^2+2m_0k+3k+2m_0-630)/2-1 \geq 235$. This method can be applied to the following systems:

$$\begin{array}{llll} \mathcal{L}(m_0+k;m_0,7^{\times 9,10,11}), & k \geq 22, & \mathcal{L}(m_0+k;m_0,8^{\times 9,10,11}), & k \geq 25, \\ \mathcal{L}(m_0+k;m_0,9^{\times 9,10,11}), & k \geq 28, & \mathcal{L}(m_0+k;m_0,10^{\times 9,10,11}), & k \geq 31. \end{array}$$

4.2. Double glueing. As before, but we must glue twice (i.e. eight points of multiplicity m into two points of multiplicity $2m+1$). As an example consider $\mathcal{L}(m_0+k;m_0,8^{\times 12})$ for $k \geq 34$, $m_0 \geq 12$. Glue twice to obtain $\mathcal{L}(m_0+k;m_0,17^{\times 2},8^{\times 4})$ in standard form. This method can be applied to the following systems:

$$\begin{array}{llll} \mathcal{L}(m_0+k;m_0,8^{\times 12}), & k \geq 34, & \mathcal{L}(m_0+k;m_0,9^{\times 12}), & k \geq 38, \\ \mathcal{L}(m_0+k;m_0,10^{\times 12}), & k \geq 42. & & \end{array}$$

4.3. Glueing and Cremona. Glue four points $\mathcal{L}(m_0+k;m_0,m^{\times r}) \to \mathcal{L}(m_0+k;m_0,2m+1,m^{\times(r-4)})$. Then use the Cremona transformation based on points with multiplicities m_0, $2m+1$ and m to obtain

$$\mathcal{L}(m_0+2k-3m-1;m_0+k-3m-1,k-m,k-2m-1,m^{\times(r-5)}).$$

The last system should be non-special and in standard form. As an example take $\mathcal{L}(m_0+k;m_0,7^{\times 9,10,11})$, $k \in \{17,\ldots,21\}$, $m_0 \geq 12$. After glueing we have $\operatorname{Cr}(\mathcal{L}(m_0+k;m_0,15,7^{\times 5,6,7})) = \mathcal{L}(m_0+2k-22;m_0+k-22,k-7,k-15,7^{\times 4,5,6})$. This system is in standard form since $(m_0+2k-22)-(m_0+k-22)-(k-7)-(k-15)=-k+22\geq 0$, $(m_0+2k-22)-(m_0+k-22)-(k-7)-7 = 0$ and $(m_0+2k-22)-(k-7)-7-7 = m_0+k-29 \geq 0$. The computation of the virtual dimension is straightforward and gives $\operatorname{vdim} = (k^2+2m_0k+3k+2m_0-630)/2-1 \geq 70$. This method can be applied to the following systems:

$$\begin{array}{ll} \mathcal{L}(m_0+k;m_0,7^{\times 9,10,11}), & k \in \{17,\ldots,21\}, \\ \mathcal{L}(m_0+k;m_0,8^{\times 9,10,11}), & k \in \{20,\ldots,24\}, \\ \mathcal{L}(m_0+k;m_0,9^{\times 9,10,11}), & k \in \{25,26,27\}, \\ \mathcal{L}(m_0+k;m_0,10^{\times 9,10,11}), & k \in \{28,29,30\}. \end{array}$$

4.4. Double glueing and Cremona. As before, but we must glue twice. As an example consider $\mathcal{L}(m_0+k;m_0,8^{\times 12})$ for $k \in \{23,\ldots,33\}$, $m_0 \geq 12$. Glue twice to obtain $\operatorname{Cr}(\mathcal{L}(m_0+k;m_0,17^{\times 2},8^{\times 4})) = \mathcal{L}(m_0+2k-34;m_0+k-34,(k-17)^{\times 2},8^{\times 4})$ in standard form. This method can be applied to the following systems:

$$\begin{array}{ll} \mathcal{L}(m_0+k;m_0,8^{\times 12}), & k \in \{23,\ldots,33\}, \\ \mathcal{L}(m_0+k;m_0,9^{\times 12}), & k \in \{25,\ldots,37\}, \\ \mathcal{L}(m_0+k;m_0,10^{\times 12}), & k \in \{30,\ldots,41\}. \end{array}$$

4.5. Glueing and Cremona(s). Glue four points $\mathcal{L}(m_0+k;m_0,m^{\times r}) \to \mathcal{L}(m_0+k;m_0,2m+1,m^{\times(r-4)})$. Then use the Cremona transformation based on points with multiplicities m_0, $2m+1$ and m to obtain $\mathcal{L}(m_0+2k-3m-1;m_0+k-3m-1,k-m,k-2m-1,m^{\times(r-5)})$. The last system should be non-special in standard form except for a finite number of cases for low values of m_0. For each of these cases we must use an additional sequence of Cremona transformations to end up with a system in standard form. As an example take $\mathcal{L}(m_0+16;m_0,7^{\times 9,10})$ for $m_0 \geq 12$. After glueing we have $\mathrm{Cr}(\mathcal{L}(m_0+16;m_0,15,7^{\times 5,6})) = \mathcal{L}(m_0+10;m_0-6,9,7^{\times 4,5},1)$. Since $m_0+10-9-7-7 = m_0-13$, the last system is in standard form for $m_0 \geq 13$. The remaining case $m_0 = 12$ can be settled as follows: $\mathrm{Cr}^\circ(\mathcal{L}(22;6,9,7^{\times 4,5},1)) = \mathcal{L}(20;7^{\times 1,2},6^{\times 5},1)$. This method can be applied to the following systems:

$$
\begin{array}{ll}
\mathcal{L}(m_0+k;m_0,7^{\times 9,10}), & k \in \{16,15,14\},\\
\mathcal{L}(m_0+k;m_0,8^{\times 9,10,11}), & k \in \{19,18\},\\
\mathcal{L}(m_0+17;m_0,8^{\times 9,10}), & \\
\mathcal{L}(m_0+16;m_0,8^{\times 9,10}), & m_0 \geq 13,\\
\mathcal{L}(m_0+k;m_0,9^{\times 9,10,11}), & k \in \{24,23,22,21\},\\
\mathcal{L}(m_0+20;m_0,9^{\times 9,10}), & \\
\mathcal{L}(m_0+20;m_0,9^{\times 11}), & m_0 \geq 14,\\
\mathcal{L}(m_0+19;m_0,9^{\times 9,10}), & m_0 \geq 13,\\
\mathcal{L}(m_0+19;m_0,9^{\times 11}), & m_0 \geq 15,\\
\mathcal{L}(m_0+18;m_0,9^{\times 9,10}), & m_0 \geq 14,\\
\mathcal{L}(m_0+k;m_0,10^{\times 9,10,11}), & k \in \{25,26,27\},\\
\mathcal{L}(m_0+24;m_0,10^{\times 9,10,11}), & \\
\mathcal{L}(m_0+23;m_0,10^{\times 9,10}), & \\
\mathcal{L}(m_0+23;m_0,10^{\times 11}), & m_0 \geq 14,\\
\mathcal{L}(m_0+22;m_0,10^{\times 9,10}), & m_0 \geq 13,\\
\mathcal{L}(m_0+22;m_0,10^{\times 11}), & m_0 \geq 15,\\
\mathcal{L}(m_0+21;m_0,10^{\times 9,10}), & m_0 \geq 14,\\
\mathcal{L}(m_0+20;m_0,10^{\times 9,10}), & m_0 \geq 16.
\end{array}
$$

4.6. Double glueing and Cremona(s). As before, but we must glue twice. As an example consider $\mathcal{L}(m_0+22;m_0,8^{\times 12})$ for $m_0 \geq 12$. Glue twice to obtain $\mathrm{Cr}(\mathcal{L}(m_0+22;m_0,17^{\times 2},8^{\times 4})) = \mathcal{L}(m_0+10;m_0-12,8^{\times 4},5^{\times 2})$. For $m_0 \geq 14$ the last system is in standard form, the remaining cases are

$$\mathrm{Cr}^\circ(\mathcal{L}(23; 8^{\times 4}, 5^{\times 2}, 1)) = \mathcal{L}(22; 8, 7^{\times 3}, 5^{\times 2}, 1),$$
$$\mathrm{Cr}^\circ(\mathcal{L}(22; 8^{\times 4}, 5^{\times 2})) = \mathcal{L}(20; 8, 6^{\times 3}, 5^{\times 2}).$$

This method can be applied to the following systems:

$$\begin{array}{llll}
\mathcal{L}(m_0+22; m_0, 8^{\times 12}), & & \mathcal{L}(m_0+21; m_0, 8^{\times 12}), & \\
\mathcal{L}(m_0+20; m_0, 8^{\times 12}), & m_0 \geq 13, & \mathcal{L}(m_0+24; m_0, 9^{\times 12}), & m_0 \geq 13, \\
\mathcal{L}(m_0+23; m_0, 9^{\times 12}), & m_0 \geq 14, & \mathcal{L}(m_0+29; m_0, 10^{\times 12}), & \\
\mathcal{L}(m_0+28; m_0, 10^{\times 12}), & m_0 \geq 13, & \mathcal{L}(m_0+27; m_0, 10^{\times 12}), & m_0 \geq 14, \\
\mathcal{L}(m_0+26; m_0, 10^{\times 12}), & m_0 \geq 15, & \mathcal{L}(m_0+25; m_0, 10^{\times 12}), & m_0 \geq 16.
\end{array}$$

4.7. Glue, Cremona(s), glue, Cremona(s). Glue four points of equal multiplicity, then perform the Cremona transformation several times to obtain a system with lower multiplicities. Then glue four points (but now the multiplicities are lower) and use Cremona transformation(s) to obtain a non-special system in standard form. As an example consider $\mathcal{L}(32; 12, 8^{\times 12})$. Glue to obtain $\mathrm{Cr}^\circ(\mathcal{L}(32; 17, 12, 8^{\times 8})) = \mathcal{L}(24; 9, 8, 7^{\times 7}, 3)$. Glue again to consider $\mathrm{Cr}^\circ(\mathcal{L}(24; 15, 9, 8, 7^{\times 3}, 3)) = \mathcal{L}(10; 6, 2^{\times 3}, 1^{\times 2})$. This method can be applied to the following systems:

$$\begin{array}{ll}
\mathcal{L}(32; 12, 8^{\times 12}), & \mathcal{L}(36; 12, 9^{\times 12}), \\
\mathcal{L}(36; 13, 9^{\times 12}), & \mathcal{L}(35; 12, 9^{\times 12}), \\
\mathcal{L}(40; 12, 10^{\times 12}), & \mathcal{L}(40; 13, 10^{\times 12}), \\
\mathcal{L}(39; 12, 10^{\times 12}), & \mathcal{L}(40; 14, 10^{\times 12}), \\
\mathcal{L}(39; 13, 10^{\times 12}), & \mathcal{L}(38; 12, 10^{\times 12}), \\
\mathcal{L}(40; 15, 10^{\times 12}), & \mathcal{L}(39; 14, 10^{\times 12}), \\
\mathcal{L}(38; 13, 10^{\times 12}). &
\end{array}$$

4.8. Cremona (even) and glueing. Consider $\mathcal{L}(m_0 + k; m_0, m^{\times 2r})$ such that $k - 2m < 0$. Perform Cremona transformations based on the first point and two points with multiplicity m. Each time the degree and the first multiplicity is changed by $k - 2m$. We end up with $\mathcal{L}(m_0 + k + r(k - 2m); m_0 + r(k - 2m), (k - m)^{\times 2r})$. For m_0 such that $m_0 + r(k - 2m) \leq 11$ the situation is known (observe that this multiplicity can be negative). Otherwise glue four points of multiplicity $k - m$ and end up with a system in standard form. As an example consider $\mathcal{L}(m_0 + 9; m_0, 7^{\times 10})$ for $m_0 \geq 12$. Use Cremona transformations to obtain $\mathcal{L}(m_0 - 16; m_0 - 25, 2^{\times 10})$. For $m_0 \geq 37$ glue points to obtain $\mathcal{L}(m_0 - 16; m_0 - 25, 5, 2^{\times 6})$ in standard form. This method can be applied to the following systems:

$$\begin{array}{ll}
\mathcal{L}(m_0+10; m_0, 7^{\times 10}), & \mathcal{L}(m_0+9; m_0, 7^{\times 10}), \\
\mathcal{L}(m_0+11; m_0, 8^{\times 10}), & \mathcal{L}(m_0+10; m_0, 8^{\times 10}),
\end{array}$$

$$\mathcal{L}(m_0+13; m_0, 9^{\times 10}), \quad \mathcal{L}(m_0+12; m_0, 9^{\times 10}),$$
$$\mathcal{L}(m_0+14; m_0, 10^{\times 10}), \quad \mathcal{L}(m_0+13; m_0, 10^{\times 10}),$$
$$\mathcal{L}(m_0+12; m_0, 10^{\times 10}).$$

4.9. Cremona (even) and multiple glueing. As before, but we must glue several times to produce a system based on at most nine points. As an example consider $\mathcal{L}(m_0+9; m_0, 7^{\times 12})$ for $m_0 \geq 12$. Use Cremona transformations to obtain $\mathcal{L}(m_0-21; m_0-30, 2^{\times 12})$. For $m_0 \geq 42$ glue twice and finish with $\mathcal{L}(m_0-21; m_0-30, 5^{\times 2}, 2^{\times 4})$ in standard form. This method can be applied to the following systems (in square brackets we indicate how many times we glue):

$$\mathcal{L}(m_0+9; m_0, 7^{\times 12}), \quad [2], \quad \mathcal{L}(m_0+10; m_0, 8^{\times 12,14}), \quad [2],$$
$$\mathcal{L}(m_0+11; m_0, 9^{\times 10,12,14}), \quad [2], \quad \mathcal{L}(m_0+11; m_0, 9^{\times 16,18,20}), \quad [4],$$
$$\mathcal{L}(m_0+12; m_0, 10^{\times 12,14}), \quad [2], \quad \mathcal{L}(m_0+12; m_0, 10^{\times 16,18}), \quad [4],$$
$$\mathcal{L}(m_0+12; m_0, 10^{\times 20,22}), \quad [5].$$

4.10. Cremona (even), glueing and Cremona(s). As before, consider $\mathcal{L}(m_0+k; m_0, m^{\times 2r})$ such that $k-2m<0$, but now the system after glueing $\mathcal{L}(m_0+k+r(k-2m); m_0+r(k-2m), 2k-2m+1, (k-m)^{\times(2r-4)})$ is not in standard form, and we must use another Cremona transformation based on points with multiplicities $m_0+r(k-2m)$, $2k-2m+1$, $k-m$. After this, the system is in standard form for m_0 large enough, but for a finite set of values of m_0 (this set may be empty) we must use an additional sequence of Cremona transformation(s). As an example $\mathcal{L}(m_0+11; m_0, 7^{\times 10})$ for $m_0 \geq 12$. Transform this system into $\mathcal{L}(m_0-4; m_0-15, 4^{\times 10})$. For $m_0 \geq 27$ glue points to $\mathcal{L}(m_0-4; m_0-15, 9, 4^{\times 6})$. The standard form is $\mathcal{L}(m_0-6; m_0-17, 7, 4^{\times 5}, 2)$. Another example is $\mathcal{L}(m_0+15; m_0, 8^{\times 10})$. This system can be transformed into $\mathcal{L}(m_0+10; m_0-5, 7^{\times 10})$. For $m_0 \geq 17$ use glueing to obtain $\mathcal{L}(m_0+10; m_0-5, 15, 7^{\times 6})$. For $m_0 \geq 19$ the standard form is $\mathcal{L}(m_0+3; m_0-12, 8, 7^{\times 5})$, the remaining cases are

$$\mathrm{Cr}^{\circ}(\mathcal{L}(20; 8, 7^{\times 5}, 5)) = \mathcal{L}(14; 5^{\times 2}, 4^{\times 5}),$$
$$\mathrm{Cr}^{\circ}(\mathcal{L}(21; 8, 7^{\times 5}, 6)) = \mathcal{L}(19; 7, 6^{\times 6}).$$

This method can be applied to the following systems:

$$\mathcal{L}(m_0+11; m_0, 7^{\times 10}), \quad \mathcal{L}(m_0+15; m_0, 8^{\times 10}),$$
$$\mathcal{L}(m_0+12; m_0, 8^{\times 10}), \quad \mathcal{L}(m_0+17; m_0, 9^{\times 10}),$$
$$\mathcal{L}(m_0+16; m_0, 9^{\times 10}), \quad \mathcal{L}(m_0+14; m_0, 9^{\times 10}),$$
$$\mathcal{L}(m_0+19; m_0, 10^{\times 10}), \quad m_0 \neq 17, \quad \mathcal{L}(m_0+17; m_0, 10^{\times 10}),$$
$$\mathcal{L}(m_0+16; m_0, 10^{\times 10}), \quad \mathcal{L}(m_0+15; m_0, 10^{\times 10}).$$

4.11. Cremona (even), multiple glueing and Cremona(s). As before, but we must glue several times. As an example take $\mathcal{L}(m_0+10; m_0, 8^{\times 16})$ for $m_0 \geq 12$. This system can be transformed into $\mathcal{L}(m_0-38; m_0-48, 2^{\times 16})$. For $m_0 \geq 60$ glue three times to obtain $\mathcal{L}(m_0-38; m_0-48, 5^{\times 3}, 2^{\times 4})$, which can be transformed into the standard form $\mathcal{L}(m_0-43; m_0-53, 2^{\times 4})$. This method can be applied to the following systems (in square brackets we indicate how many times we glue):

$$\begin{array}{llll} \mathcal{L}(m_0+10; m_0, 8^{\times 16}), & [3], & \mathcal{L}(m_0+12; m_0, 9^{\times 12}), & [2], \\ \mathcal{L}(m_0+14; m_0, 10^{\times 12}), & [2], & \mathcal{L}(m_0+13; m_0, 10^{\times 12,14}), & [2]. \end{array}$$

4.12. Cremona (odd), glueing and Cremona(s). Consider $\mathcal{L}(m_0+k; m_0, m^{\times 2r+1})$ such that $k-2m<0$. This system can be transformed into $\mathcal{L}(m_0+k+r(k-2m); m_0+r(k-2m), m, (k-m)^{\times 2r})$. For m_0 such that $m_0+r(k-2m) \leq 11$ the situation is known. Otherwise glue four points of multiplicity $k-m$ to obtain $\mathcal{L}(m_0+k+r(k-2m); m_0+r(k-2m), 2k-2m+1, m, (k-m)^{\times(2r-4)})$. Use another Cremona transformation based on points with multiplicities $m_0+r(k-2m)$, $2k-2m+1$, m to obtain the system $\mathcal{L}(m_0+r(k-2m)+m-1; m_0+m-k+r(k-2m)-1, k-m, 2m-k-1, (k-m)^{\times(2r-4)})$ in standard form for m_0 large enough. For the remaining values of m_0 we must use additional Cremona transformation(s) to end up in standard form. As an example consider $\mathcal{L}(m_0+13; m_0, 7^{\times 9})$ for $m_0 \geq 12$. This system can be transformed into $\mathcal{L}(m_0+9; m_0-4, 7, 6^{\times 8})$. For $m_0 < 16$ the situation is known. For $m_0 \geq 16$ glue points to $\mathcal{L}(m_0+9; m_0-4, 13, 7, 6^{\times 4})$. The standard form of the last system is $\mathcal{L}(m_0+2; m_0-11, 6^{\times 5})$. This method can be applied to the following systems:

$$\begin{array}{ll} \mathcal{L}(m_0+k; m_0, 7^{\times 9}), & k \in \{13, 12, 10\}, \\ \mathcal{L}(m_0+k; m_0, 7^{\times 9,11}), & k \in \{11, 9\}, \\ \mathcal{L}(m_0+k; m_0, 8^{\times 9}), & k \in \{12, \dots, 15\}, \\ \mathcal{L}(m_0+k; m_0, 8^{\times 9,11}), & k \in \{11, 10\}, \\ \mathcal{L}(m_0+k; m_0, 9^{\times 9}), & k \in \{17, 16, 15\}, \\ \mathcal{L}(m_0+k; m_0, 9^{\times 9,11}), & k \in \{14, 13, 12\}, \\ \mathcal{L}(m_0+19; m_0, 10^{\times 9}), & m_0 \neq 16, \\ \mathcal{L}(m_0+18; m_0, 10^{\times 9}), & \\ \mathcal{L}(m_0+k; m_0, 10^{\times 9,11}), & k \in \{12, \dots, 17\}. \end{array}$$

4.13. Cremona (odd), multiple glueing and Cremona(s). As before, but we must glue several times to obtain the system with at most nine multiplicities. As an example take $\mathcal{L}(m_0+9; m_0, 7^{\times 13})$ for $m_0 \geq 12$. Transform our system into $\mathcal{L}(m_0-21; m_0-30, 7, 2^{\times 12})$. For $m_0 \geq 42$ glue twice to obtain $\mathrm{Cr}(\mathcal{L}(m_0-21;m_0-30,7,5,5,2^{\times 4}))=\mathcal{L}(m_0-24;m_0-33,5,4,2^{\times 5})$ in standard form. This method can be applied to the following systems (in square brackets we indicate how many times we glue):

$$
\begin{array}{llll}
\mathcal{L}(m_0+9;m_0,7^{\times 13}), & [2], & \mathcal{L}(m_0+10;m_0,8^{\times 13}), & [2],\\
\mathcal{L}(m_0+10;m_0,8^{\times 15,17}), & [3], & \mathcal{L}(m_0+12;m_0,9^{\times 13}), & [2],\\
\mathcal{L}(m_0+11;m_0,9^{\times 9,11,13}), & [2], & \mathcal{L}(m_0+11;m_0,9^{\times 15}), & [3],\\
\mathcal{L}(m_0+11;m_0,9^{\times 17,19}), & [4], & \mathcal{L}(m_0+11;m_0,9^{\times 21}), & [5],\\
\mathcal{L}(m_0+13;m_0,10^{\times 13}), & [2], & \mathcal{L}(m_0+13;m_0,10^{\times 15}), & [3],\\
\mathcal{L}(m_0+12;m_0,10^{\times 13}), & [2], & \mathcal{L}(m_0+12;m_0,10^{\times 15,17}), & [3],\\
\mathcal{L}(m_0+12;m_0,10^{\times 19}), & [4], & \mathcal{L}(m_0+12;m_0,10^{\times 21,23}), & [5].
\end{array}
$$

4.14. Negative glueing and Cremona. We use glueing to show that the system $\mathcal{L}(m_0+k;m_0,m^{\times r})$ with negative virtual dimension is empty. Therefore we glue four points of multiplicity m to one point of multiplicity $2m$, then we use Cremona transformation(s) to show that the resulting system is empty. As an example consider $\mathcal{L}(32;13,9^{\times 11})$. Use glueing to consider $\mathcal{L}(32;18,13,9^{\times 7})$, which can be transformed into $\mathcal{L}(0;2^{\times 3},1,(-1)^{\times 5},-2,-4)$. The last system is empty. This method can be applied to the following systems:

$$
\begin{array}{ll}
\mathcal{L}(32;12,9^{\times 11}), & \mathcal{L}(32;13,9^{\times 11}),\\
\mathcal{L}(31;12,9^{\times 11}), & \mathcal{L}(31;13,9^{\times 10}),\\
\mathcal{L}(30;12,9^{\times 10}), & \mathcal{L}(35;12,10^{\times 11}),\\
\mathcal{L}(35;13,10^{\times 11}), & \mathcal{L}(34;12,10^{\times 11}),\\
\mathcal{L}(34;13,10^{\times 10}), & \mathcal{L}(33;12,10^{\times 10}),\\
\mathcal{L}(34;14,10^{\times 10}), & \mathcal{L}(33;13,10^{\times 10}),\\
\mathcal{L}(32;12,10^{\times 10}). &
\end{array}
$$

4.15. Low multiplicities. Consider $\mathcal{L}(m_0+k;m_0,m^{\times r})$ for $k-m\leq 1$. As before, this system can be transformed (by a sequence of Cremona transformations) into a system in standard form with at most two arbitrary “high” multiplicities, the other being strictly less than 2. Let $L=\mathcal{L}(d;m_1,m_2,m^{\times s})$ be such a system, $m\leq 1$. If $m\leq -2$ then L is -1-special if and only if L is non-empty, which is equivalent to the non-emptiness of $\mathcal{L}(d;m_1,m_2)$. For $m=-1$ or $m=0$ it is enough to consider $\mathcal{L}(d;m_1,m_2)$ based on at most two points. For $m=1$ we have two cases. If $m_1\geq -1$, $m_2\geq -1$ then L is non-special since multiplicity 1 always imposes an independent condition. For the opposite case we must decide whether L is non-empty. Dropping negative multiplicities we end up with a system with at most one multiplicity not equal to 1. As an example consider $\mathcal{L}(m_0+8; m_0, 7^{\times(2r+1)})$. This system can be transformed to $\mathcal{L}(m_0+k-6r; m_0-6r, 7, 1^{\times(2r)})$ in standard form. If $m_0-6r \geq -1$ then the system is non-special. If $m_0-6r < -1$ then the system is -1-special if and only if $\mathcal{L}(m_0+8-6r; 7, 1^{\times(2r)})$ is non-empty, which holds for $\binom{m_0-6r+10}{2} \geq 28+2r$. In fact we have $\binom{m_0-6r+10}{2} \leq \binom{8}{2} = 28$, so $r=0$ and our system is non-special. This method can be applied to

$$\begin{aligned} &\mathcal{L}(m_0+k; m_0, 7^{\times r}), && k \leq 8,\ r \geq 9,\\ &\mathcal{L}(m_0+k; m_0, 8^{\times r}), && k \leq 9,\ r \geq 9,\\ &\mathcal{L}(m_0+k; m_0, 9^{\times r}), && k \leq 10,\ r \geq 9,\\ &\mathcal{L}(m_0+k; m_0, 10^{\times r}), && k \leq 11,\ r \geq 9. \end{aligned}$$

4.16. Additional methods. We use non-standard glueing, the reduction algorithm, etc.

- $\mathcal{L}(28; 12, 8^{\times 9})$ Glue three points (using non-special system $\mathcal{L}(15; 8^{\times 3})$) to obtain $\mathcal{L}(28; 16, 12, 8^{\times 6})$. The standard form of the last system is $\mathcal{L}(4; 4)$.
- $\mathcal{L}(31; 12, 9^{\times 9})$ This system is non-special due to the reduction algorithm. We begin with the diagram $(\overline{32})$, use 12-reduction followed by nine 9-reductions. The last diagram is equal to $(\overline{6}, 6, 6, 5, 5, 2)$.
- $\mathcal{L}(31; 13, 9^{\times 9})$ It is enough to show that $\mathcal{L}(30; 13, 9^{\times 9})$ is non-special (observe that the last system has virtual dimension -1). We have $\mathrm{Cr}^\circ(\mathcal{L}(30; 13, 9^{\times 9})) = \mathcal{L}(26; 9^{\times 2}, 8^{\times 8})$. Since all multiplicities are bounded by 11 we can use Theorem 10.
- $\mathcal{L}(34; 12, 10^{\times 9})$ This system has positive virtual dimension. Since $\mathcal{L}(19; 10^{\times 3})$ is non-empty and non-special we use glueing to consider

$$\mathrm{Cr}^\circ(\mathcal{L}(34; 20, 12, 10^{\times 6})) = \mathcal{L}(4; 2).$$

- $\mathcal{L}(35; 15, 10^{\times 9})$ This system has positive virtual dimension. As above, we glue to consider $\mathrm{Cr}^\circ(\mathcal{L}(35; 20, 15, 10^{\times 6})) = \mathcal{L}(5; 5)$.
- $\mathcal{L}(32; 12, 10^{\times 9})$ This system is empty due to the reduction algorithm. We begin with the diagram $(\overline{33})$, use 12-reduction followed by eight 10-reductions. The last diagram is equal to $(\overline{6}, 6, 6, 5, 5)$, which can be enlarged to $(\overline{10})$ and reduced to the empty diagram.
- $\mathcal{L}(35; 16, 10^{\times 9})$ This system can be transformed into $\mathcal{L}(31; 12, 10, 9^{\times 8})$. It is enough to show that the system $\mathcal{L}(30; 12, 10, 9^{\times 8})$ is non-empty and non-special. The last system can be transformed into $\mathcal{L}(29; 11, 9^{\times 8}, 8)$, which is non-special due to Theorem 10.

4.17. Direct computation. Sometimes we are forced to compute the rank of the matrix associated to a system. To make this task possible, we specialize to random points and compute over $\mathbb{F}_p$ for some prime p. If the rank is maximal for specialized points over $\mathbb{F}_p$ then it is maximal over $\mathbb{Q}$ (and hence over any field of characteristic zero) and for points in general position. Alternatively, we may use the diagram cutting method of [Dum 07a] (the author has checked that it is possible in all cases). This method must be applied to the following systems:

$$\begin{array}{ll}
\mathcal{L}(28; 12, 8^{\times 10}), & \mathcal{L}(33; 13, 9^{\times 11}), \\
\mathcal{L}(33; 14, 9^{\times 11}), & \mathcal{L}(31; 12, 9^{\times 10}), \\
\mathcal{L}(32; 14, 9^{\times 10}), & \mathcal{L}(30; 12, 9^{\times 9}), \\
\mathcal{L}(37; 12, 10^{\times 12}), & \mathcal{L}(36, 13, 10^{\times 11}), \\
\mathcal{L}(36; 14, 10^{\times 11}), & \mathcal{L}(34; 12, 10^{\times 10}), \\
\mathcal{L}(34; 13, 10^{\times 9}), & \mathcal{L}(33; 12, 10^{\times 9}), \\
\mathcal{L}(35; 15, 10^{\times 10}), & \mathcal{L}(34; 14, 10^{\times 9}), \\
\mathcal{L}(33; 13, 10^{\times 9}), & \mathcal{L}(36; 17, 10^{\times 10}).
\end{array}$$

5. List of special systems. To produce the list of all special quasi-homogeneous systems $\mathcal{L}(d; m_0, m^{\times r})$ with $m = 7, 8, 9, 10$ we will use Cremona transformations. From [Gim 89] (see the discussion after Definition 9) and Theorem 4 it follows that the system as above is special if and only if

- while performing Cremona transformations some doubly negative (i.e. ≤ -2) multiplicities appear, and
- its standard form has nonnegative virtual dimension.

Fix $m \in \{7, 8, 9, 10\}$. We will begin with systems $\mathcal{L}(m_0 + k; m_0, m^{\times 2r})$ for $k = 0, \ldots, m - 2$. By a sequence of Cremona transformations we can transform $\mathcal{L}(m_0 + k; m_0, m^{\times 2r})$ into

$$\mathcal{L}(m_0 + k + r(k - 2m); m_0 + r(k - 2m), (k - m)^{\times 2r}).$$

Since $k - m \leq -2$, we know that $\mathcal{L}(m_0 + k; m_0, m^{\times 2r})$ is special if and only if $\mathcal{L}(m_0 + k + r(k - 2m); m_0 + r(k - 2m))$ is non-empty. The system with one base point cannot be special, so non-emptiness is equivalent to

$$m_0 + k + r(k - 2m) \geq 0.$$

To deal with $\mathcal{L}(m_0 + k; m_0, m^{\times (2r+1)})$ for $k = 0, \ldots, m - 2$ we proceed as above with one more Cremona at the end (based on points with multiplicities $m_0 + r(k - 2m), m, 0)$. The condition for being special is

$$m_0 + 2k + r(k - 2m) - m \geq 0.$$

Now let k be either $m - 1$ or m. We proceed as above by transforming $\mathcal{L}(m_0 + k; m_0, m^{\times 2r})$ into $\mathcal{L}(m_0 + k + r(k - 2m); m_0 + r(k - 2m), (k - m)^{\times 2r})$, but now $k - m$ is either -1 or zero and can be dropped out. Therefore

$\mathcal{L}(m_0+k; m_0, m^{\times 2r})$ is special if and only if $\mathcal{L}(m_0+k+r(k-2m); m_0+r(k-2m), (k-m)^{\times 2r})$ is special. The last is equivalent to

$$m_0+k+r(k-2m) \geq 0, \qquad m_0+r(k-2m) \leq -2,$$

which can be written as $0 \leq m_0+k+r(k-2m) \leq k-2$. The case $\mathcal{L}(m_0+k; m_0, m^{\times(2r+1)})$ can be handled similarly, and the condition for speciality is $0 \leq m_0+2k+r(k-2m)-m \leq k-2$.

Let $k \in \{m+1, \dots, 2m-1\}$. Transform $\mathcal{L}(m_0+k; m_0, m^{\times 2r})$ into $\mathcal{L}(m_0+k+r(k-2m); m_0+r(k-2m), (k-m)^{\times 2r}) =: L$. Observe that L is empty for $m_0+k+r(k-2m) < k-m$, in standard form for $m_0+k+r(k-2m) \geq 3(k-m)$ and without doubly negative multiplicities for $m_0+k+r(k-2m) \geq -1$. Hence L can be special only for

$$m_0+r(k-2m) \in \{-m, \dots, \max\{3(k-m)-k-1, -2\}\}.$$

Observe that for r large enough L must be empty, since $k-m \geq 1$. This gives a finite number of cases that should be dealt with separately, for example by a suitable computer program.

The same applies to $\mathcal{L}(m_0+k; m_0, m^{\times(2r+1)})$, but now we must search through systems satisfying

$$m_0+r(k-2m) \in \{m-k, \dots, \max\{k-m-1, -2\}\}.$$

For $k \in \{2m, \dots, 3m-1\}$ the system $\mathcal{L}(m_0+k; m_0, m^{\times r})$ is in standard form for $m_0+k \geq 3m$, so we must check a finite number of cases for

$$m_0 \in \{0, \dots, 3m-k-1\}.$$

For $k \geq 3m$ the system $\mathcal{L}(m_0+k; m_0, m^{\times r})$ is always in standard form and hence non-special.

The author did all the computations mentioned above for $m = 7, 8, 9, 10$ and $k \geq m+1$. The full list for a fixed m contains an infinite family of special systems for $k \leq m$ (which can be easily described) and additional 81 (resp. 128, 179, 257) systems for $m = 7$ (resp. $m = 8$, $m = 9$, $m = 10$). This list can be presented in a shorter form by gathering some systems into small families (e.g. $\mathcal{L}(5+6r; -3+6r, 7^{\times 2r})$ for $r = 1, \dots, 10$). The list for $m = 7$ is presented in Table 3. Similar lists for multiplicities 8, 9, 10 are available online at [Dum 09a].

6. A note on Seibert's proof for $m = 4$. In [Sei 01] all special systems of the form $\mathcal{L}(d; m, 4^{\times r})$ have been classified. For all non-special cases but one the proof involved techniques avoiding computation of the rank of the interpolation matrix. For $\mathcal{L}(13; 5, 4^{\times 9})$ Seibert used a Maple program to compute the rank of a 105×105 matrix. The rank appeared to be maximal, so the system is non-special. Using the diagram cutting method (introduced in [Dum-Jar 07]) we propose a much nicer proof of this fact, which can be easily checked by hand. The cutting is presented in Fig. 1. The order of cutting is indicated by the numbers in the diagram. Now, Seibert's proof does not rely on computations that cannot be done by hand.

2													
2	2												
2	2	2											
2	2	2	2										
6	6	6	6										
6	6	6											
6	6	8				5							
6	8	8	9				5						
8	8	8	8	9	9	9	5	5					
1	8	8	8	9	9	9	9	5	5				
1	1	4	7	7	7	9	9	5	5	3			
1	1	1	4	4	7	7	7	7	5	3	3		
1	1	1	1	4	4	7	7	7	5	3	3	3	
1	1	1	1	1	4	4	4	4	7	3	3	3	3

Fig. 1. Diagram cutting for $\mathcal{L}(13; 5, 4^{\times 9})$

7. Tables and figures

Table 1. Results of InitialCases. It follows that $\mathcal{L}(d; m_0, m^{\times r})$ is non-special for $d = m_0 + k$ ($d \geq m_0 + k$ for type B) and $r \geq r_0$.

Type	m	k	s	r_0	Type	m	k	s	r_0
B	7	17		12	A	9	19	4	12
A	7	16	1	11	A	9	18	5	11
A	7	15	2	11	A	9	17	6	11
A	7	14	3	11	A	9	16	7	11
A	7	13	4	10	A	9	15	8	10
A	7	12	5	10	A	9	14	14	12
A	7	11	11	12	A	9	13	17	12
A	7	10	13	11	A	9	12	29	14
A	7	9	24	14	A	9	11	62	22
B	8	20		13	B	10	25		13
A	8	19	1	12	A	10	24	1	12
A	8	18	2	12	A	10	23	2	12
A	8	17	3	11	A	10	22	3	12
A	8	16	5	11	A	10	21	4	11
A	8	15	5	11	A	10	20	5	11
A	8	14	6	10	A	10	19	6	11
A	8	13	7	10	A	10	18	7	10
A	8	12	13	11	A	10	17	13	12

Type	m	k	s	r_0	Type	m	k	s	r_0
A	8	11	19	12	A	10	16	15	12
A	8	10	41	18	A	10	15	17	12
B	9	23		13	A	10	14	26	13
A	9	22	1	12	A	10	13	41	16
A	9	21	2	12	A	10	12	79	24
A	9	20	3	12					

Table 2. Cases to be considered separately

$\mathcal{L}(m_0+k; m_0, 7^{\times r})$ $k \geq 17$, $r \in \{9, 10, 11\}$	$\mathcal{L}(m_0+k; m_0, 7^{\times r})$ $k \in \{16, 15, 14\}$, $r \in \{9, 10\}$
$\mathcal{L}(m_0+k; m_0, 7^{\times 9})$ $k \in \{13, 12\}$	$\mathcal{L}(m_0+11; m_0, 7^{\times r})$, $r \in \{9, 10, 11\}$
$\mathcal{L}(m_0+10; m_0, 7^{\times r})$ $r \in \{9, 10\}$	$\mathcal{L}(m_0+9; m_0, 7^{\times r})$, $r \in \{9, 10, 11, 12, 13\}$
$\mathcal{L}(m_0+k; m_0, 7^{\times r})$ $k \in \{0, \dots, 8\}$, $r \geq 9$	
$\mathcal{L}(m_0+k; m_0, 8^{\times r})$ $k \geq 20$, $r \in \{9, \dots, 12\}$	$\mathcal{L}(m_0+k; m_0, 8^{\times r})$ $k \in \{17, 16, 15, 12\}$, $r \in \{9, 10\}$
$\mathcal{L}(m_0+k; m_0, 8^{\times r})$ $k \in \{19, 18, 11\}$, $r \in \{9, 10, 11\}$	$\mathcal{L}(m_0+k; m_0, 8^{\times 9})$ $k \in \{14, 13\}$
$\mathcal{L}(m_0+10; m_0, 8^{\times r})$ $r \in \{9, \dots, 17\}$	$\mathcal{L}(m_0+k; m_0, 8^{\times r})$ $k \in \{0, \dots, 9\}$, $r \geq 9$
$\mathcal{L}(m_0+k; m_0, 9^{\times r})$ $k \geq 23$, $r \in \{9, \dots, 12\}$	$\mathcal{L}(m_0+k; m_0, 9^{\times r})$ $k \in \{22, \dots, 19, 14, 13\}$, $r \in \{9, 10, 11\}$
$\mathcal{L}(m_0+k; m_0, 9^{\times r})$ $k \in \{18, 17, 16\}$, $r \in \{9, 10\}$	$\mathcal{L}(m_0+15; m_0, 9^{\times 9})$
$\mathcal{L}(m_0+12; m_0, 9^{\times r})$ $r \in \{9, \dots, 13\}$	$\mathcal{L}(m_0+11; m_0, 9^{\times r})$, $r \in \{9, \dots, 21\}$
$\mathcal{L}(m_0+k; m_0, 9^{\times r})$ $k \in \{0, \dots, 10\}$, $r \geq 9$	
$\mathcal{L}(m_0+k; m_0, 10^{\times r})$ $k \geq 25$, $r \in \{9, \dots, 12\}$	$\mathcal{L}(m_0+k; m_0, 10^{\times r})$ $k \in \{24, 23, 22, 17, 16, 15\}$, $r \in \{9, 10, 11\}$
$\mathcal{L}(m_0+k; m_0, 10^{\times r})$ $k \in \{21, 20, 19\}$, $r \in \{9, 10\}$	$\mathcal{L}(m_0+18; m_0, 10^{\times 9})$
$\mathcal{L}(m_0+14; m_0, 10^{\times r})$, $r \in \{9, \dots, 12\}$	$\mathcal{L}(m_0+13; m_0, 10^{\times r})$, $r \in \{9, \dots, 15\}$
$\mathcal{L}(m_0+12; m_0, 10^{\times r})$ $r \in \{9, \dots, 23\}$	$\mathcal{L}(m_0+k; m_0, 10^{\times r})$ $k \in \{0, \dots, 11\}$, $r \geq 9$

Table 3. Quasi-homogeneous special systems for $m = 7$

$\mathcal{L}(m_0 + k; m_0, 7^{\times 2r})$	$k \leq 5, m_0 + k + r(k - 14) \geq 0$
$\mathcal{L}(m_0 + k; m_0, 7^{\times (2r+1)})$	$k \leq 5, m_0 + 2k + r(k - 14) - 7 \geq 0$
$\mathcal{L}(m_0 + 7; m_0, 7^{\times 2r})$	$0 \leq m_0 + 7 - 7r \leq 5$
$\mathcal{L}(m_0 + 7; m_0, 7^{\times (2r+1)})$	$0 \leq m_0 + 7 - 7r \leq 5$
$\mathcal{L}(m_0 + 6; m_0, 7^{\times 2r})$	$0 \leq m_0 + 6 - 6r \leq 4$
$\mathcal{L}(m_0 + 6; m_0, 7^{\times (2r+1)})$	$0 \leq m_0 + 5 - 6r \leq 4$
$\mathcal{L}(8 + 6k; 6k, 7^{\times (2+2k)})$	$k \in \{0, 1\}$
$\mathcal{L}(9 + 6k; 1 + 6k, 7^{\times (2+2k)})$	$k \in \{0, \dots, 3\}$
$\mathcal{L}(10 + 6k; 2 + 6k, 7^{\times (2+2k)})$	$k \in \{0, \dots, 6\}$
$\mathcal{L}(11 + 6k; 3 + 6k, 7^{\times (2+2k)})$	$k \in \{0, \dots, 9\}$
$\mathcal{L}(12 + 6k; 4 + 6k, 7^{\times (2+2k)})$	$k \in \{0, \dots, 12\}$
$\mathcal{L}(9 + 5k; 5k, 7^{\times (2+2k)})$	$k \in \{0, 1\}$
$\mathcal{L}(10 + 5k; 1 + 5k, 7^{\times (2+2k)})$	$k \in \{0, 1, 2\}$
$\mathcal{L}(11 + 5k; 2 + 5k, 7^{\times (2+2k)})$	$k \in \{0, \dots, 3\}$
$\mathcal{L}(12 + 5k; 3 + 5k, 7^{\times (2+2k)})$	$k \in \{0, \dots, 4\}$
$\mathcal{L}(12; 3, 7^{\times 3})$	
$\mathcal{L}(10 + 4k; 4k, 7^{\times (2+2k)})$	$k \in \{0, 1\}$
$\mathcal{L}(11 + 4k; 1 + 4k, 7^{\times (2+2k)})$	$k \in \{0, 1\}$
$\mathcal{L}(12 + 4k; 2 + 4k, 7^{\times (2+2k)})$	$k \in \{0, 1, 2\}$
$\mathcal{L}(11 + k; 1 + k, 7^{\times 3})$	$k \in \{0, 1\}$
$\mathcal{L}(22; 12, 7^{\times 7})$	
$\mathcal{L}(11 + 3k; 3k, 7^{\times (2+2k)})$	$k \in \{0, 1\}$
$\mathcal{L}(12 + 3k; 1 + 3k, 7^{\times (2+2k)})$	$k \in \{0, 1\}$
$\mathcal{L}(11 + k; k, 7^{\times 3})$	$k \in \{0, 1\}$
$\mathcal{L}(16 + 5k; 5 + 5k, 7^{\times (5+2k)})$	$k \in \{0, 1\}$
$\mathcal{L}(12 + 2k; 2k, 7^{\times (2+2k)})$	$k \in \{0, 1\}$
$\mathcal{L}(12 + 4k; 4k, 7^{\times (3+2k)})$	$k \in \{0, 1\}$
$\mathcal{L}(15 + k; 2 + k, 7^{\times 5})$	$k \in \{0, 1\}$
$\mathcal{L}(14 + k; k, 7^{\times 5})$	$k \in \{0, 1, 2\}$
$\mathcal{L}(15 + k; k, 7^{\times 5})$	$k \in \{0, 1\}$
$\mathcal{L}(16; 7^{\times 5})$	

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