Bundle functors on all foliated manifold morphisms have locally finite order

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Abstract. We prove that any bundle functor $F : \mathcal{F}_{\text{ol}} \to \mathcal{F}_{\text{FM}}$ on the category $\mathcal{F}_{\text{ol}}$ of all foliated manifolds without singularities and all leaf respecting maps is of locally finite order.

Let $\mathcal{M}_{f_m}$ be the category of $m$-dimensional manifolds and their embeddings and $\mathcal{F}_{\text{FM}}$ be the category of all fibred manifolds and their fibred maps. In [9], R. Palais and C. Terng showed that any natural bundle in the sense of A. Nijenhuis [8] (bundle functor) $F : \mathcal{M}_{f_m} \to \mathcal{F}_{\text{FM}}$ has finite order $\text{ord}(F) \leq 2^{f} + 1$, where $f = \dim(F_0 \mathbb{R}^m)$. (We remark that a bundle functor $F : \mathcal{M}_{f_m} \to \mathcal{F}_{\text{FM}}$ is of order $r$ if for any $\mathcal{M}_{f_m}$-maps $\varphi, \psi : M \to N$ and any $x \in M$, from $j_x^\varphi = j_x^\psi$ it follows that $F\varphi = F\psi$ on the fiber of $FM$ over $x$.) In [1], D. Epstein and W. Thurston showed that $\text{ord}(F) \leq 2^f + 1$. In [11], A. Zajtz presented the best inequality

$$\text{ord}(F) \leq \max \left( \frac{f}{m-1}, \frac{f}{m} + 1 \right)$$

if $m > 1$. In [2], I. Kolář, P. Michor and J. Slovák extended the result from [11] to bundle functors $F : \mathcal{F}_{\mathcal{M}_{m,n}} \to \mathcal{F}_{\mathcal{FM}}$, where $\mathcal{F}_{\mathcal{M}_{m,n}}$ is the category of fibred manifolds with $m$-dimensional bases and $n$-dimensional fibers and their fibred embeddings, and obtained the estimate $\text{ord}(F) \leq 2^f + 1$ for all $m, n$, and

$$\text{ord}(F) \leq \max \left( \frac{f}{m-1}, \frac{f}{m} + 1, \frac{f}{n-1}, \frac{f}{n} + 1 \right)$$

if $m > 1$ and $n > 1$, where $f = \dim(F_{0,0}(\mathbb{R}^m \times \mathbb{R}^n))$ (the definition of the order of bundle functors on $\mathcal{F}_{\mathcal{M}_{m,n}}$ is a direct generalization of the one for bundle functors on $\mathcal{M}_{f_m}$). From [2] it follows that every product preserving bundle functor $F : \mathcal{M}_{f} \to \mathcal{F}_{\mathcal{FM}}$, where $\mathcal{M}_{f}$ is the category of all

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manifolds and all maps, is of finite order $\text{ord}(F) = \text{ord}(F|\mathcal{M}f_1)$. In [6], the second author presented an example of a vector bundle functor $\mathcal{M}f \to \mathcal{VB}$ of strictly infinite order.

**Example 1 ([6])**. We recall that $T^{(r)}M = (J^r(M, \mathbb{R}))^*$ denotes the $r$th order vector tangent bundle of a manifold $M$. Let $d_r = \dim(T^{(r)}_0\mathbb{R})$. We set $GM = \bigoplus_{k=1}^{\infty} \bigwedge^{d_k} T^{(k)}M$. Then $GM$ is a finite-dimensional vector bundle for every manifold $M$ because for $k > \dim(M)$ the vector bundle $\bigwedge^{d_k} T^{(k)}M$ is the zero-bundle. Hence the direct sum in the definition of $GM$ is in fact a finite sum. For a mapping $f: M \to N$ the induced mapping $Ff: GM \to GN$ is defined in the natural way from $T^{(k)}f: T^{(k)}M \to T^{(k)}N$.

The vector bundle functor $G$ is of strictly infinite order because its restriction to the category $Mf_k$ is of order at least $k$.

In [7], the second author proved that every bundle functor $F: Mf \to \mathcal{F}M$ has locally finite order in the following sense.

**Proposition 1 ([7])**. Let $F: \mathcal{M}f \to \mathcal{F}M$ be a bundle functor. Let $r_m := \text{ord}(F|\mathcal{M}f_m)$. For all maps $f_1, f_2: M \to N$ and $x \in M$, from $j_x^{\dim(M)+1} f_1 = j_x^{\dim(M)+1} f_2$ it follows that $Ff_1 = Ff_2$ on the fiber over $x$.

In [2], the above result is extended to bundle functors $F: \mathcal{F}M_m \to \mathcal{F}M$, where $\mathcal{F}M_m$ is the category of fibred manifolds with $m$-dimensional bases and their fibred maps covering embeddings. Namely, the following proposition is proved.

**Proposition 2 ([2])**. Let $F: \mathcal{F}M_m \to \mathcal{F}M$ be a bundle functor. Let $r_n = \text{ord}(F|\mathcal{F}M_{m,n})$. For all $\mathcal{F}M_m$-maps $f_1, f_2: Y \to Z$ and $x \in Y$, from $j_x^{\dim(Y)-m+1} f_1 = j_x^{\dim(Y)-m+1} f_2$ it follows that $Ff_1 = Ff_2$ on the fiber over $x$.

From [4] it follows that any product preserving bundle functor $F: \mathcal{F}M \to \mathcal{F}M$ has finite order $\text{ord}(F) = \text{ord}(F|\mathcal{F}M_{1,1})$. In [3], a fiber-product preserving bundle functor $F: \mathcal{F}M \to \mathcal{F}M$ of strictly infinite order is given. From [3] it follows that any fiber-product preserving bundle functor $F: \mathcal{F}M \to \mathcal{F}M$ is of locally finite order in the following sense: for all $\mathcal{F}M$-maps $f_1, f_2: Y \to Z$ and $x \in Y$ with $Y \in \mathcal{F}M_{m,n}$, from $j_x^{r_m} f_1 = j_x^{r_m} f_2$ it follows that $Ff_1 = Ff_2$ over $x$, where $r_m = \max(\text{ord}(F|\mathcal{F}M_{m,0}), \text{ord}(F|\mathcal{F}M_{m,1}))$. So, we have the following natural question.

**Question 1**. Is any bundle functor $F: \mathcal{F}M \to \mathcal{F}M$ of locally finite order?

In this paper we give an affirmative answer to the above question. Since the category $\mathcal{F}M$ has the same skeleton as the category $\mathcal{Fol}$ of all foliated manifolds without singularities and all leaf respecting maps, it is sufficient to study the order of bundle functors on $\mathcal{Fol}$. 
Bundle functors

We recall (see [2]) that a bundle functor on \( \mathcal{Fol} \) is a covariant functor \( F : \mathcal{Fol} \to \mathcal{FM} \) satisfying:

(i) (Base preservation) \( B_{\mathcal{FM}} \circ F = B_{\mathcal{Fol}} \), where \( B_{\mathcal{FM}} : \mathcal{FM} \to \mathcal{MF} \) is the base functor and \( B_{\mathcal{Fol}} : \mathcal{Fol} \to \mathcal{MF} \) is the functor \((M, \mathcal{F}) \to M\). Hence the induced projections form a natural transformation \( \pi : F \to B_{\mathcal{Fol}} \).

(ii) (Localization) For every inclusion \( i_{(U, \mathcal{F}[U])} : (U, \mathcal{F}[U]) \to (M, \mathcal{F}) \) of an open subset, \( F(U, \mathcal{F}[U]) \) is the restriction \( \pi^{-1}(U) \) of \( \pi : F(M, \mathcal{F}) \to M \) over \( U \) and \( F|_{i_{(U, \mathcal{F}[U])}} \) is the inclusion \( \pi^{-1}(U) \to F(M, \mathcal{F}) \).

(iii) (Regularity) \( F \) transforms smoothly parametrized families of \( \mathcal{Fol} \)-maps into smoothly parametrized families of fibred maps.

**Example 2.** A well-known example of a bundle functor \( F : \mathcal{Fol} \to \mathcal{FM} \) is the normal bundle functor \( N : \mathcal{Fol} \to \mathcal{FM} \) transforming any foliated manifold \((M, \mathcal{F})\) into its normal bundle \( N(M, \mathcal{F}) = TM/T\mathcal{F} \) and any \( \mathcal{Fol} \)-map \( f : (M, \mathcal{F}) \to (M_1, \mathcal{F}_1) \) into the quotient map \( Nf = [Tf] : N(M, \mathcal{F}) \to N(M_1, \mathcal{F}_1) \). This bundle functor \( N \) is product preserving. Another product preserving bundle functor \( \mathcal{Fol} \to \mathcal{FM} \) can be found in [10]. (In [5], the second author described all product preserving bundle functors \( F : \mathcal{Fol} \to \mathcal{FM} \) in terms of Weil algebra homomorphisms \( \mu : A \to B \).)

**Example 3.** Let \( F = T \otimes V : \mathcal{FM} \to \mathcal{VB} \) be the vector bundle functor sending any fibred manifold \( p : Y \to M \) into the tensor product \( FY = TM \otimes_Y VY \) of the tangent bundle \( TM \) with the vertical bundle \( VY \to Y \) of \( Y \to M \), and any \( \mathcal{FM} \)-map \( f : Y \to Y_1 \) covering \( f : M \to M_1 \) into \( Ff = Tf \otimes Vf : FY \to FY_1 \). This bundle functor \( F \) is fibre product preserving but is not product preserving. Using the standard “gluing” argument one can uniquely extend \( F \) to \( \widehat{F} : \mathcal{Fol} \to \mathcal{FM} \). In this way we obtain a vector bundle functor which is not product preserving. (In [3], I. Kolář and the second author described all fibre product preserving vector bundle functors \( F : \mathcal{FM} \to \mathcal{VB} \). The functors are of the form \( F = G \otimes V : \mathcal{FM} \to \mathcal{VB} \) \((FY = GM \otimes_Y VY, Ff = Gf \otimes Vf)\) for some vector bundle functor \( G : \mathcal{MF} \to \mathcal{VB} \). Taking \( G \) of strictly infinite order (see Example 1), we produce \( F : \mathcal{FM} \to \mathcal{FM} \) of strictly infinite order. Then using the standard “gluing” argument we produce \( \widehat{F} : \mathcal{Fol} \to \mathcal{FM} \) of strictly infinite order.)

**Example 4.** Let \( S \) be a manifold. We have a trivial bundle functor \( F = \text{id}_{\mathcal{Fol}} \times \text{id}_S : \mathcal{Fol} \to \mathcal{FM} \), \( F(M, \mathcal{F}) = M \times S \), \( Ff = f \times \text{id}_S \). This \( F \) is not a product preserving bundle functor if \( S \) is not one point. If \( S \) is not a vector bundle, then \( F \) is not a vector bundle functor.

We recall that a bundle functor \( F : \mathcal{Fol} \to \mathcal{FM} \) is of *locally finite order* if for any \( m, n \) there exists a finite number \( r_{m, n} \) such that for any foliated \((m+n)-dimensional \) manifold \( M \) with \( n \)-dimensional foliation \( \mathcal{F} \) and
any $\mathcal{F}ol$-maps $f, g : (M, \mathcal{F}) \rightarrow (N, \mathcal{F}_1)$ (into an arbitrary foliated manifold $(N, \mathcal{F}_1)$) and any $x \in M$, from $j^{r(m, n)}_x f = j^{r(m, n)}_x g$ it follows that $F f = F g$ on the fiber of $F(M, \mathcal{F})$ over $x$.

The purpose of the present note is to prove the following theorem which gives an affirmative answer to Question 1.

**Theorem 1.** Any bundle functor $F : \mathcal{F}ol \rightarrow \mathcal{F}M$ has locally finite order in the following sense: Let $m, n$ be positive integers, $(M, \mathcal{F})$ be an $(m + n)$-dimensional foliated manifold $M$ with $n$-dimensional foliation $\mathcal{F}$, and $x \in M$ be a point. Then for all $\mathcal{F}ol$-maps $f_1, f_2 : (M, \mathcal{F}) \rightarrow (M_1, \mathcal{F}_1)$, from $j^{r(m, n)}_x f_1 = j^{r(m, n)}_x f_2$ it follows that $F f_1 = F f_2$ on the fiber over $x$, where $r(m, n) = \max(\text{ord}(F|\mathcal{F}M_{m+1,n}), \text{ord}(F|\mathcal{F}M_{m,n+1})))$.

**Proof.** Let $f_1, f_2 : (M, \mathcal{F}) \rightarrow (M_1, \mathcal{F}_1)$ be $\mathcal{F}ol$-maps such that $j^{r(m, n)}_x f_1 = j^{r(m, n)}_x f_2$ for some $x \in M$. We show that $F f_1 = F f_2$ over $x$.

(I) First we assume that $p \geq m$ and $q \geq n$. Because of the regularity of $F$ we can assume that $d_x f_1$ is of rank $m + n$. Then by the rank theorem we can assume $(M, \mathcal{F}) = (\mathbb{R}^m \times \mathbb{R}^n, \{a\} \times \mathbb{R}^n)_{a \in \mathbb{R}^m}$, $x = (0, 0)$, $(M_1, \mathcal{F}_1) = (\mathbb{R}^p \times \mathbb{R}^q, \{c\} \times \mathbb{R}^q)_{c \in \mathbb{R}^p}$, $f_1(0, 0) = f_2(0, 0) = (0, 0)$ and

$$f_1(x, y) = ((x, 0), (y, 0))$$

for any $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$. Let $f_i(x, y) = (\varphi_i(x), \psi_i(x, y))$ for any $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$, $i = 1, 2$. Define $\mathcal{F}ol$-maps $\Phi_i : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m \times \mathbb{R}^q$ by

$$\Phi_i(x, y) = (x, \psi_i(x, y)), \quad (x, y) \in \mathbb{R}^m \times \mathbb{R}^n,$$

and $\Psi_i : \mathbb{R}^m \times \mathbb{R}^q \rightarrow \mathbb{R}^p \times \mathbb{R}^q$,

$$\Psi_i(x, z) = (\varphi_i(x), z), \quad (x, z) \in \mathbb{R}^m \times \mathbb{R}^q.$$

Then $f_i = \Psi_i \circ \Phi_i$, $i = 1, 2$.

Define a bundle functor $G : \mathcal{F}M_m \rightarrow \mathcal{F}M$ by $G = F|\mathcal{F}M_m$. Of course, the $\Phi_i$ are $\mathcal{F}M_m$-maps and $j^{r(m, n)}_{(0, 0)} \Phi_1 = j^{r(m, n)}_{(0, 0)} \Phi_2$. Then by Proposition 2 we have $G \Phi_1 = G \Phi_2$ over $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^n$. So $F \Phi_1 = F \Phi_2$ on the fibre $F_{(0, 0)}$ of $F(\mathbb{R}^m \times \mathbb{R}^n, \{a\} \times \mathbb{R}^n)_{a \in \mathbb{R}^m}$ over $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^n$.

Hence it remains to show that $F \Psi_1 = F \Psi_2$ over $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^n$.

We define $\mathcal{F}ol$-maps $\tilde{\Psi}_i := \varphi_i \times \text{id}_{\mathbb{R}^n} : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^p \times \mathbb{R}^n$ and $I_s : \mathbb{R}^s \times \mathbb{R}^n \rightarrow \mathbb{R}^s \times \mathbb{R}^q$, $I_s(w, y) = (w, (y, 0))$. Then $\tilde{\Psi}_i \circ I_m = I_p \circ \tilde{\Psi}_i$ and $I_m = \Phi_1$. Clearly, $FI_s$ is an embedding because $I_s$ is (see [2]). Then it suffices to show that $F \tilde{\Psi}_1 = F \tilde{\Psi}_2$ over $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^n$.

Define a bundle functor $H : \mathcal{M} f \rightarrow \mathcal{F}M$ by $HM = F(M \times \mathbb{R}^n, \{a\} \times \mathbb{R}^n)$, $H \varphi = F(\varphi \times \text{id}_{\mathbb{R}^n})$. Clearly, $j_0^{r(m, n)} \varphi_1 = j_0^{r(m, n)} \varphi_2$. Then by Proposition 1, $H \varphi_1 = H \varphi_2$ over $0 \in \mathbb{R}^m$. Therefore $F \tilde{\Psi}_1 = F \tilde{\Psi}_2$ over $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^n$, as well, which implies $F f_1 = F f_2$ over $x \in M$ under the assumption $p \geq m$ and $q \geq n$. 


(II) Now let $p$ and $q$ be arbitrary. We may assume $(M, \mathcal{F}) = (\mathbb{R}^m \times \mathbb{R}^n, \{\{a\} \times \mathbb{R}^n\}_{a \in \mathbb{R}^m}, x = (0, 0), (M_1, \mathcal{F}_1) = (\mathbb{R}^p \times \mathbb{R}^q, \{\{c\} \times \mathbb{R}^q\}_{c \in \mathbb{R}^p}, f_1(0, 0) = f_2(0, 0) = (0, 0)$. Let $\tilde{p} \geq \max(m, p)$ and $\tilde{q} \geq \max(n, q)$. Let $J : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^\tilde{p} \times \mathbb{R}^\tilde{q}$ be the $\text{Fol}$-embedding given by $J(u, w) = ((u, 0), (w, 0))$. Then $j^{r(m,n)}_{(0,0)}(J \circ f_1) = j^{r(m,n)}_{(0,0)}(J \circ f_2)$. Hence, by (I) for $J \circ f_i$ instead of $f_i$ and $(\tilde{p}, \tilde{q})$ instead of $(p, q)$, we have $F(J \circ f_1) = F(J \circ f_2)$ over $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^n$. But $FJ$ is an embedding because $J$ is. Then $Ff_1 = Ff_2$ over $x$ as well.

From Theorem 1 we immediately obtain the following corollary.

Corollary 1. Any bundle functor $F : \mathcal{FM} \to \mathcal{FM}$ has locally finite order in the following sense: Let $m, n$ be positive integers. Let $Y \to M$ be an $\mathcal{FM}_{m,n}$-object and $x \in Y$ be a point. Then for all $\mathcal{FM}$-morphisms $f_1, f_2 : Y \to Y_1$, from $j^{r(m,n)}_x f_1 = j^{r(m,n)}_x f_2$ it follows that $Ff_1 = Ff_2$ over $x$, where $r(m, n)$ is defined as in Theorem 1.

Example 5. Let $G : Mf \to \mathcal{VB}$ be the vector bundle functor of strictly infinite order as in Example 1. We define a bundle functor $F = G : \text{Fol} \to \mathcal{VB}$, $F(M, \mathcal{F}) = GM, Ff = Gf$. This bundle functor is of strictly infinite order. It is of locally finite order, but in this case we cannot replace $r(m, n)$ in Theorem 1 by an $r(m)$ depending only on $m$ (in contrast to Example 3).

References


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