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On the Noether exponent

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Abstract. We obtain, in a simple way, an estimate for the Noether exponent of an ideal I without embedded components (i.e. we estimate the smallest number μ such that $(\operatorname{rad} I)^{\mu} \subset I$).

1. Introduction. Let $\mathbf{k}[X]$ be the polynomial ring of n variables over an algebraically closed field. The Nullstellensatz guarantees that for a given ideal $I \subset \mathbf{k}[X]$ there is a number μ such that $(\operatorname{rad} I)^{\mu} \subset I$. The smallest such μ is called the *Noether exponent* of the ideal I and will be denoted by μ_I. An estimate of this exponent was obtained by Kollár in [K]. Subsequently several authors contributed to this problem, especially in the easier case when the ideal I has only isolated components in its primary decomposition (see e.g. [CP], [FPT], [JOW], [STV], [AM]).

In this note we also consider the case without embedded components. We give a very simple method to obtain an estimate for the Noether exponent (Theorem 7) which is sharper than the results obtained in [FPT] and [JOW]. More precisely for the ideal $I = (f_1, \dots, f_k)$, where $\deg f_2 \geq \dots \geq \deg f_k \geq \deg f_1$ we show that

$$\mu_I \leq \max_{i \in \{r_1, \dots, r_m\}} \left\{ \frac{\deg f_1 \cdot \ \dots \ \cdot \deg f_i}{d_i} \right\},$$

where $r_1, \dots, r_m$ are all possible codimensions of irreducible components of the zero set of the ideal I, and d_i is the minimal degree of irreducible components of codimension i of the variety given by the ideal I.

We conjecture that this estimate is also valid in the general case.

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2. Preliminaries. We denote by $\mathbb{A}^n$ the affine space of dimension n and by $\mathbf{k}[X] = \mathbf{k}[x_1, \ldots, x_n]$ the polynomial ring over the algebraically closed field $\mathbf{k}$. The zero set of an ideal I is denoted by $V(I)$. For an algebraic set $Z \subset \mathbb{A}^n$ we consider the ideal $I(Z) = \{f \in \mathbf{k}[X] \mid f_{|Z} = 0\}$.

Let $Z_1, \ldots, Z_r \subset \mathbb{A}^n$ be hypersurfaces and let V be an irreducible component of $Z_1 \cap \ldots \cap Z_r$. We say that $Z_1, \ldots, Z_r$ *meet properly along* V or that this intersection is proper along V if $\dim V = n - r$. Recall

Definition 1. Let $Z_1 = \{F_1 = 0\}, \ldots, Z_r = \{F_r = 0\} \subset \mathbb{A}^n$ be hypersurfaces meeting properly along the variety V. Then we define the *index of intersection* of $Z_1, \ldots, Z_r$ along V to be the number

$$i(Z_1 \cdot \ldots \cdot Z_r; V) := e_{\mathbf{k}[X]_p}((F_1, \ldots, F_r)_p),$$

where

$$\frac{e_{\mathbf{k}[X]_p}((F_1, \ldots, F_r)_p)}{n!} t^n + \ldots$$

is the Hilbert–Samuel polynomial of $\mathbf{k}[X]_p/(F_1, \ldots, F_r)_p$ and $p = I(V)$ is the ideal of the variety V.

Remark 2. Since $\mathbf{k}[X]_p$ is a Cohen–Macaulay ring we have

$$i(Z_1 \cdot \ldots \cdot Z_r; V) = \operatorname{length}(\mathbf{k}[X]_p/(F_1, \ldots, F_r)_p).$$

See e.g. [F, Example 7.1.10, p. 123].

We will need the following facts:

Theorem 3 (Associativity formula from [Na, (24.7)]). *Let* $(R, \mathfrak{m})$ *be a local ring and let* I *be an* $\mathfrak{m}$*-primary ideal generated by a system of parameters* $f_1, \ldots, f_n \in \mathfrak{m}$. *Let* $\mathfrak{b}$ *be the ideal generated by* $f_1, \ldots, f_r$ *for some* $r \le n$, *and let* p_i *be the minimal prime ideals of* $\mathfrak{b}$ *such that* $\operatorname{length}(p_i) = k$ *and* $\dim R/p_i = n - r$. *Then*

$$e(I) = \sum_i e((I + p_i)p_i) \cdot e(\mathfrak{b}R_{p_i}).$$

Theorem 4. *Let* $\Phi : \mathbf{k}^n \to \mathbf{k}^n$ *be a generically finite polynomial mapping of geometric degree* $\operatorname{gdeg} \Phi$ (*by geometric degree we mean the number of points in a generic fiber*). *Then for each* $y \in \mathbf{k}^n$ *the number of isolated points in the fiber* $\Phi^{-1}(y)$ *is not greater than* $\operatorname{gdeg} \Phi$.

Proof. The statement is obvious for a quasi-finite mapping. The general case follows from the Stein factorization applied to the compactification of Φ. Indeed, let $\Gamma = \operatorname{graph} \Phi$ and let $\overline{\Gamma} \subset \mathbb{P}^n \times \mathbf{k}^n$ be its closure. Consider the projection $\overline{f} : \overline{\Gamma} \to \mathbf{k}^n$. Due to the Stein factorization theorem there exist a normal variety W and two morphisms, $q : \overline{\Gamma} \to W$ which has connected fibers and $u : W \to \mathbf{k}^n$ which is finite, such that $\overline{f} = u \circ q$. Moreover, $\operatorname{gdeg} u = \operatorname{gdeg} \Phi$. Consequently, every fiber $\overline{f}^{-1}(y)$ has no more than $\operatorname{gdeg} u = \operatorname{gdeg} \Phi$

connected components. This implies that the number of isolated points in a fiber $\Phi^{-1}(y)$ is not greater than $\operatorname{gdeg}\Phi$. ■

We have a useful characterization of the index of proper intersection:

PROPOSITION 5. *Let* $Z_1 = \{F_1 = 0\}, \ldots, Z_r = \{F_r = 0\} \subset \mathbb{A}^n$ *be hypersurfaces given by polynomials* F_i *which meet along* V *properly. Let* $H_j = V(\alpha_j)$ *be the hyperplane given by a linear form* α_j *for* $j \in \{1, \ldots, n-r\}$. *Define* $\Phi := (F_1, \ldots, F_r, \alpha_1, \ldots, \alpha_{n-r})$. *If the intersection* $Z_1 \cap \ldots \cap Z_r \cap H_1 \cap \ldots \cap H_{n-r}$ *is proper at a point* Q, *then*

$$i(Z_1 \cdot \ldots \cdot Z_r; V) = i(Z_1 \cdot \ldots \cdot Z_r \cdot L; Q) = \mu_Q(\Phi)$$

for every linear subspace L *of dimension* $n-r$ *which meets* $\bigcap_{i=1}^r Z_i$ *transversely at* Q. *Here* $\mu_Q(\Phi)$ *denotes the multiplicity index of* Φ *at* Q.

Proof. We follow [No]. Set $F = \{F_1, \ldots, F_r\}$ and $H = \{\alpha_1, \ldots, \alpha_{n-r}\}$. Clearly $F_1, \ldots, F_r$ form a system of parameters of the localization $\mathbf{k}[X]_p$, where p is the ideal of the variety V. Since the hyperplanes H_j for $j \in \{1, \ldots, n-r\}$ meet V transversely at Q, it follows that $\{F, H\}$ is a system of local parameters in the ring $\mathbf{k}[X]_{\mathfrak{m}_Q}$ and hence from the associativity formula we get

$$e((F, H)) = e(((F, H) + p)/p) \cdot e(\mathbf{k}[X]_p/(F)_p).$$

Since (F, H) generate the maximal ideal in the local ring $\mathbf{k}[X]_{\mathfrak{m}_Q}$, we get $e(((F, H) + p)/p) = 1$ and

$$e(F, H) = e(\mathbf{k}[X]_p/(F)_p),$$

which proves that $i(Z_1 \cdot \ldots \cdot Z_r; V) = i(Z_1 \cdot \ldots \cdot Z_r \cdot L; Q)$. ■

We will also need the following version of the Bézout Theorem.

THEOREM 6 (Bézout Theorem, an affine version). *Let* $Z_1 = \{F_1 = 0\}, \ldots, Z_r = \{F_r = 0\} \subset \mathbb{A}^n$ *be hypersurfaces given by polynomials* F_i *which meet along* $V_1, \ldots, V_s$ *properly. Then*

$$\sum_{i=1}^{s} i(Z_1 \cdot \ldots \cdot Z_r; V_i) \deg V_i \le \prod_{i=1}^{r} \deg F_i.$$

Proof. According to the previous proposition the index of intersection is independent of the choice of a generic point P. Take generic hyperplanes H_j given by linear forms α_j $(j = 1, \ldots, n-r)$ and set $d_i := \deg V_i$ for $i = 1, \ldots, s$.

Clearly the intersection $S := (\bigcup_{i=1}^s V_i) \cap \bigcap_{j=1}^{n-r} H_j$ is a finite set. Let $\{P_1^i, \ldots, P_{d_i}^i\} := V_i \cap S$ for $i \in \{1, \ldots, s\}$. Consider the map $\Phi := (F_1, \ldots, F_r, \alpha_1, \ldots, \alpha_{n-r})$. Note that it is generically finite and hence for a generic fiber $\Phi^{-1}(y)$ we have, by Theorem 4 and by the inequality of Rusek–Winiarski

[Ł, p. 319]),

$$\sum_{P\in\Phi^{-1}(y)} \mu_P(\Phi) = \operatorname{gdeg}\Phi \le \prod_{i=1}^{r} \deg F_i,$$

where $\mu_P(\Phi)$ is the multiplicity of Φ at P. Since there are exactly d_i points in $V_i \cap S$ we have

$$\sum_{P\in\Phi^{-1}(y)} \mu_P(\Phi) = \sum_{i=1}^{s} \mu_{P_1^i}(\Phi)\cdot d_i,$$

and finally due to Proposition 5 we get

$$\sum_{i=1}^{s} i(Z_1\cdot\ldots\cdot Z_r; V_i)\deg V_i \le \prod_{i=1}^{r} \deg F_i. \ \blacksquare$$

3. Main result. Our main result is the following estimate.

THEOREM 7. *Let $I = (f_1,\ldots,f_k)$ be an ideal generated by polynomials $f_j \in \mathbf{k}[X]$, where $\deg f_2 \ge \ldots \ge \deg f_k \ge \deg f_1$. Assume that there is a primary decomposition $I = \bigcap_{i=1}^m q_i$ without embedded components, where q_i are p_i-primary ideals. Set $r_i := \operatorname{codim} V(q_i)$, and define $d_t := \min\{\deg V(q_i) \mid \operatorname{codim} V(q_i) = t\}$ for $t \in \{r_1,\ldots,r_m\}$. Then*

$$\mu_I \le \max_{t\in\{r_1,\ldots,r_m\}} \left\{\frac{\deg f_1\cdot\ldots\cdot\deg f_t}{d_t}\right\}.$$

To prove this theorem we will proceed by reduction to the case where the intersection along components of $V(I)$ is proper. The proof will be given in the next section.

Observe that for an ideal I without embedded components we are able to find in this ideal a finite set F of polynomials such that each component of $V(I)$ can be represented as a proper intersection of some hypersurfeces given by polynomials which lie in the set F. In fact we have the following

LEMMA 8. *Let $I = (f_1,\ldots,f_k)$ be an ideal with only isolated p_i-prime components q_i, say $(f_1,\ldots,f_k) = \bigcap_{i=1}^m q_i$, and define $r_i := \operatorname{codim} V(q_i)$. Then there exists a family of polynomials*

$$\begin{cases} F_1 := f_1, \\ F_u := a_u^u f_u + \ldots + a_k^u f_k \quad \text{for } u \ge 2, \end{cases}$$

where $a_j^u \in \mathbf{k}$, such that for each $i \in \{1,\ldots,m\}$ the intersection $V(F_1)\cap\ldots\cap V(F_{r_i})$ along $V(q_i)$ is proper.

Proof. We will construct such a family inductively. Obviously for i such that $\operatorname{codim} V(q_i) = 1$ the statement is true. Assume that for all $s < l \le \max\{r_1,\ldots,r_m\}$ we have polynomials F_s such that the intersection $W_s := V(F_1)\cap\ldots\cap V(F_s)$ is proper along each component which does not lie in the set $V(I)$. Moreover, suppose that if there is $i \in \{1, \ldots, m\}$ such that $r_i = s$ then the hypersurfaces $V(F_1), \ldots, V(F_s)$ meet properly along $V(q_i)$. Consider a decomposition $W_{l-1} = W_{l-1}^1 \cup \ldots \cup W_{l-1}^{s_{l-1}}$ into irreducible components. Take points $x_{l-1}^p \in W_{l-1}^p \setminus V(I)$ (for non-empty $W_{l-1}^p \setminus V(I)$). Since not all f_j for $j = l, \ldots, k$ vanish at x_{l-1}^p, for generic a_j^l the intersection $W_{l-1} \cap V(F_l)$ along all components of codimension l is proper. Thus if there is $V(q_j)$ such that $r_j = l$ then clearly it must be contained in $W_{l-1} \cap V(F_l)$ and hence $V(F_1), \ldots, V(F_l)$ meet properly along this $V(q_j)$. Continuing this process, in a finite number of steps we obtain a family of polynomials F_u such that for each i the intersection $V(F_1) \cap \ldots \cap V(F_{r_i})$ is proper along $V(q_i)$. ■

4. Estimates. In this section we give the proof of the main result, hence we work throughout under the assumptions and notation of Theorem 7. First, consider the rings $\mathbf{k}[X]_{p_i}/I_{p_i}$ for $i \in \{1, \ldots, m\}$. Since the ideal I has only isolated components we obtain for each i an isomorphism

$$\mathbf{k}[X]_{p_i}/I_{p_i} \cong \mathbf{k}[X]_{p_i}(q_i)_{p_i} =: R_i.$$

Consider in the ring R_i an increasing family of modules

$$M_s^i(h) := ((q_i)_{p_i} : h^s) = \{g \in R_i \mid gh^s \in (q_i)_{p_i}\},$$

for some $h \in \mathbf{k}[X]$ with $h_{|V(I)} = 0$. We have

LEMMA 9. *If s is such that*

$$M_0^i(h) \subsetneq \ldots \subsetneq M_s^i(h) = M_{s+1}^i(h) = \ldots$$

then $h^s \in q_i$.

Proof. Take the smallest $n \in \mathbb{N}$ such that $h^n \in (q_i)_{p_i}$ and assume that $n > s$. Clearly $h^{n-s-1} \in M_{s+1}^i(h)$ and hence also $h^{n-s-1} \in M_s^i(h)$, but this means that $h^{n-1} \in (q_i)_{p_i}$, contrary to the minimality of n. Thus $\frac{h^s}{1} = \frac{a}{b}$ for some $a \in q_i$ and $b \notin \operatorname{rad} q_i$. This means that $bh^s \in q_i$ and since the ideal q_i is primary, $h^s \in q_i$. ■

Define

$$s_i(h) := \min\{s \mid M_s^i(h) = M_{s+1}^i(h)\}, \qquad s_i := \max\{s_i(h) \mid h_{|V(I)} = 0\}.$$

This is well defined since R_i is the Artin ring. Finally define s as the maximum of s_i for $i \in \{1, \ldots, m\}$. Then we have the following inequalities:

LEMMA 10. *$\mu_I \le s \le \operatorname{length}(R_i)$ for i such that $s_i = s$. Consequently, $\mu_I \le \max_{i \in \{1,\ldots,m\}}\{\operatorname{length}(R_i)\}$.*

Proof. Take h such that $h_{|V(I)} = 0$. Then $h = 0$ on each V_i and hence $h^s \in q_i$ for each $i \in \{1, \ldots, m\}$ by Lemma 9. This proves the first inequality. The second one is a consequence of the definition of the length. ■

Proof of Theorem 7. We choose polynomials F_u as in Lemma 8. Denote by Z_u the hypersurface $V(F_u)$. Since for each $i = 1, \ldots, m$ the intersection $Z_1 \cap \ldots \cap Z_{r_i}$ is proper along $V(q_i)$, Remark 2 yields

$$\operatorname{length}(\mathbf{k}[X]_{p_i}/(F_1, \ldots, F_{r_i})_{p_i}) = i(Z_1 \cdot \ldots \cdot Z_{r_i}; V_i).$$

Using the affine version of the Bézout Theorem we get

$$i(Z_1 \cdot \ldots \cdot Z_{r_i}; V_i) \leq \frac{\deg F_1 \cdot \ldots \cdot \deg F_{r_i}}{d_{r_i}} \leq \frac{\deg f_1 \cdot \ldots \cdot \deg f_{r_i}}{d_{r_i}}.$$

Clearly

$$\operatorname{length}(R_i) \leq \operatorname{length}(\mathbf{k}[X]_{p_i}/(F_1, \ldots, F_{r_i})_{p_i}),$$

hence finally due to Lemma 10 we obtain

$$\mu_I \leq \max_{i \in \{r_1, \ldots, r_m\}} \left\{ \frac{\deg f_1 \cdot \ldots \cdot \deg f_i}{d_i} \right\}. \ \blacksquare$$

Note that for a set-theoretic complete intersection we have at once (see [PT])

COROLLARY 11. *If an ideal $I = (f_1, \ldots, f_r)$ is a set-theoretic complete intersection, then*

$$\mu_I \leq \frac{\deg f_1 \cdot \ldots \cdot \deg f_r}{\min_{i \in \{1, \ldots, s\}} \{\deg X_i\}},$$

where $V(I) = \bigcup_{i=1}^{s} X_i$.

The next corollary is a generalization of a result from [CP].

COROLLARY 12. *Let $f = (f_1, \ldots, f_n)$ be a polynomial mapping such that $f^{-1}(0)$ is a finite, non-empty set. Set $\mu := \max\{\mu_a(f) \mid a \in f^{-1}(0)\}$, where $\mu_a(f)$ is the local multiplicity of the map f at the point a. Then for each polynomial $g \in \mathbb{C}[X]$ such that $g_{|f^{-1}(0)} = 0$, we have $g^{\mu} \in (f_1, \ldots, f_n)$.*

Finally let us state the following

CONJECTURE. *Let $I = (f_1, \ldots, f_k)$ be an ideal generated by polynomials $f_j \in \mathbf{k}[X]$, where $\deg f_2 \geq \ldots \geq \deg f_k \geq \deg f_1$. Let $\bigcap_{i=1}^{m} q_i = I$ be a primary decomposition. Set $r_i := \operatorname{codim} V(q_i)$ and define $d_t := \min\{\deg V(q_j) \mid \operatorname{codim} V(q_j) = t\}$ for $t \in \{r_1, \ldots, r_m\}$. Then*

$$\mu_I \leq \max_{t \in \{r_1, \ldots, r_m\}} \left\{ \frac{\deg f_1 \cdot \ldots \cdot \deg f_t}{d_t} \right\}.$$

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