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# Convergence in capacity

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**Abstract.** We prove that if $\mathcal{E}(\Omega) \ni u_j \to u \in \mathcal{E}(\Omega)$ in $C_n$-capacity then $\liminf_{j\to\infty}(dd^c u_j)^n \geq 1_{\{u>-\infty\}}(dd^c u)^n$. This result is used to consider the convergence in capacity on bounded hyperconvex domains and compact Kähler manifolds.

**1. Introduction.** Let $\Omega$ be an open set in $\mathbb{C}^n$. We denote by $\mathrm{PSH}(\Omega)$ the set of plurisubharmonic (psh) functions on $\Omega$. In [BT1, 2] the authors established the comparison principle and used it to study the Dirichlet problem in $\mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega)$. Recently, Cegrell introduced a general class $\mathcal{E}(\Omega)$ of psh functions on which the complex Monge–Ampère operator can be defined. He obtained many important results of pluripotential theory in $\mathcal{E}(\Omega)$, for example, on the comparison principle and solvability of the Dirichlet problem (see [Ce1,2]). In [Bł1, 2] Błocki proved that the class $\mathcal{E}(\Omega)$ has local property. In [ÅCCH] the authors studied Monge–Ampère measure of functions in $\mathcal{E}(\Omega)$ on pluripolar sets and solved a general Dirichlet problem.

The aim of the present paper is to continue the study of convergence in capacity. In Section 2 we introduce some definitions and known results. In Section 3, we first prove that if $\mathcal{E}(\Omega) \ni u_j \to u \in \mathcal{E}(\Omega)$ in $C_n$-capacity then $\liminf_{j\to\infty}(dd^c u_j)^n \geq 1_{\{u>-\infty\}}(dd^c u)^n$. This result is then used to investigate when $(dd^c u_j)^n \to (dd^c u)^n$ weakly as $j \to \infty$.

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**2. Preliminaries.** First we recall some elements of pluripotential theory that will be used throughout the paper. All this can be found in [BT1, 2], [Ce1, 2], [GZ], [H1–3], [Kl], [Ko], [Xi1, 2].

**2.1.** Unless otherwise specified, $\Omega$ will be a bounded hyperconvex domain in $\mathbb{C}^n$, meaning that there exists a negative exhaustive psh function for $\Omega$.

**2.2.** The $C_n$-*capacity* in the sense of Bedford and Taylor on $\Omega$ is the set function given by

$$C_n(E) = C_n(E,\Omega) = \sup\Big\{ \int_E (dd^c u)^n : u \in \mathrm{PSH}(\Omega),\ -1 \le u \le 0\Big\}$$

for every Borel set $E$ in $\Omega$. It is known [BT2] that

$$C_n(E) = \int_\Omega (dd^c h^*_{E,\Omega})^n,$$

where $h^*_{E,\Omega}$ is the relative extremal psh function for $E$ (relative to $\Omega$) defined as the smallest upper semicontinuous majorant of $h_{E,\Omega}$, where

$$h_{E,\Omega}(z) = \sup\{u(z) : u \in \mathrm{PSH}(\Omega),\ -1 \le u \le 0,\ u \le -1 \text{ on } E\}.$$

The following definition was introduced in [Xi1]: A sequence $u_j \in \mathrm{PSH}^-(\Omega)$ converges to $u$ *in* $C_n$-*capacity* if

$$C_n(K \cap \{|u_j - u| > \delta\}) \to 0 \quad \text{as } j \to \infty,\ \forall K \subset\subset \Omega,\ \forall \delta > 0.$$

**2.3.** The following classes of psh functions were introduced by Cegrell in [Ce1, 2]:

$$\mathcal{E}_0 = \mathcal{E}_0(\Omega) = \Big\{\varphi \in \mathrm{PSH}^-(\Omega) \cap L^\infty(\Omega) : \lim_{z\to\partial\Omega} \varphi(z) = 0,\ \int_\Omega (dd^c\varphi)^n < \infty\Big\},$$

$$\mathcal{F} = \mathcal{F}(\Omega) = \Big\{\varphi \in \mathrm{PSH}^-(\Omega) : \exists \mathcal{E}_0(\Omega) \ni \varphi_j \searrow \varphi,\ \sup_{j\ge1} \int_\Omega (dd^c\varphi_j)^n < \infty\Big\},$$

$$\mathcal{E} = \mathcal{E}(\Omega) = \{\varphi \in \mathrm{PSH}^-(\Omega) : \exists \varphi_K \in \mathcal{F}(\Omega),\ \varphi_K = \varphi \text{ on } K,\ \forall K \subset\subset \Omega\},$$

$$\mathcal{E}^a = \mathcal{E}^a(\Omega) = \{\varphi \in \mathcal{E}(\Omega) : (dd^c\varphi)^n \text{ vanishes on all pluripolar sets}\}.$$

**2.4.** Let $u, v \in \mathcal{E}(\Omega)$. We say that $(u, v) \in \mathcal{A}(\Omega)$ if for every $z \in \Omega$ there is a neighborhood $G$ of $z$ in $\Omega$ and $\psi_G \in \mathcal{E}^a(G)$ such that $u + \psi_G \le v$ on $G$.

Next we introduce some results needed for our work:

**2.5.** Proposition. *Let $\mathcal{E}(\Omega) \ni v \le u_j \in \mathcal{E}(\Omega)$ for $j \ge 1$, and $\varphi \in PSH \cap L^\infty_{\mathrm{loc}}(\Omega)$. Assume that $u_j$ converges to some $u \in \mathcal{E}(\Omega)$ in $C_n$-capacity. Then $\varphi(dd^c u_j)^n \to \varphi(dd^c u)^n$ weakly as $j \to \infty$.*

*Proof.* We can assume that $\varphi \in \mathrm{PSH}^-(\Omega)$. Let $D \subset\subset \Omega$. By the remark following Definition 4.6 in [Ce2] we can find $w \in \mathcal{F}(\Omega)$ such that $w|_D = v|_D$

and $w \geq v$ on $\Omega$. We set

$$\widetilde{u}_j = \max(u_j, w), \qquad \widetilde{u} = \max(u, w),$$
$$\widetilde{\varphi} = \sup\{\psi \in \mathrm{PSH}^-(\Omega) : \psi \leq \varphi \text{ on } D\} \in \mathcal{E}_0(\Omega).$$

We have $\mathcal{F}(\Omega) \ni \widetilde{u}_j \to \widetilde{u} \in \mathcal{F}(\Omega)$ in $C_n$-capacity and $\widetilde{u}_j|_D = u_j|_D$, $\widetilde{u}|_D = u|_D$, $\widetilde{\varphi}|_D = \varphi|_D$. We only have to prove that $\widetilde{\varphi}(dd^c\widetilde{u}_j)^n \to \widetilde{\varphi}(dd^c\widetilde{u})^n$ weakly as $j \to \infty$. We can assume that $\widetilde{\varphi}(dd^c\widetilde{u}_j)^n \to \mu$ weakly as $j \to \infty$. Let $C^-(\Omega) \ni \widetilde{\varphi}_k \searrow \widetilde{\varphi}$ and $f \in C_0^\infty(\Omega)$ with $f \geq 0$. By Theorem 1.1 in [Ce3] we have $(dd^c\widetilde{u}_j)^n \to (dd^c\widetilde{u})^n$ weakly as $j \to \infty$. So, we obtain

$$\int_\Omega f\,d\mu = \lim_{j\to\infty} \int_\Omega f\widetilde{\varphi}(dd^c\widetilde{u}_j)^n \leq \limsup_{k\to\infty} \Big(\lim_{j\to\infty} \int_\Omega f\widetilde{\varphi}_k(dd^c\widetilde{u}_j)^n\Big)$$
$$= \limsup_{k\to\infty} \int_\Omega f\widetilde{\varphi}_k(dd^c\widetilde{u})^n = \int_\Omega f\widetilde{\varphi}(dd^c\widetilde{u})^n.$$

Therefore $\mu \leq \widetilde{\varphi}(dd^c\widetilde{u})^n$. On the other hand, by the proof of Theorem 1.1 in [Ce3] we have

$$\int_\Omega \widetilde{\varphi}(dd^c\widetilde{u})^n = \lim_{j\to\infty} \int_\Omega \widetilde{\varphi}(dd^c\widetilde{u}_j)^n \leq \mu(\Omega).$$

Hence $\mu = \widetilde{\varphi}(dd^c\widetilde{u})^n$.

**2.6.** Proposition.

(i) *If $u, v \in \mathcal{E}(\Omega)$, $u \geq v$ then*

$$1_{\{u=-\infty\}}(dd^c u)^n \leq 1_{\{v=-\infty\}}(dd^c v)^n.$$

(ii) *If $u \in \mathcal{E}(\Omega)$ and $v \in \mathcal{E}^a(\Omega)$ then*

$$1_{\{u+v=-\infty\}}(dd^c(u+v))^n = 1_{\{u=-\infty\}}(dd^c u)^n.$$

*where $1_E$ is the characteristic function of the set $E$.*

*Proof.* (i) See Lemma 4.3 in [ÅCCH].
(ii) See Lemma 4.8 in [ÅCCH].

**2.7.** Proposition. *Let $\mu$, $\nu$ be non-negative measures on $\Omega$. Assume that $\mu(\Omega) + \nu(\Omega) < \infty$ and $\int_\Omega -\varphi\,d\mu \geq \int_\Omega -\varphi\,d\nu$ for all $\varphi \in \mathcal{E}_0(\Omega)$. Then $\mu(K) \geq \nu(K)$ for all complete pluripolar subsets $K$ in $\Omega$.*

*Proof.* By Theorem 2.1 in [Ce2] we have

$$\int_\Omega -\varphi\,d\mu \geq \int_\Omega -\varphi\,d\nu \qquad \forall \varphi \in \mathrm{PSH}^- \cap L^\infty(\Omega).$$

Let $\psi \in \mathrm{PSH}^-(\Omega)$ be such that $K = \{\psi = -\infty\}$. We have

$$\int_\Omega -\max(\varepsilon\psi, -1)\,d\mu \geq \int_\Omega -\max(\varepsilon\psi, -1)\,d\nu$$

for all $\varepsilon > 0$. Letting $\varepsilon \to 0$ we get $\mu(K) \geq \nu(K)$.

**2.8.** Proposition. *Let $K$ be a compact subset of $E_1 \times \cdots \times E_n$ with $E_1, \ldots, E_n$ polar in $\mathbb{C}$. Then there exists a function $\varphi \in \mathrm{PSH}(\mathbb{C}^n)$ such that $K = \{\varphi = -\infty\}$.*

*Proof.* We can assume that $E_1, \ldots, E_n$ are complete polar in $\mathbb{C}$. Let $a = (a_1, \ldots, a_n) \notin K$. Since $E_1 \setminus \{a_1\}, \ldots, E_n \setminus \{a_n\}$ are complete polar in $\mathbb{C}$ we can find $u_1, \ldots, u_n \in \mathrm{PSH}(\mathbb{C})$ such that $E_j \setminus \{a_j\} = \{u_j = -\infty\}$ for $j = 1, \ldots, n$. Set

$$u(z_1, \ldots, z_n) = u_1(z_1) + \cdots + u_n(z_n) \in \mathrm{PSH}(\mathbb{C}^n).$$

Then $u(a) > -\infty$ and $K \subset E_1 \times \cdots \times E_n \setminus \{(a_1, \ldots, a_n)\} \subset \{u = -\infty\}$. By [Ze], $K$ is complete pluripolar in $\mathbb{C}^n$.

We set

$$\mathcal{K}(\Omega) = \{u \in \mathcal{E}(\Omega) : 1_{\{u=-\infty\}}(dd^c u)^n(\Omega \setminus E_1 \times \cdots \times E_n) = 0 \quad \text{for some } E_1, \ldots, E_n \text{ polar in } \mathbb{C}\}.$$

**2.9.** Proposition.

(i) *If $u \in \mathrm{PSH}^-(\Omega)$, $v \in \mathcal{K}(\Omega)$ and $u \ge v$ then $u \in \mathcal{K}(\Omega)$.*
(ii) *If $u, v \in \mathcal{K}(\Omega)$ then $u + v \in \mathcal{K}(\Omega)$.*
(iii) *If $u_1 \in \mathcal{K}(\Omega_1)$ and $u_2 \in \mathcal{K}(\Omega_2)$ then $\max(u_1, u_2) \in \mathcal{K}(\Omega_1 \times \Omega_2)$.*

*Proof.* (i) Follows directly from Proposition 2.6.
(ii) & (iii) Follow from [ÅCCH].

**3. Convergence in capacity.** We start with the first main result:

**3.1.** Theorem. *Let $\mathcal{E}(\Omega) \ni u_j \to u \in \mathcal{E}(\Omega)$ in $C_n$-capacity. Then*

$$\liminf_{j\to\infty} (dd^c u_j)^n \ge 1_{\{u>-\infty\}}(dd^c u)^n.$$

*Proof.* Let $f \in C_0^\infty(\Omega)$ and $\Omega' \subset\subset \Omega$ with $f \ge 0$ and $\mathrm{supp} f \subset\subset \Omega'$. We only have to prove that

$$\liminf_{j\to\infty} \Big[ \int_\Omega f(dd^c u_j)^n - \int_\Omega 1_{\{u>-\infty\}} f(dd^c u)^n \Big] \ge 0.$$

For each $s > 0$ we have

$$\int_\Omega f(dd^c u_j)^n - \int_\Omega 1_{\{u>-\infty\}} f(dd^c u)^n = A_{js} + B_{js} + C_s,$$

where

$$A_{js} = \int\limits_{\Omega} f[(dd^c u_j)^n - (dd^c \max(u_j, -s))^n] + \int\limits_{\Omega} 1_{\{u=-\infty\}} f(dd^c u)^n,$$
$$B_{js} = \int\limits_{\Omega} f[(dd^c \max(u_j, -s))^n - (dd^c \max(u, -s))^n],$$
$$C_s = \int\limits_{\Omega} f[(dd^c \max(u, -s))^n - (dd^c u)^n].$$

By Theorem 4.1 in [KH] we get

$$\begin{aligned} A_{js} &= \int\limits_{\{u_j \le -s\}} f[(dd^c u_j)^n - (dd^c \max(u_j, -s))^n] + \int\limits_{\Omega} 1_{\{u=-\infty\}} f(dd^c u)^n \\ &\ge -\int\limits_{\{u_j \le -s\}} f(dd^c \max(u_j, -s))^n + \int\limits_{\Omega} 1_{\{u=-\infty\}} f(dd^c u)^n \\ &\ge -\int\limits_{\{u_j \le -s\} \cap \{|u_j - u| \le 1\}} f(dd^c \max(u_j, -s))^n \\ &\quad - \int\limits_{\{|u_j - u| > 1\}} f(dd^c \max(u_j, -s))^n] + \int\limits_{\Omega} 1_{\{u=-\infty\}} f(dd^c u)^n \\ &\ge -\int\limits_{\{u < -s+2\}} f(dd^c \max(u_j, -s))^n - s^n C_n(\{|u_j - u| > 1\} \cap \Omega') \\ &\quad + \int\limits_{\Omega} 1_{\{u=-\infty\}} f(dd^c u)^n \\ &\ge \int\limits_{\Omega} h_{\{u<-s+2\}\cap\Omega',\Omega} f(dd^c \max(u_j, -s))^n - s^n C_n(\{|u_j - u| > 1\} \cap \Omega') \\ &\quad + \int\limits_{\Omega} 1_{\{u=-\infty\}} f(dd^c u)^n. \end{aligned}$$

Letting $j \to \infty$ by Proposition 2.5 we have

$$\liminf_{j\to\infty} A_{js} \ge \int\limits_{\Omega} h_{\{u<-s+2\}\cap\Omega',\Omega} f(dd^c \max(u, -s))^n + \int\limits_{\Omega} 1_{\{u=-\infty\}} f(dd^c u)^n.$$

Thus by Proposition 2.5 we get

$$\begin{aligned} &\liminf_{s\to\infty} (\liminf_{j\to\infty} A_{js}) \\ &\qquad \ge \liminf_{s\to\infty} \int\limits_{\Omega} h_{\{u<-s+2\}\cap\Omega',\Omega} f(dd^c \max(u, -s))^n + \int\limits_{\Omega} 1_{\{u=-\infty\}} f(dd^c u)^n \\ &\qquad \ge \liminf_{s\to\infty} \int\limits_{\Omega} h_{\{u<-t\}\cap\Omega',\Omega} f(dd^c \max(u, -s))^n + \int\limits_{\Omega} 1_{\{u=-\infty\}} f(dd^c u)^n \\ &\qquad = \int\limits_{\Omega} h_{\{u<-t\}\cap\Omega',\Omega} f(dd^c u)^n + \int\limits_{\Omega} 1_{\{u=-\infty\}} f(dd^c u)^n \end{aligned}$$

for all $t > 0$. Since $\{u < -t\} \cap \Omega' \searrow \{u = -\infty\} \cap \Omega'$ as $t \to \infty$ and $C_n(\{u = -\infty\} \cap \Omega') = 0$, it follows that $h_{\{u<-t\}\cap\Omega',\Omega} \nearrow 0$ on $\Omega \setminus E$ as $t \to \infty$ for some subset $E$ of $\Omega$ with $C_n(E) = 0$. Letting $t \to \infty$ by the decomposition theorem of Cegrell (Theorem 5.11 in [Ce2]) we get

$$\liminf_{s\to\infty}(\liminf_{j\to\infty} A_{js}) \geq \int_\Omega -1_E f(dd^c u)^n + \int_\Omega 1_{\{u=-\infty\}} f(dd^c u)^n \geq 0.$$

Moreover by Proposition 2.5 we get

$$\liminf_{j\to\infty} \Big[ \int_\Omega f(dd^c u_j)^n - \int_\Omega 1_{\{u>-\infty\}} f(dd^c u)^n \Big] \geq \liminf_{s\to\infty}(\liminf_{j\to\infty} A_{js}) + \liminf_{s\to\infty} C_s \geq 0.$$

**3.2.** Corollary. *Let $\mathcal{E}(\Omega) \ni u_j \to u \in \mathcal{E}(\Omega)$ in $C_n$-capacity. Assume that $(u_j, u) \in \mathcal{A}(\Omega)$ for all $j \geq 1$. Then*

$$\liminf_{j\to\infty} (dd^c u_j)^n \geq (dd^c u)^n.$$

*Proof.* By Definition of $\mathcal{A}(\Omega)$ and Proposition 2.6 we have

$$(dd^c u_j)^n \geq 1_{\{u_j=-\infty\}}(dd^c u_j)^n \geq 1_{\{u=-\infty\}}(dd^c u)^n.$$

Hence Theorem 3.1 yields the assertion.

**3.3.** Corollary. *Let $\mathcal{F}(\Omega) \ni u_j \to u \in \mathcal{F}(\Omega)$ in $C_n$-capacity. Assume that $(u_j, u) \in \mathcal{A}(\Omega)$ for all $j \geq 1$ and*

$$\lim_{j\to\infty} \int_\Omega (dd^c u_j)^n = \int_\Omega (dd^c u)^n.$$

*Then $(dd^c u_j)^n \to (dd^c u)^n$ weakly as $j \to \infty$.*

*Proof.* We can assume that $(dd^c u_j)^n \to \mu$ weakly as $j \to \infty$. By Corollary 3.2 we get $\mu \geq (dd^c u)^n$. On the other hand,

$$\mu(\Omega) \leq \liminf_{j\to\infty} \int_\Omega (dd^c u_j)^n = \int_\Omega (dd^c u)^n$$

Therefore $\mu = (dd^c u)^n$.

The second main result is a generalization of Theorem 1.1 in [Ce3] for the class $\mathcal{K}(\Omega)$.

**3.4.** Theorem. *Let $u_j, v \in \mathcal{E}(\Omega)$, $u \in \mathcal{K}(\Omega)$, and $D \subset\subset \Omega$ be such that $u_j \geq v$ on $\Omega \setminus D$ for all $j \geq 1$ and $u_j \to u$ in $C_n$-capacity. Then $(dd^c u_j)^n \to (dd^c u)^n$ weakly as $j \to \infty$.*

*Proof.* Let $E_1, \ldots, E_n$ be polar subsets in $\mathbb{C}$ such that

$$1_{\{u=-\infty\}}(dd^c u)^n(\Omega \setminus E_1 \times \cdots \times E_n) = 0.$$

We set

$$\widetilde{u}_j = \max(u_j, v), \qquad \widetilde{u} = \max(u, v).$$

Then $\mathcal{E}(\Omega) \ni \widetilde{u}_j \to \widetilde{u} \in \mathcal{E}(\Omega)$ in $C_n$-capacity and $\widetilde{u}_j|_{\Omega\setminus D} = u_j|_{\Omega\setminus D}$, $\widetilde{u}|_{\Omega\setminus D} = u|_{\Omega\setminus D}$. By Proposition 2.5, $(dd^c\widetilde{u}_j)^n \to (dd^c\widetilde{u})^n$ weakly as $j \to \infty$. Let $\Omega'$ be a hyperconvex domain such that $D \subset\subset \Omega' \subset\subset \Omega$. By Stokes' theorem we have

$$\limsup_{j\to\infty} \int_{\Omega'} (dd^c u_j)^n = \limsup_{j\to\infty} \int_{\Omega'} (dd^c\widetilde{u}_j)^n \le \int_{\bar{\Omega}'} (dd^c\widetilde{u})^n < \infty.$$

So, we can assume that $(dd^c u_j)^n \to \mu$ weakly as $j \to \infty$. We only have to prove that $\mu = (dd^c u)^n$ on $\Omega'$. Let $\varphi \in \mathcal{E}_0(\Omega')$. By Stokes' theorem we get

$$\int_{\Omega'} -\varphi d\mu = \lim_{j\to\infty} \int_{\Omega'} -\varphi(dd^c u_j)^n \ge \lim_{j\to\infty} \int_{\Omega'} -\varphi(dd^c\widetilde{u}_j)^n \ge \int_{\Omega'} -\varphi(dd^c\widetilde{u})^n.$$

Moreover by Propositions 2.7 and 2.8 we get

$$\mu(K) \ge (dd^c u)^n(K)$$

for all compact subsets $K$ of $E_1 \times \cdots \times E_n$. Therefore $\mu \ge 1_{\{u=-\infty\}}(dd^c u)^n$. Thus by Theorem 3.1 we have

$$\mu \ge (dd^c u)^n \quad \text{on } \Omega'. \tag{1}$$

Let $\Omega''$ be a domain such that $D \subset\subset \Omega'' \subset\subset \Omega'$. By Stokes' theorem we have

$$\begin{aligned}\mu(\Omega'') &\le \liminf_{j\to\infty} \int_{\Omega''} (dd^c u_j)^n = \liminf_{j\to\infty} \int_{\Omega''} (dd^c\widetilde{u}_j)^n \\ &\le \int_{\bar{\Omega}''} (dd^c\widetilde{u})^n \le \int_{\Omega'} (dd^c\widetilde{u})^n = \int_{\Omega'} (dd^c u)^n.\end{aligned}$$

Hence

$$\mu(\Omega') \le (dd^c u)^n(\Omega'). \tag{2}$$

It follows from (1) and (2) that $\mu = (dd^c u)^n$ on $\Omega'$.

**3.5.** Example. We set $u_j(z_1, z_2) = \max(j \ln|z_1|, j^{-1}\ln|z_2|)$ on $\Delta^2$, the unit polydisk in $\mathbb{C}^2$. Then $\mathcal{F}(\Delta^2) \ni u_j \to 0$ in $C_n$-capacity but $(dd^c u_j)^n = \delta_{\{0\}} \not\to 0$ weakly as $j \to \infty$.

Let $X$ be a compact Kähler manifold with a fundamental form $\omega = \omega_X$ such that $\int_X \omega^n = 1$. An upper semicontinuous function $\varphi : X \to [-\infty, \infty)$ is called *$\omega$-plurisubharmonic* ($\omega$-psh) if $\varphi \in L^1(X)$ and $\omega + dd^c\varphi \ge 0$. We consider the Cegrell class

$$\begin{aligned}\mathcal{E}(X,\omega) = \{\varphi \in \mathrm{PSH}(X,\omega) : \forall z \in X,\ &\text{there is a neighborhood } U \text{ of } z \\ &\text{and a potential } \theta \text{ of } \omega \text{ on } U \text{ such that } (\varphi+\theta)|_U \in \mathcal{E}(U)\}.\end{aligned}$$

In [Ko] Kołodziej introduced the capacity $C_{X,\omega}$ on $X$ by

$$C_X(E) = C_{X,\omega}(E) = \sup\Big\{ \int_E \omega_\varphi^n : \varphi \in \mathrm{PSH}(X,\omega),\ -1 \le \varphi \le 0\Big\},$$

where $\omega_\varphi^n = (\omega + dd^c\varphi)^n$ and $n = \dim X$. In [GZ] Guedj and Zeriahi proved that $C_X$ is a Choquet capacity on $X$ and

$$C_X(E) = \int_X (-h_{E,\omega}^*)\omega_{h_{E,\omega}^*}^n,$$

where $h_{E,\omega}^*$ denotes the upper semicontinuous regularization of $h_{E,\omega}$ given by

$$h_{E,\omega}(z) = \sup\{\varphi(z) : \varphi \in \mathrm{PSH}^-(X,\omega),\ \varphi|_E \le -1\}.$$

From Corollary 3.2 we deduce the following

**3.6.** COROLLARY. *Let $\mathcal{E}(X,\omega) \ni u_j \to u \in \mathcal{E}(X,\omega)$ in $C_X$-capacity. Assume that $(u_j,u) \in \mathcal{A}(X)$ for all $j \ge 1$. Then $\omega_{u_j}^n \to \omega_u^n$ weakly as $j \to \infty$.*

*Proof.* We can assume that $\omega_{u_j}^n \to \mu$ weakly as $j \to \infty$ with $\mu(X) = \omega_u^n(X) = 1$. On the other hand, by Corollary 3.2 we have $\mu \ge \omega_u^n$. Hence $\mu = \omega_u^n$.

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