ANNALES
POLONICI MATHEMATICI
93.1 (2008)

On the mean-value property of superharmonic functions

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Abstract. We complement a previous result concerning a converse of the mean-value property for smooth superharmonic functions. The case of harmonic functions was treated by Kuran and an improvement was given by Armitage and Goldstein.

Recall that a function u is *harmonic* (resp. *superharmonic*) on an open set $U \subset \mathbb{R}^n$ $(n \geq 1)$ if $u \in C^2(U)$ and $\Delta u = 0$ (resp. $\Delta u \leq 0$) on U. Denote by $H(U)$ the space of harmonic functions on U and by $SH(U)$ the subset of $C^2(U)$ consisting of superharmonic functions on U. Notice that superharmonic functions are usually defined in a more general sense (see [5] and Remark 3).

If $A \subset \mathbb{R}^n$ is Lebesgue measurable, $L^1(A)$ denotes the space of Lebesgue integrable functions on A. If A has finite measure we denote by $|A|$ the Lebesgue measure of A.

In [3] we proved the following theorem.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a bounded open set. Suppose that there exists $x_0 \in \Omega$ such that*

$$u(x_0) \geq \frac{1}{|\Omega|} \int_{\Omega} u(x)\, dx$$

for every $u \in SH(\Omega) \cap L^1(\Omega) \setminus H(\Omega)$. Then Ω is a ball with center x_0.

Theorem 1 extends a result obtained by Epstein and Schiffer [4] for harmonic functions. The final step concerning harmonic functions was achieved by Kuran [6] who proved the following theorem.

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a connected open set of finite measure. Suppose that there exists $x_0 \in \Omega$ such that*

$$u(x_0) = \frac{1}{|\Omega|} \int_{\Omega} u(x)\, dx \tag{1}$$

for every $u \in H(\Omega) \cap L^1(\Omega)$. Then Ω is a ball with center x_0.

2000 *Mathematics Subject Classification*: Primary 31B05.
Key words and phrases: superharmonic functions, mean-value property.

Remark 1. The proof of Theorem 2 given in [2] for $n = 2$, which extends to $n \geq 2$, shows that the connectedness assumption is superfluous.

Remark 2. The hypothesis of Theorem 2 can be weakened to require only that (1) holds for all positive harmonic functions that are integrable over Ω: see [1].

We first give a proof of the following result.

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be an open set of finite measure. Suppose that there exists $x_0 \in \Omega$ such that*

$$u(x_0) \geq \frac{1}{|\Omega|} \int_{\Omega} u(x)\,dx \tag{2}$$

for every $u \in SH(\Omega) \cap L^1(\Omega) \setminus H(\Omega)$. Then Ω is a ball with center x_0.

Proof. We shall show that Ω satisfies the assumptions of Theorem 2. Let $h \in H(\Omega) \cap L^1(\Omega)$. Let $y \in \mathbb{R}^n \setminus \Omega$ and $a_n \in (-n, 0)$. We define $v : \mathbb{R}^n \setminus \{y\} \to \mathbb{R}$ by

$$v(x) = -\|x - y\|^{a_n}, \qquad x \in \mathbb{R}^n \setminus \{y\}.$$

We have $\Delta v < 0$ in Ω. Moreover, $v \in L^1(\Omega)$. Indeed, let $B(y, r)$ denote the open ball of fixed radius $r > 0$. Clearly $v \in L^1(B(y, r))$. Since Ω has finite measure and v is bounded on $\Omega \setminus B(y, r)$, $v \in L^1(\Omega \setminus B(y, r))$ and the result follows. Now for $m \in \mathbb{N}^\star = \{1, 2, 3, \ldots\}$ we set

$$u_m = h + \frac{1}{m} v.$$

Then $u_m \in SH(\Omega) \cap L^1(\Omega) \setminus H(\Omega)$ and we have

$$u_m(x_0) \geq \frac{1}{|\Omega|} \int_{\Omega} u_m(x)\,dx,$$

that is,

$$h(x_0) + \frac{1}{m} v(x_0) \geq \frac{1}{|\Omega|} \int_{\Omega} h(x)\,dx + \frac{1}{m|\Omega|} \int_{\Omega} v(x)\,dx.$$

Letting $m \to +\infty$ we obtain

$$h(x_0) \geq \frac{1}{|\Omega|} \int_{\Omega} h(x)\,dx. \tag{3}$$

As (3) holds for every $h \in H(\Omega) \cap L^1(\Omega)$, replacing h by $-h$ in (3) we conclude that

$$h(x_0) = \frac{1}{|\Omega|} \int_{\Omega} h(x)\,dx$$

for all $h \in H(\Omega) \cap L^1(\Omega)$. Then Theorem 2 implies that Ω is a ball centered at x_0.

Now we have the following theorem.

THEOREM 4. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be an open set of finite measure. Suppose that, for all $y \in \partial\Omega$, there exists a sequence (y_j) in $\mathbb{R}^n \setminus \overline{\Omega}$ such that $y_j \to y$ as $j \to +\infty$. If* (2) *holds for every positive $u \in SH(\Omega) \cap L^1(\Omega) \setminus H(\Omega)$, then Ω is a ball with center x_0.*

We shall need three lemmas.

LEMMA 1. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be an open set of finite measure. Suppose that* (2) *holds for all positive $u \in SH(\Omega) \cap L^1(\Omega) \setminus H(\Omega)$. Then* (2) *holds for all $u \in SH(\Omega) \cap L^1(\Omega) \setminus H(\Omega)$ that are bounded from below.*

Proof. Let $u \in SH(\Omega) \cap L^1(\Omega) \setminus H(\Omega)$ be bounded from below. There exists $c \in \mathbb{R}$ such that $u > c$ on Ω. Then $u - c \in SH(\Omega) \cap L^1(\Omega) \setminus H(\Omega)$ since Ω has finite measure, and $u - c$ is positive on Ω. By hypothesis, $u - c$ has the mean value property (2). The constant function c has the mean value property (1). Hence u satisfies (2).

LEMMA 2. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be an open set of finite measure and let $\alpha \in (-n, 0)$.*

(i) *Suppose that there exists a sequence (y_j) in $\mathbb{R}^n \setminus \overline{\Omega}$ such that $y_j \to y \in \partial\Omega$ as $j \to +\infty$.*

(ia)
$$\lim_{j \to +\infty} \int_{\Omega} \|x - y_j\|^{\alpha}\, dx = \int_{\Omega} \|x - y\|^{\alpha}\, dx.$$

(ib) *If (b_j) is a sequence in $\mathbb{R}^n$ such that $b_j \to b$ as $j \to +\infty$, then*
$$\lim_{j \to +\infty} \int_{\Omega} b_j.(x - y_j)\|x - y_j\|^{\alpha - 1}\, dx = \int_{\Omega} b.(x - y)\|x - y\|^{\alpha - 1}\, dx.$$

(ii) *Suppose that there exists a sequence (z_j) in $\mathbb{R}^n \setminus \overline{\Omega}$ such that $\|z_j\| \to +\infty$ as $j \to +\infty$. Then*
$$\lim_{j \to +\infty} \int_{\Omega} \|x - z_j\|^{\alpha}\, dx = 0.$$

Proof. (i) Since the arguments are similar, we only prove (ia). Since Ω has finite measure the function $x \mapsto \|x - y_j\|^{\alpha}$ is in $L^1(\Omega)$ for every $j \in \mathbb{N}$ and we have seen in the proof of Theorem 3 that $x \mapsto \|x - y\|^{\alpha}$ is also in $L^1(\Omega)$. Let $r > 0$ be fixed. There exists $j(r) \in \mathbb{N}$ such that $y_j \in B(y, r/3)$ for all $j \geq j(r)$. For $j \geq j(r)$ we can write

$$\int_{\Omega \cap B(y,r)} \|x - y_j\|^{\alpha}\, dx = \int_{\Omega \cap B(y_j, r/3)} \|x - y_j\|^{\alpha}\, dx + \int_{\Omega \cap B(y,r) \setminus B(y_j, r/3)} \|x - y_j\|^{\alpha}\, dx.$$

We have

$$\int_{\Omega\cap B(y_j,r/3)} \|x-y_j\|^\alpha\,dx \le C_n r^{n+\alpha}, \qquad \int_{\Omega\cap B(y,r)} \|x-y\|^\alpha\,dx \le C_n r^{n+\alpha}$$

and

$$\int_{\Omega\cap B(y,r)\setminus B(y_j,r/3)} \|x-y_j\|^\alpha\,dx \le (r/3)^\alpha|\Omega\cap B(y,r)\setminus B(y_j,r/3)| \le C_n r^{n+\alpha},$$

where $C_n > 0$ is independent of j and r. On the other hand, the Lebesgue dominated convergence theorem implies that

$$\lim_{j\to+\infty} \int_{\Omega\setminus B(y,r)} \|x-y_j\|^\alpha\,dx = \int_{\Omega\setminus B(y,r)} \|x-y\|^\alpha\,dx. \tag{4}$$

Now let $f_j(x) = \|x-y_j\|^\alpha - \|x-y\|^\alpha$, $x\in\Omega$. For $j \ge j(r)$ we write

$$\begin{aligned}\Big|\int_\Omega f_j(x)\,dx\Big| &\le \Big|\int_{\Omega\cap B(y,r)} f_j(x)\,dx\Big| + \Big|\int_{\Omega\setminus B(y,r)} f_j(x)\,dx\Big| \\ &\le 3C_n r^{n+\alpha} + \Big|\int_{\Omega\setminus B(y,r)} f_j(x)\,dx\Big|.\end{aligned}$$

Let $\varepsilon > 0$. Take $r > 0$ such that $3C_n r^{n+\alpha} \le \varepsilon/2$. By (4) there exists $j_0 \ge j(r)$ such that

$$\Big|\int_{\Omega\setminus B(y,r)} f_j(x)\,dx\Big| \le \varepsilon/2 \quad \forall j \ge j_0,$$

and (ia) follows.

(ii) For $k\in\mathbb{N}^\star$ define $\Omega_k = \Omega\setminus B(0,k)$. Let $k\in\mathbb{N}^\star$ and $r>0$ be fixed. We write

$$\int_\Omega \|x-z_j\|^\alpha\,dx = \int_{\Omega\cap B(0,k)} \|x-z_j\|^\alpha\,dx + \int_{\Omega_k} \|x-z_j\|^\alpha\,dx.$$

We have

$$\int_{\Omega_k\setminus B(z_j,r)} \|x-z_j\|^\alpha\,dx \le r^\alpha|\Omega_k|$$

and

$$\int_{\Omega_k\cap B(z_j,r)} \|x-z_j\|^\alpha\,dx \le C_n r^{n+\alpha},$$

where $C_n > 0$ is independent of j, k and r. Now let $\varepsilon > 0$. Take $r > 0$ such that $C_n r^{n+\alpha} \le \varepsilon/3$. Since $|\Omega_k| \to 0$ as $k\to+\infty$, there exists $k\in\mathbb{N}$ such that $r^\alpha|\Omega_k| \le \varepsilon/3$. By the Lebesgue dominated convergence theorem there exists $j_0\in\mathbb{N}$ such that

$$\int_{\Omega\cap B(0,k)} \|x-z_j\|^\alpha\,dx \le \varepsilon/3 \quad \forall j\ge j_0,$$

and (ii) follows.

Lemma 3. *In the setting of Theorem 4 there exists a sequence* (z_j) *in* $\mathbb{R}^n \setminus \overline{\Omega}$ *such that* $\|z_j\| \to +\infty$ *as* $j \to +\infty$.

Proof. Suppose first that $\partial\Omega$ is bounded. Since Ω has finite measure, we deduce that $\mathbb{R}^n \setminus \overline{\Omega}$ is unbounded and the lemma follows. Now, if $\partial\Omega$ is unbounded, there exists a sequence (y_j) in $\partial\Omega$ such that $\|y_j\| \to +\infty$. By hypothesis, for each $j \in \mathbb{N}$ there exists $z_j \in (\mathbb{R}^n \setminus \overline{\Omega}) \cap B(y_j, 1)$. Clearly $\|z_j\| \to +\infty$.

Proof of Theorem 4. Since Ω has finite measure there exists a largest open ball B centered at x_0 of radius r which lies in Ω. We will show that $\Omega = B$. There exists $y \in \partial\Omega \cap \partial B$ such that $\|y - x_0\| = r$. Let (y_j) in $\mathbb{R}^n \setminus \overline{\Omega}$ be such that $y_j \to y$ as $j \to +\infty$ and let (z_j) be as in Lemma 3. Define

$$h(x) = r^{n-2}(\|x - x_0\|^2 - r^2)\|x - y\|^{-n}, \qquad x \in \mathbb{R}^n \setminus \{y\},$$

$$h_j(x) = \|y_j - x_0\|^{n-2}(\|x - x_0\|^2 - \|y_j - x_0\|^2)\|x - y_j\|^{-n}, \qquad x \in \mathbb{R}^n \setminus \{y_j\},$$

and

$$v_j(x) = -\|x - z_j\|^{a_n}, \qquad x \in \mathbb{R}^n \setminus \{z_j\},$$

where $a_n \in (-n, 0)$. Clearly $h \in H(\mathbb{R}^n \setminus \{y\})$, $h_j \in H(\mathbb{R}^n \setminus \{y_j\})$ and $\Delta v_j < 0$ in Ω. Moreover h, h_j and v_j are in $L^1(\Omega)$, $h(x_0) = -1$ and $h > 0$ on $\mathbb{R}^n \setminus \overline{B}$. Let $u_j = 1 + h_j + v_j$. Then $u_j \in SH(\Omega) \cap L^1(\Omega) \setminus H(\Omega)$ and u_j is bounded from below on Ω for every j. Therefore Lemma 1 implies that

$$u_j(x_0) \geq \frac{1}{|\Omega|} \int_{\Omega} u_j(x)\, dx \qquad \forall j \in \mathbb{N}. \tag{5}$$

By Lemma 2 we can let $j \to +\infty$ in (5) to obtain

$$1 + h(x_0) \geq \frac{1}{|\Omega|} \int_{\Omega} (1 + h(x))\, dx. \tag{6}$$

Since $1 + h \in H(\Omega) \cap L^1(\Omega)$ we have

$$0 = 1 + h(x_0) = \int_{B} (1 + h(x))\, dx. \tag{7}$$

Now with the help of (6) and (7) we can write

$$\begin{aligned}
0 &\geq \frac{1}{|\Omega|} \int_{\Omega} (1 + h(x))\, dx \\
&= \frac{1}{|\Omega|} \int_{\Omega \setminus B} (1 + h(x))\, dx + \frac{1}{|\Omega|} \int_{B} (1 + h(x))\, dx \\
&= \frac{1}{|\Omega|} \int_{\Omega \setminus B} (1 + h(x))\, dx \geq \frac{|\Omega \setminus B|}{|\Omega|} \geq \frac{|\Omega \setminus \overline{B}|}{|\Omega|}.
\end{aligned}$$

This implies that $|\Omega \setminus \overline{B}| = 0$. Then the open set $\Omega \setminus \overline{B}$ must be empty, hence $\Omega \subset \overline{B}$. Since Ω is open and $B \subset \Omega \subset \overline{B}$, we deduce that $\Omega = B$.

Remark 3. The assumption in Theorem 2 imposes a certain geometric restriction on the open set Ω. We give an example that shows that this hypothesis cannot be omitted completely. Let $\Omega = B \setminus \{x\}$ where B denotes an open ball centered at the origin in $\mathbb{R}^n$ $(n \geq 2)$ and $x \in B \setminus \{0\}$. We claim that if $u \in L^1(\Omega)$ is a positive superharmonic function on Ω, then (2) holds. Indeed such a function u has a (unique) superharmonic extension v on B (see [5, Theorem 7.7, p. 130]). Then we have

$$u(0) = v(0) \geq \frac{1}{|B|} \int_B v(y)\,dy = \frac{1}{|\Omega|} \int_\Omega u(y)\,dy.$$

In fact, using Theorem 7.7 in [5], we can take $\Omega = B \setminus Z$ where Z is a relatively closed polar subset of B such that $0 \notin Z$. For instance in $\mathbb{R}^3$, Z could be a line segment (see [5, Example 4, p. 127]).

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Received 6.9.2007
and in final form 8.1.2008 (1815)