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On extremal holomorphically contractible families

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Abstract. We prove (Theorem 1.2) that the category of generalized holomorphically contractible families (Definition 1.1) has maximal and minimal objects. Moreover, we present basic properties of these extremal families.

1. Introduction. Main results. First recall the standard definition of a holomorphically contractible family (cf. [Jar-Pfl 1993, §4.1]). A family $(d_G)_G$ of functions

$$d_G : G \times G \to \mathbb{R}_+ := [0, \infty),$$

where G runs over all domains $G \subset \mathbb{C}^n$ with arbitrary $n \in \mathbb{N}$, is said to be *holomorphically contractible* if the following two conditions are satisfied:

- for the unit disc E we have $d_E(a, z) = m_E(a, z) := \left|\frac{z-a}{1-\bar{a}z}\right|$ for $a, z \in E$,
- for any domains $G \subset \mathbb{C}^n$ and $D \subset \mathbb{C}^m$, every holomorphic mapping $F : G \to D$ is a contraction with respect to d_G and d_D, i.e.

$$d_D(F(a), F(z)) \le d_G(a, z), \qquad a, z \in G.$$

Let us recall some important holomorphically contractible families:

- *Möbius pseudodistance*:

$$\begin{aligned} c^*_G(a, z) &:= \sup\{m_E(f(a), f(z)) : f \in \mathcal{O}(G, E)\} \\ &= \sup\{|f(z)| : f \in \mathcal{O}(G, E),\ f(a) = 0\}, \end{aligned}$$

- *higher order Möbius function*:

$$m^{(k)}_G(a, z) := \sup\{|f(z)|^{1/k} : f \in \mathcal{O}(G, E),\ \mathrm{ord}_a f \ge k\}, \qquad k \in \mathbb{N},$$

where $\mathrm{ord}_a f$ denotes the order of zero of f at a,

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• *pluricomplex Green function*:

$$g_G(a,z) := \sup\{u(z) : \ u : G \to [0,1),\ \log u \in \mathcal{PSH}(G),\\ \exists_{C=C(u)>0}\,\forall_{w\in G} : u(w) \le C\|w-a\|\},$$

where $\mathcal{PSH}(G)$ denotes the family of all functions plurisubharmonic on G,

• *Lempert function*:

$$\begin{aligned}\widetilde{k}^*_G(a,z) &:= \inf\{m_E(\lambda,\mu) : \ \exists_{\varphi\in\mathcal{O}(E,G)} : \varphi(\lambda)=a,\ \varphi(\mu)=z\}\\ &= \inf\{|\mu| : \ \exists_{\varphi\in\mathcal{O}(E,G)} : \varphi(0)=a,\ \varphi(\mu)=z\}.\end{aligned}$$

It is well known that

$$c^*_G = m^{(1)}_G \le m^{(k)}_G \le g_G \le \widetilde{k}^*_G,$$

and for any holomorphically contractible family $(d_G)_G$ we have

$$(*) \qquad c^*_G \le d_G \le \widetilde{k}^*_G,$$

i.e. the Möbius family is minimal and the Lempert family is maximal.

The Green function g_G may be generalized as follows. Let $\boldsymbol{p} : G \to \mathbb{R}_+$ be an arbitrary function. Define

$$g_G(\boldsymbol{p},z) := \sup\{u(z) : \ u : G \to [0,1),\ \log u \in \mathcal{PSH}(G),\\ \forall_{a\in G}\,\exists_{C=C(u,a)>0}\,\forall_{w\in G} : u(w) \le C\|w-a\|^{\boldsymbol{p}(a)}\}, \quad z\in G \ (^1);$$

obviously the above growth condition is trivially satisfied at all points $a\in G$ such that $\boldsymbol{p}(a)=0$. We have $g_G(\boldsymbol{0},\cdot)\equiv 1$. The function $g_G(\boldsymbol{p},\cdot)$ is called the *generalized pluricomplex Green function with poles* (*weights*) $\boldsymbol{p}$. Observe that if the set

$$|\boldsymbol{p}| := \{a\in G : \boldsymbol{p}(a) > 0\}$$

is not pluripolar, then $g_G(\boldsymbol{p},\cdot)\equiv 0$.

In the case where $|\boldsymbol{p}|$ is finite, the function $g_G(\boldsymbol{p},\cdot)$ was introduced by P. Lelong in [Lel 1989].

For $\boldsymbol{p} = \chi_A$ = the characteristic function of a set $A\subset G$, we put $g_G(A,\cdot) := g_G(\chi_A,\cdot)$. Obviously, $g_G(\{a\},\cdot) = g_G(a,\cdot)$ for $a\in G$.

The generalized Green function was recently studied by many authors, e.g. [Car-Wie 2003], [Com 2000], [Edi 2002], [Edi-Zwo 1998], [Lár-Sig 1998].

Using the same idea, one can generalize the Möbius function. For

$$\boldsymbol{p} : G \to \mathbb{Z}_+ := \{0,1,2,\dots\}$$

we put

$$m_G(\boldsymbol{p},z) := \sup\{|f(z)| : f\in\mathcal{O}(G,E),\ \mathrm{ord}_a f \ge \boldsymbol{p}(a),\ a\in G\}, \quad z\in G.$$

The function $m_G(\boldsymbol{p},\cdot)$ is called the *generalized Möbius function with weights* $\boldsymbol{p}$. Clearly $m_G(\boldsymbol{0},\cdot)\equiv 1$. Observe that if the set $|\boldsymbol{p}|$ is not thin,

(^{1}) Here $0^0 := 1$.

then $m_G(\boldsymbol{p},\cdot)\equiv 0$. Similarly to the case of the generalized Green function we put $m_G(A,\cdot):=m_G(\chi_A,\cdot)$, $A\subset G$. We get $m_G(\{a\},\cdot)=c^*_G(a,\cdot)$, $a\in G$. Moreover, if $|\boldsymbol{p}|=\{a\}$ and $\boldsymbol{p}(a)=k$, then $m_G(\boldsymbol{p},\cdot)=[m^{(k)}_G(a,z)]^k$.

It is clear that $m_G(\boldsymbol{p},\cdot)\le g_G(\boldsymbol{p},\cdot)$ (for any $\boldsymbol{p}:G\to\mathbb{Z}_+$). Some other properties of $g_G(\boldsymbol{p},\cdot)$ and $m_G(\boldsymbol{p},\cdot)$ will be presented in §2.

Definition 1.1. A family $\underline{d}=(d_G)_G$ of functions

$$d_G:\mathbb{R}^G_+\times G\to\mathbb{R}_+$$

is said to be *a generalized holomorphically contractible family* if the following three axioms are satisfied:

(E) $\prod_{a\in E}[m_E(a,z)]^{\boldsymbol{p}(a)}\le d_E(\boldsymbol{p},z)\le\inf_{a\in E}[m_E(a,z)]^{\boldsymbol{p}(a)}$ for every $(\boldsymbol{p},z)\in\mathbb{R}^E_+\times E$ (2),

(H) for any $F\in\mathcal{O}(G,D)$ and $\boldsymbol{q}:D\to\mathbb{R}_+$ we have

$$d_D(\boldsymbol{q},F(z))\le d_G(\boldsymbol{q}\circ F,z)\quad\text{ for every } z\in G,$$

(M) for any $\boldsymbol{p},\boldsymbol{q}:G\to\mathbb{R}_+$, if $\boldsymbol{p}\le\boldsymbol{q}$, then $d_G(\boldsymbol{q},\cdot)\le d_G(\boldsymbol{p},\cdot)$.

If in the above definition one considers only integer-valued weights (as in the case of the generalized Möbius function), then we get the definition of a *generalized holomorphically contractible family with integer-valued weights*.

Put $d_G(A,\cdot):=d_G(\chi_A,\cdot)$, $A\subset G$, $d_G(a,\cdot):=d_G(\{a\},\cdot)$, $a\in G$.

One can prove that the generalized Green and Möbius functions satisfy all the above axioms (cf. §2).

The main result of the paper is the following theorem.

Theorem 1.2. *In the category of generalized holomorphically contractible families there exists a minimal and a maximal object. They are given by the following formulae*:

$$\begin{aligned}
d^{\min}_G(\boldsymbol{p},z)&:=\sup\Big\{\prod_{\mu\in f(G)}[m_E(\mu,f(z))]^{\sup\boldsymbol{p}(f^{-1}(\mu))}:f\in\mathcal{O}(G,E)\Big\}\\
&=\sup\Big\{\prod_{\mu\in f(G)}|\mu|^{\sup\boldsymbol{p}(f^{-1}(\mu))}:f\in\mathcal{O}(G,E),\ f(z)=0\Big\},\\
d^{\max}_G(\boldsymbol{p},z)&:=\inf\{[\widetilde{k}^*_G(a,z)]^{\boldsymbol{p}(a)}:a\in G\}\\
&=\inf\{|\mu|^{\boldsymbol{p}(\varphi(\mu))}:\varphi\in\mathcal{O}(E,G),\ \varphi(0)=z,\ \mu\in E\}.
\end{aligned}$$

Observe that if $|\boldsymbol{p}|=\{a\}$ and $\boldsymbol{p}(a)=k$, then $d^{\min}_G(\boldsymbol{p},\cdot)=[c^*_G(a,\cdot)]^k$ and $d^{\max}_G(\boldsymbol{p},\cdot)=[\widetilde{k}^*_G(a,\cdot)]^k$. Moreover, for $A\subset G$ we get

(2) We put $\prod_{a\in A}h(a):=\inf\{\prod_{a\in B}h(a):B\subset A,\ \#B<\infty\}$ for $h:A\to[0,1]$.

$$d_G^{\min}(A,z)=\sup\Big\{\prod_{\mu\in f(A)} m_E(\mu,f(z)) : f\in\mathcal{O}(G,E)\Big\}$$
$$=\sup\Big\{\prod_{\mu\in f(A)}|\mu| : f\in\mathcal{O}(G,E),\ f(z)=0\Big\}\ (^3),$$
$$d_G^{\max}(A,z)=\inf\{\widetilde{k}_G^*(a,z): a\in A\}.$$

The function $d_G^{\min}$ (resp. $d_G^{\max}$) may be considered as a generalization of the Möbius function c_G^* (resp. Lempert function $\widetilde{k}_G^*$). The proof of Theorem 1.2 will be given in §3. Some properties of $d_G^{\min}$ and $d_G^{\max}$ will be presented in §4.

2. Basic properties of g_G and m_G. Directly from the definitions we conclude that the systems $(g_G)_G$ and $(m_G)_G$ satisfy (H) and (M) and the following conditions (to simplify formulations we will write d_G if a given property holds simultaneously for m_G and g_G):

PROPERTY 2.1. *We have*

$$d_G(\boldsymbol{p},\cdot)d_G(\boldsymbol{q},\cdot)\le d_G(\boldsymbol{p}+\boldsymbol{q},\cdot)\le\min\{d_G(\boldsymbol{p},\cdot),d_G(\boldsymbol{q},\cdot)\}.$$

In particular, $g_G(\boldsymbol{p},\cdot)\le\inf_{a\in G}[g_G(a,\cdot)]^{\boldsymbol{p}(a)}\le d_G^{\max}(\boldsymbol{p},\cdot)$.

PROPERTY 2.2. *If the set* $|\boldsymbol{p}|$ *is finite*, *then*

$$\prod_{a\in G}[d_G(a,\cdot)]^{\boldsymbol{p}(a)}\le d_G(\boldsymbol{p},\cdot).$$

PROPERTY 2.3. *We have*

$$g_G(\boldsymbol{p},z)=\sup\{u(z):\ u:G\to[0,1),\ \log u\in\mathcal{PSH}(G),\ u(\cdot)\le\inf_{a\in G}[g_G(a,\cdot)]^{\boldsymbol{p}(a)}\},\qquad z\in G.$$

PROPERTY 2.4. $m_G(\boldsymbol{p},\cdot)\in\mathcal{C}(G)$.

Proof. The family $\{f\in\mathcal{O}(G,E):\operatorname{ord}_a f\ge\boldsymbol{p}(a),\ a\in G\}$ is equicontinuous. ∎

PROPERTY 2.5. *If* $\boldsymbol{p}\not\equiv 0$, *then for any* $z_0\in G$ *there exists an extremal function for* $m_G(\boldsymbol{p},z_0)$, *i.e. a function* $f_{z_0}\in\mathcal{O}(G,E)$ *with* $\operatorname{ord}_a f_{z_0}\ge\boldsymbol{p}(a)$, $a\in G$, *and* $m_G(\boldsymbol{p},z_0)=|f_{z_0}(z_0)|$.

PROPERTY 2.6. $\log d_G(\boldsymbol{p},\cdot)\in\mathcal{PSH}(G)$.

Proof. Argue as in the one-pole case (cf. [Jar-Pfl 1993, §§2.5, 4.2]). ∎

PROPERTY 2.7. *If* $G_k\nearrow G$ *and* $\boldsymbol{p}_k\nearrow\boldsymbol{p}$, *then* $d_{G_k}(\boldsymbol{p}_k,z)\searrow d_G(\boldsymbol{p},z)$ *for* $z\in G$.

(3) In fact, $d_G^{\min}(A,\cdot)=m_G(A,\cdot)$ (cf. Corollary 3.1(c)).

Proof. It is clear that the sequence is monotone and the limit function u satisfies $u \geq d_G(\boldsymbol{p}, \cdot)$.

In the case of the generalized Green function, using 2.6, we have $u \in \mathcal{PSH}(G)$. By 2.3 it remains to observe that $u(z) \leq \inf_{a \in G}[g_G(a, z)]^{\boldsymbol{p}(a)}$ for $z \in G$ (because $g_{G_k}(a, z) \searrow g_G(a, z)$ for every $(a, z) \in G \times G$).

The case of the generalized Möbius function is simpler and it follows from 2.5 and a Montel argument. ∎

Property 2.8. $g_G(\boldsymbol{p}, \cdot) = \inf\{g_G(\boldsymbol{q}, \cdot) : \boldsymbol{q} \leq \boldsymbol{p},\ \#|\boldsymbol{q}| < \infty\}$.

Proof. Let $u := \inf\{g_G(\boldsymbol{q}, \cdot) : \boldsymbol{q} \leq \boldsymbol{p},\ \#|\boldsymbol{q}| < \infty\}$. Obviously $g_G(\boldsymbol{p}, \cdot) \leq u$. By 2.3, to prove the opposite inequality we only need to show that $\log u$ is plurisubharmonic. Observe that

$$g_G(\max\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N\}, \cdot) \leq \min\{g_G(\boldsymbol{q}_1, \cdot), \ldots, g_G(\boldsymbol{q}_N, \cdot)\}.$$

We finish the proof by applying the following general result.

Lemma 2.9. *Let $(v_i)_{i \in A} \subset \mathcal{PSH}(\Omega)$ ($\Omega \subset \mathbb{C}^n$) be such that for any $i_1, \ldots, i_N \in A$ there exists an $i_0 \in A$ such that $v_{i_0} \leq \min\{v_{i_1}, \ldots, v_{i_N}\}$. Then $v := \inf_{i \in A} v_i \in \mathcal{PSH}(\Omega)$.*

Proof. It suffices to consider the case $n = 1$. Take a disc $\Delta_a(r) \Subset \Omega$, $\varepsilon > 0$, and a continuous function $w \in \mathcal{C}(\partial \Delta_a(r))$ such that $w \geq v$ on $\partial \Delta_a(r)$. We want to show that $v(a) \leq (2\pi)^{-1} \int_0^{2\pi} w(a + re^{i\theta})\, d\theta + \varepsilon$. For any point $b \in \partial \Delta_a(r)$ there exists an $i = i(b) \in A$ such that $v_i(b) < w(b) + \varepsilon$. Hence there exists an open arc $I = I(b) \subset \partial \Delta_a(r)$ with $b \in I$ such that $v_i(\lambda) < w(\lambda) + \varepsilon$ for $\lambda \in I$. By a compactness argument, we find $b_1, \ldots, b_N \in \partial \Delta_a(r)$ such that $\partial \Delta_a(r) = \bigcup_{j=1}^N I(b_j)$. By assumption, there exists an $i_0 \in A$ such that $v_{i_0} \leq \min\{v_{i(b_1)}, \ldots, v_{i(b_N)}\}$. Then

$$\begin{aligned} v(a) \leq v_{i_0}(a) &\leq \frac{1}{2\pi} \int_0^{2\pi} v_{i_0}(a + re^{i\theta})\, d\theta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} w(a + re^{i\theta})\, d\theta + \varepsilon. \ \blacksquare\ \blacksquare \end{aligned}$$

Property 2.10. *We have*

$$\prod_{a \in G} [g_G(a, \cdot)]^{\boldsymbol{p}(a)} \leq g_G(\boldsymbol{p}, \cdot).$$

Proof. Use 2.2 and 2.8. ∎

Property 2.11. *If $G \subset \mathbb{C}$, then*

$$g_G(\boldsymbol{p}, z) = \prod_{a \in G} [g_G(a, z)]^{\boldsymbol{p}(a)}, \qquad z \in G.$$

In particular, $g_E(\boldsymbol{p}, z) = \prod_{a \in E} [m_E(a, z)]^{\boldsymbol{p}(a)}$ for $z \in E$.

Proof. By 2.8 we may assume that the set $|\boldsymbol{p}|$ is finite, and by 2.7, that $G \Subset \mathbb{C}$ is regular with respect to the Dirichlet problem. Let $u := \prod_{a\in|\boldsymbol{p}|}[g_G(a,\cdot)]^{\boldsymbol{p}(a)}$. Then $\log u$ is subharmonic on G and harmonic on $G\setminus|\boldsymbol{p}|$. The function $v := \log g_G(\boldsymbol{p},\cdot) - \log u$ is locally bounded from above in G and $\limsup_{z\to\zeta} v(z) \le 0$ for $\zeta \in \partial G$. Consequently, v extends to a subharmonic function on G, and by the maximum principle, $v \le 0$ on G, i.e. $g_G(\boldsymbol{p},\cdot) \le u$ on G. The opposite inequality follows from 2.10. ∎

PROPERTY 2.12. *For any* $\boldsymbol{p} : G \to \mathbb{Z}_+$,

$$m_G(\boldsymbol{p},\cdot) = \inf\{m_G(\boldsymbol{q},\cdot) : \ \ \boldsymbol{q} : G \to \mathbb{Z}_+,\ \boldsymbol{q} \le \boldsymbol{p},\ \#|\boldsymbol{q}| < \infty\}.$$

In particular, *for any* $\boldsymbol{p} : E \to \mathbb{Z}_+$,

$$m_E(\boldsymbol{p}, z) = g_E(\boldsymbol{p}, z) = \prod_{a\in E}[m_E(a, z)]^{\boldsymbol{p}(a)}, \qquad z \in E.$$

Proof. The case where $|\boldsymbol{p}|$ is finite is trivial; the case where it is countable follows from 2.7. In the general case let $A_k := \{a \in G : \boldsymbol{p}(a) = k\}$ and let B_k be a countable (or finite) dense subset of A_k for $k \in \mathbb{Z}_+$. Put $B := \bigcup_{k=0}^{\infty} B_k$ and $\boldsymbol{p}' := \boldsymbol{p}\cdot\chi_B$. Then $\boldsymbol{p}' \le \boldsymbol{p}$, the set $|\boldsymbol{p}'|$ is countable, and $m_G(\boldsymbol{p},\cdot) \equiv m_G(\boldsymbol{p}',\cdot)$. Consequently, the result reduces to the countable case. ∎

PROPOSITION 2.13 ([Edi-Zwo 1998], [Lár-Sig 1998]). *Let* $G, D \subset \mathbb{C}^n$ *be domains and let* $F : G \to D$ *be a proper holomorphic mapping. Let* $\boldsymbol{q} : D \to \mathbb{R}_+$. *Assume that* $\det F'(a) \ne 0$ *for any* $a \in G$ *such that* $\boldsymbol{q}(F(a)) > 0$. *Then*

$$g_D(\boldsymbol{q}, F(z)) = g_G(\boldsymbol{q}\circ F, z), \qquad z \in G.$$

In particular, *if* $B \subset D$ *is such that* $\det F'(a) \ne 0$ *for any* $a \in F^{-1}(B)$, *then*

$$g_D(B, F(z)) = g_G(F^{-1}(B), z), \qquad z \in G.$$

COROLLARY 2.14. *Let* $A_1,\dots,A_n \subset E$ *be finite sets. Put*

$$F_j(\lambda) := \prod_{a\in A_j}\frac{\lambda - a}{1-\overline{a}\lambda}, \qquad \lambda \in E,\ j = 1,\dots,n,$$

$$F(z) := (F_1(z_1),\dots,F_n(z_n)), \qquad z = (z_1,\dots,z_n) \in E^n.$$

Then

$$\begin{aligned} m_{E^n}(A_1\times\ldots\times A_n, z) &\le g_{E^n}(A_1\times\ldots\times A_n, z)\\ &= g_{E^n}(0, F(z)) = \max\{|F_j(z_j)| : j = 1,\dots,n\}\\ &= \max\{m_E(A_1, z_1),\dots,m_E(A_n, z_n)\}\\ &\le m_{E^n}(A_1\times\ldots\times A_n, z), \qquad z = (z_1,\dots,z_n) \in E^n. \end{aligned}$$

Proposition 2.15 ([Car-Wie 2003]). *Let* $\boldsymbol{p} : E^n \to \mathbb{R}_+$ *be such that* $|\boldsymbol{p}| = \{a_1, \dots, a_N\} \subset E \times \{0\}^{n-1}$. *Put* $k_j := \boldsymbol{p}(a_j)$, $j = 1, \dots, N$, *and assume that* $k_1 \geq \dots \geq k_N$. *Then*

$$g_{E^n}(\boldsymbol{p}, z) = \prod_{j=1}^{N} u_j^{k_j - k_{j+1}}(z), \qquad z \in E^n,$$

where $k_{N+1} := 0$ *and*

$$\begin{aligned} u_j(z) &:= \max\{m_E(a_{1,1}, z_1) \dots m_E(a_{j,1}, z_1), |z_2|, \dots, |z_n|\} \\ &= \max\{m_E(\{a_{1,1}, \dots, a_{j,1}\}, z_1), |z_2|, \dots, |z_n|\} \\ &= g_{E^n}(\{a_1, \dots, a_j\}, z), \qquad j = 1, \dots, N. \end{aligned}$$

If $k_1, \dots, k_N \in \mathbb{N}$, *then* $m_{E^n}(\boldsymbol{p}, \cdot) = g_{E^n}(\boldsymbol{p}, \cdot)$.

Observe that if $k_1 = \dots = k_N = 1$, then the above formula coincides with that from Corollary 2.14.

Notice that even for the simplest case not covered by Proposition 2.15: $n = N = 2$, $a_1 = (0,0)$, $a_2 \in (E_*)^2$, $k_1 = k_2 = 1$, an effective formula for $g_{E^n}(\boldsymbol{p}, \cdot)$ is not known.

Recall that by the Lempert theorem (cf. [Jar-Pfl 1993, Ch. 8]), if $G \subset \mathbb{C}^n$ is convex, then $c_G^* = \widetilde{k}_G^*$, and consequently, by $(*)$, all holomorphically contractible families coincide on G. The following example shows that this is not true in the category of generalized holomorphically contractible families.

Example 2.16 (due to W. Zwonek). Let $D := \{(z,w) \in \mathbb{C}^2 : |z| + |w| < 1\}$, $A_t := \{(t, \sqrt{t}), (t, -\sqrt{t})\}$, $0 < t \ll 1$. Then

$$m_D(A_t, (0,0)) < g_D(A_t, (0,0)) < d_D^{\max}(A_t, (0,0))$$

for small t.

Indeed, let $G := \{(z,w) \in \mathbb{C}^2 : |z| + \sqrt{|w|} < 1\}$ and let $F : D \to G$, $F(z,w) := (z, w^2)$. Note that F is proper and locally biholomorphic in a neighborhood of A_t. Moreover, $A_t = F^{-1}(t,t)$.

Using Proposition 2.13, we conclude that $g_D(A_t, (0,0)) = g_G((t,t), (0,0))$.

Observe that $m_D(A_t, (0,0)) = m_G((t,t), (0,0))$. In fact, the inequality "$\geq$" follows from (H) (applied to F). The opposite inequality may be proved as follows. Let $f \in \mathcal{O}(D, E)$ be such that $f|_{A_t} = 0$. Define

$$\widetilde{f}(z,w) := \tfrac{1}{2}(f(z, \sqrt{w}) + f(z, -\sqrt{w})), \qquad (z,w) \in G.$$

Note that $\widetilde{f}$ is well defined, $|\widetilde{f}| < 1$, $\widetilde{f}(t,t) = 0$, $\widetilde{f}$ is continuous, and $\widetilde{f}$

is holomorphic on $D \cap \{w \neq 0\}$. In particular, $\widetilde{f}$ is holomorphic on D. Consequently, $|f(0,0)| = |\widetilde{f}(0,0)| \leq m_G((t,t),(0,0))$.

Suppose that $m_D(A_{t_k},(0,0)) = g_D(A_{t_k},(0,0))$ for a sequence $t_k \searrow 0$. Then

$$\begin{aligned} g_G((t_k,t_k),(0,0)) &= g_D(A_{t_k},(0,0)) = m_D(A_{t_k},(0,0)) \\ &= m_G((t_k,t_k),(0,0)) \leq g_G((t_k,t_k),(0,0)), \quad k=1,2,\ldots \end{aligned}$$

Thus $m_G((t_k,t_k),(0,0)) = g_G((t_k,t_k),(0,0))$, $k=1,2,\ldots$

Consequently, using [Jar-Pfl 1993, §2.5], and [Zwo 2000a, Corollary 4.4] (or [Zwo 2000b, Corollary 4.2.3]), we conclude that

$$\gamma_G((0,0);(1,1)) = A_G((0,0);(1,1)),$$

where γ_G (resp. A_G) denotes the Carathéodory–Reiffen (resp. Azukawa) metric of G (cf. [Jar-Pfl 1993, §§2.1, 4.2]). Hence, by Propositions 4.2.7 and 2.2.1(d) from [Jar-Pfl 1993], using the fact that D is the convex envelope of G, we get

$$2 = h_D(1,1) = \gamma_G((0,0);(1,1)) = A_G((0,0);(1,1)) = h_G(1,1) = \frac{2}{3-\sqrt{5}},$$

where h_D (resp. h_G) denotes the Minkowski function for D (resp. G); contradiction.

To prove the inequality $g_D(A_t,(0,0)) < d_D^{\max}(A_t,(0,0))$, we may argue as follows. We already know that

$$\begin{aligned} g_D(A_t,(0,0)) &= g_G((t,t),(0,0)) \\ &\approx g_G((0,0),(t,t)) = h_G(t,t) = \frac{2t}{3-\sqrt{5}}, \quad t \approx 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} d_D^{\max}(A_t,(0,0)) &= \min\{\widetilde{k}_D^*((t,-\sqrt{t}),(0,0)), \widetilde{k}_D^*((t,\sqrt{t}),(0,0))\} \\ &= \min\{h_D(t,-\sqrt{t}), h_D(t,\sqrt{t})\} = t+\sqrt{t}. \end{aligned}$$

It remains to observe that $2t/(3-\sqrt{5}) < t+\sqrt{t}$ for small $t>0$.

Let $\delta_D(A_t,\cdot)$ denote the *Coman function for D with poles at A_t*, i.e.

$$\begin{aligned} \delta_D(A_t,(z,w)) = \inf\{|\mu_1\mu_2| :\ & \exists_{\varphi\in\mathcal{O}(E,D)} : \\ & \varphi(0)=(z,w),\ \varphi(\mu_1)=(t,\sqrt{t}),\ \varphi(\mu_2)=(t,-\sqrt{t})\}, \quad (z,w)\in D \end{aligned}$$

(cf. [Com 2000]). It is known that $g_D(A_t,\cdot) \leq \delta_D(A_t,\cdot)$. Taking $\varphi(\lambda) := (\lambda^2/4, \lambda/2)$, we easily see that $\delta_D(A_t,(0,0)) \leq 4t < t+\sqrt{t} = d_D^{\max}(A_t,(0,0))$, $0<t\ll 1$. *We do not know whether $g_D(A_t,(0,0)) < \delta_D(A_t,(0,0))$ for small $t>0$.*

3. Proof of Theorem 1.2

STEP 1. *If* $(d_G)_G$ *satisfies* (H) *and*

(E$^+$) $d_E(\boldsymbol{p},\lambda) \le d_E^{\max}(\boldsymbol{p},\lambda) = \inf\{[m_E(\mu,\lambda)]^{\boldsymbol{p}(\mu)} : \mu \in E\}$, $(\boldsymbol{p},\lambda) \in \mathbb{R}_+^E \times E$, *then* $d_G \le d_G^{\max}$ *for any* G. *The same remains true in the category of contractible families with integer-valued weights.*

Proof. We have

$$\begin{aligned} d_G(\boldsymbol{p},z) &\overset{\text{(H)}}{\le} \inf\{d_E(\boldsymbol{p}\circ\varphi,0) : \varphi \in \mathcal{O}(E,G),\, \varphi(0)=z\} \\ &\overset{(\mathrm{E}^+)}{\le} \inf\{|\mu|^{\boldsymbol{p}(\varphi(\mu))} : \varphi \in \mathcal{O}(E,G),\, \varphi(0)=z,\, \mu\in E\} \\ &= d_G^{\max}(\boldsymbol{p},z), \qquad (\boldsymbol{p},z)\in\mathbb{R}_+^G\times G. \ \blacksquare \end{aligned}$$

STEP 2. *The system* $(d_G^{\max})_G$ *satisfies* (E), (H), *and* (M).

Proof. (E) and (M) are obvious. To prove (H) let $F : G \to D$ be holomorphic and let $\boldsymbol{q} : D \to \mathbb{R}_+$. Then

$$\begin{aligned} d_D^{\max}(\boldsymbol{q},F(z)) &= \inf\{[\widetilde{k}_D^*(b,F(z))]^{\boldsymbol{q}(b)} : b\in D\} \\ &\le \inf\{[\widetilde{k}_D^*(F(a),F(z))]^{\boldsymbol{q}(F(a))} : a\in G\} \\ &\le \inf\{[\widetilde{k}_G^*(a,z)]^{\boldsymbol{q}(F(a))} : a\in G\} = d_G^{\max}(\boldsymbol{q}\circ F,z), \qquad z\in G. \ \blacksquare \end{aligned}$$

STEP 3. *If* $(d_G)_G$ *satisfies* (H), (M), *and*

(E$^-$) $$\prod_{\mu\in E}[m_E(\mu,\lambda)]^{\boldsymbol{p}(\mu)} \le d_E(\boldsymbol{p},\lambda), \qquad (\boldsymbol{p},\lambda)\in\mathbb{R}_+^E\times E,$$

then $d_G^{\min} \le d_G$ *for any* G. *The same remains true in the category of contractible families with integer-valued weights.*

Proof. Indeed,

$$\begin{aligned} &d_G(\boldsymbol{p},z) \\ &\overset{\text{(M)}}{\ge} \sup\{d_G(\boldsymbol{q}\circ f,z) : f\in\mathcal{O}(G,E),\, \boldsymbol{q}:E\to\mathbb{R}_+,\, f(z)=0,\, \boldsymbol{p}\le\boldsymbol{q}\circ f\} \\ &\overset{\text{(H)}}{\ge} \sup\{d_E(\boldsymbol{q},0) : f\in\mathcal{O}(G,E),\, \boldsymbol{q}:E\to\mathbb{R}_+,\, f(z)=0,\, \boldsymbol{p}\le\boldsymbol{q}\circ f\} \\ &\overset{(\mathrm{E}^-)}{\ge} \sup\Big\{\prod_{\mu\in E}|\mu|^{\boldsymbol{q}(\mu)} : f\in\mathcal{O}(G,E),\, \boldsymbol{q}:E\to\mathbb{R}_+,\, f(z)=0,\, \boldsymbol{p}\le\boldsymbol{q}\circ f\Big\} \\ &\ge \sup\Big\{\prod_{\mu\in f(G)}|\mu|^{\sup \boldsymbol{p}(f^{-1}(\mu))} : f\in\mathcal{O}(G,E),\, f(z)=0\Big\} \\ &= d_G^{\min}(\boldsymbol{p},z), \qquad (\boldsymbol{p},z)\in\mathbb{R}_+^G\times G. \ \blacksquare \end{aligned}$$

STEP 4. *The system* $(d_G^{\min})_G$ *satisfies* (E), (H), *and* (M).

Proof. (E) and (M) are elementary. To prove (H) let $F : G \to D$ be holomorphic and let $\boldsymbol{q} : D \to \mathbb{R}_+$. Then

$$\begin{aligned} d_D^{\min}(\boldsymbol{q}, F(z)) &= \sup\Big\{ \prod_{\mu\in g(D)} [m_E(\mu, g(F(z)))]^{\sup \boldsymbol{q}(g^{-1}(\mu))} : g \in \mathcal{O}(D,E)\Big\} \\ &\overset{f=g\circ F}{\leq} \sup\Big\{ \prod_{\mu\in f(G)} [m_E(\mu, f(z))]^{\sup(\boldsymbol{q}\circ F)(f^{-1}(\mu))} : f \in \mathcal{O}(G,E)\Big\} \\ &= d_G^{\min}(\boldsymbol{q}\circ F, z), \quad z \in G. \ \blacksquare \end{aligned}$$

COROLLARY 3.1. (a) $d_G^{\min} \leq g_G \leq d_G^{\max}$ *and* $d_G^{\min} \leq m_G \leq g_G \leq d_G^{\max}$ (*for integer-valued weights*).

(b) $d_E^{\min}(\boldsymbol{p}, \lambda) = g_E(\boldsymbol{p}, \lambda) = \prod_{\mu\in E}[m_E(\mu,\lambda)]^{\boldsymbol{p}(\mu)}$ *for* $(\boldsymbol{p},\lambda) \in \mathbb{R}_+^E \times E$.

(c) $d_G^{\min}(A, \cdot) = m_G(A, \cdot)$ *for any* $A \subset G$.

Proof. (a) follows from Theorem 1.2.

(b) Using (a) and 2.11 we get

$$\prod_{\mu\in E}[m_E(\mu,\lambda)]^{\boldsymbol{p}(\mu)} \leq d_E^{\min}(\boldsymbol{p},\lambda) \leq g_G(\boldsymbol{p},\lambda) = \prod_{\mu\in E}[m_E(\mu,\lambda)]^{\boldsymbol{p}(\mu)}.$$

(c) Let $A \subset G$. Then

$$\begin{aligned} m_G(A, z) &\geq d_G^{\min}(A, z) \\ &\geq \sup\Big\{ \prod_{\mu\in f(A)} m_E(\mu, f(z)) : f \in \mathcal{O}(G,E),\ f|_A = 0\Big\} \\ &= m_G(A, z), \quad z \in G. \ \blacksquare \end{aligned}$$

EXAMPLE 3.2. Let $G := E^2$, $a_- := \big(-\frac{1}{2}, 0\big)$, $a_+ := \big(\frac{1}{2}, 0\big)$, $b := \big(0, \frac{1}{3}\big)$, $|\boldsymbol{p}| = \{a_-, a_+\}$, $\boldsymbol{p}(a_-) = 2$, $\boldsymbol{p}(a_+) = 1$. Then $d_{E^2}^{\min}(\boldsymbol{p}, b) < m_{E^2}(\boldsymbol{p}, b)$ (cf. Corollary 3.1(c)).

Indeed, by Proposition 2.15,

$$m_{E^2}(\boldsymbol{p}, b) = u_1(b)u_2(b) = \max\big\{\tfrac{1}{2}, \tfrac{1}{3}\big\} \max\big\{\tfrac{1}{2}\cdot\tfrac{1}{2}, \tfrac{1}{3}\big\} = \tfrac{1}{2}\cdot\tfrac{1}{3} = \tfrac{1}{6}.$$

On the other hand,

$$\begin{aligned} &d_{E^2}^{\min}(\boldsymbol{p}, b) \\ &\quad = \max\{\sup\{|f(a_-)|^2|f(a_+)| : f \in \mathcal{O}(E^2, E),\ f(b) = 0,\ f(a_-) \neq f(a_+)\}, \\ &\qquad\qquad \sup\{|f(b)|^2 : f \in \mathcal{O}(E^2, E),\ f(a_-) = f(a_+) = 0\}\} \\ &\quad \leq \max\{[m_{E^2}(a_-, b)]^2 m_{E^2}(a_+, b), [m_{E^2}(\{a_-, a_+\}, b)]^2\} \\ &\quad = \max\Big\{\big[\max\big\{\tfrac{1}{2}, \tfrac{1}{3}\big\}\big]^2 \max\big\{\tfrac{1}{2}, \tfrac{1}{3}\big\}, \big[m_{E^2}\big(\big\{-\tfrac{1}{2}, \tfrac{1}{2}\big\} \times \{0\}, b\big)\big]^2\Big\} \\ &\quad = \max\big\{\tfrac{1}{8}, \big[\max\big\{\tfrac{1}{2}\cdot\tfrac{1}{2}, \tfrac{1}{3}\big\}\big]^2\big\} = \tfrac{1}{8}. \end{aligned}$$

4. Basic properties of $d_G^{\min}$ and $d_G^{\max}$

PROPERTY 4.1. *If $D \subset \mathbb{C}^m$ is a Liouville domain, then*

$$d_{G\times D}^{\min}(\boldsymbol{p},(z,w)) = d_G^{\min}(\boldsymbol{p}',z), \qquad (z,w) \in G \times D,$$

where $\boldsymbol{p}'(z) := \sup\{\boldsymbol{p}(z,w) : w \in D\}$, $z \in G$, and $d_G^{\min}(\boldsymbol{p}',\cdot) := 0$ if there exists a $z_0 \in G$ with $\boldsymbol{p}'(z_0) = \infty$.

PROPERTY 4.2. (a) *The functions $d_G^{\min}(\boldsymbol{p},\cdot)$ and $d_G^{\max}(\boldsymbol{p},\cdot)$ are upper semicontinuous.*

(b) *If $\boldsymbol{p} : G \to \mathbb{Z}_+$, then $d_G^{\min}(\boldsymbol{p},\cdot) \in \mathcal{C}(G)$* (cf. 2.4).

Proof. (a) The case of $d_G^{\max}(\boldsymbol{p},\cdot)$ is obvious. To prove the upper semicontinuity of $d_G^{\min}(\boldsymbol{p},\cdot)$, fix a $z_0 \in G$ and suppose that

$$d_G^{\min}(\boldsymbol{p},z_k) \to \alpha > \beta > d_G^{\min}(\boldsymbol{p},z_0)$$

for some sequence $z_k \to z_0$. Let $f_k \in \mathcal{O}(G,E)$ be such that $f_k(z_k) = 0$ and $\prod_{\mu\in f_k(G)} |\mu|^{\sup \boldsymbol{p}(f_k^{-1}(\mu))} \to \alpha$. By a Montel argument we may assume that $f_k \to f_0$ locally uniformly in G with $f_0 \in \mathcal{O}(G,E)$, $f_0(z_0) = 0$. Since $\prod_{\mu\in f_0(G)} |\mu|^{\sup \boldsymbol{p}(f_0^{-1}(\mu))} < \beta$, we can find a finite set $A \subset G$ such that $f_0|_A$ is injective and $\prod_{a\in A} |f_0(a)|^{\boldsymbol{p}(a)} < \beta$. Consequently, $\prod_{a\in A} |f_k(a)|^{\boldsymbol{p}(a)} < \beta$ and $f_k|_A$ is injective for $k \gg 1$. Finally, $\prod_{\mu\in f_k(G)} |\mu|^{\sup \boldsymbol{p}(f_k^{-1}(\mu))} < \beta$ for $k \gg 1$; contradiction.

(b) In view of (a), it suffices to prove that for every $f \in \mathcal{O}(G,E)$ the function $u_f(z) := \prod_{\mu\in f(G)} [m_E(\mu,f(z))]^{\sup \boldsymbol{p}(f^{-1}(\mu))}$, $z \in G$, is continuous on G. Observe that $u_f(z) = \inf_M \prod_{\mu\in M} [m_E(\mu,f(z))]^{k_f(\mu)}$, where M runs over all finite sets $M \subset f(|\boldsymbol{p}|)$ such that $k_f(\mu) := \sup \boldsymbol{p}(f^{-1}(\mu)) < \infty$, $\mu \in M$. Thus $u_f = \inf_M |h_M|$, where $h_M \in \mathcal{O}(G,E)$. Consequently, since the family $(h_M)_M$ is equicontinuous, the function u_f is continuous on G. ∎

EXAMPLE 4.3. Let $\boldsymbol{p} : E \times \mathbb{C} \to \mathbb{R}_+$ be defined by $\boldsymbol{p}(1/k,k) := 1/k^2$ for $k = 2,3,\ldots$, and $\boldsymbol{p}(z,w) := 0$ otherwise. Notice that $|\boldsymbol{p}|$ is discrete. Then by 4.1 and Corollary 3.1(b),

$$d_{E\times\mathbb{C}}^{\min}(\boldsymbol{p},(z,w)) = d_E^{\min}(\boldsymbol{p}',z) = \prod_{k=2}^{\infty} [m_E(1/k,z)]^{1/k^2}, \qquad (z,w) \in E \times \mathbb{C}.$$

In particular, $d_{E\times\mathbb{C}}^{\min}(\boldsymbol{p},\cdot)$ is discontinuous at $(0,w) \in E \times \mathbb{C} \setminus |\boldsymbol{p}|$.

PROPERTY 4.4 (cf. 2.5). *If $\#|\boldsymbol{p}| < \infty$, then for any $z_0 \in G$ there exists an extremal function for $d_G^{\min}(\boldsymbol{p},z_0)$, i.e. a function $f_{z_0} \in \mathcal{O}(G,E)$ with $f_{z_0}(z_0) = 0$ and*

$$\prod_{\mu\in f_{z_0}(G)} |\mu|^{\sup \boldsymbol{p}(f_{z_0}^{-1}(\mu))} = d_G^{\min}(\boldsymbol{p},z_0).$$

Proof. Fix a $z_0 \in G$ and let $f_k \in \mathcal{O}(G, E)$ with $f_k(z_0) = 0$ be such that

$$\alpha_k := \prod_{\mu \in f_k(G)} |\mu|^{\sup \boldsymbol{p}(f_k^{-1}(\mu))} \to \alpha := d_G^{\min}(\boldsymbol{p}, z_0).$$

Let $A_k \subset |\boldsymbol{p}|$ be such that $f_k|_{A_k}$ is injective, $f_k(A_k) = f_k(|\boldsymbol{p}|)$, and $\boldsymbol{p}(a) = \sup \boldsymbol{p}(f_k^{-1}(f_k(a)))$ for $a \in A_k$. Thus $\alpha_k = \prod_{a \in A_k} |f_k(a)|^{\boldsymbol{p}(a)}$. We may assume that $A_k = B$ is independent of k and for any $a \in B$ the fiber $B_a := f_k^{-1}(f_k(a)) \cap |\boldsymbol{p}|$ is also independent of k. Moreover, we may assume that $f_k \to f_0$ locally uniformly in G. Then $f_0 \in \mathcal{O}(G, E)$, $f_0(z_0) = 0$, and $\prod_{a \in B} |f_0(a)|^{\boldsymbol{p}(a)} = \alpha$. Observe that $f_0(B) = f_0(|\boldsymbol{p}|)$. Let $B_0 \subset B$ be such that $f_0|_{B_0}$ is injective and $f_0(B_0) = f_0(B)$. We have

$$\begin{aligned}\alpha &\geq \prod_{\mu \in f_0(|\boldsymbol{p}|)} |\mu|^{\sup \boldsymbol{p}(f_0^{-1}(\mu))} = \prod_{\mu \in f_0(B_0)} |\mu|^{\sup \boldsymbol{p}(f_0^{-1}(\mu))} \\ &= \prod_{a \in B_0} |f_0(a)|^{\max\{\boldsymbol{p}(b) : b \in B,\, f_0(b) = f_0(a)\}} \geq \prod_{a \in B} |f_0(a)|^{\boldsymbol{p}(a)} = \alpha. \ \blacksquare\end{aligned}$$

PROPERTY 4.5. $\log d_G^{\min}(\boldsymbol{p}, \cdot) \in \mathcal{PSH}(G)$ (cf. 2.6).

Proof. By 4.2(a), we only need to show that for any $f \in \mathcal{O}(G, E)$ the function $u_f(z) := \prod_{\mu \in f(G)} [m_E(\mu, f(z))]^{\sup \boldsymbol{p}(f^{-1}(\mu))}$, $z \in G$, is log-plurisubharmonic on G. The proof of 4.2 shows that $u_f = \inf_M v_M$, where v_M is a log-plurisubharmonic function given by the formula

$$v_M := \prod_{\mu \in M} [m_E(\mu, f(z))]^{k_f(\mu)}$$

and M runs over a family of finite sets as in the proof of 4.2. Observe that $v_{M_1 \cup M_2} \leq \min\{v_{M_1}, v_{M_2}\}$. It remains to apply Lemma 2.9. $\blacksquare$

PROPERTY 4.6. *If $G_k \nearrow G$ and $\boldsymbol{p}_k \nearrow \boldsymbol{p}$, then*

$$d_{G_k}^{\min}(\boldsymbol{p}_k, z) \searrow d_G^{\min}(\boldsymbol{p}, z), \qquad d_{G_k}^{\max}(\boldsymbol{p}_k, z) \searrow d_G^{\max}(\boldsymbol{p}, z), \qquad z \in G.$$

Proof. By (H) and (M) the sequences are monotone and for the limit functions u we have $u \geq d_G^{\min}(\boldsymbol{p}, \cdot)$ (resp. $u \geq d_G^{\max}(\boldsymbol{p}, \cdot)$). Fix a $z_0 \in G$.

In the case of the minimal family suppose that $u(z_0) > \alpha > d_G^{\min}(G, z_0)$. Let $f_k \in \mathcal{O}(G_k, E)$ be such that $f_k(z_0) = 0$ and

$$\prod_{\mu \in f_k(G_k)} |\mu|^{\sup \boldsymbol{p}_k(f_k^{-1}(\mu))} \to u(z_0).$$

By a Montel argument we may assume that $f_k \to f_0$ locally uniformly in G with $f_0 \in \mathcal{O}(G, E)$, $f_0(z_0) = 0$. Since $\prod_{\mu \in f_0(G)} |\mu|^{\sup \boldsymbol{p}(f_0^{-1}(\mu))} < \alpha$, we can find a finite set $A \subset G$ such that $f|_A$ is injective and $\prod_{a \in A} |f_0(a)|^{\boldsymbol{p}(a)} < \alpha$. Consequently, $\prod_{a \in A} |f_k(a)|^{\boldsymbol{p}_k(a)} < \alpha$ and $f_k|_A$ is injective for $k \gg 1$. Finally, $\prod_{\mu \in f_k(G_k)} |\mu|^{\sup \boldsymbol{p}_k(f_k^{-1}(\mu))} < \alpha$ for $k \gg 1$; contradiction.

In the case of the maximal family for any $a \in G$ and $\varepsilon > 0$ there exists a $k(a,\varepsilon) \in \mathbb{N}$ such that $z_0, a \in G_k$, $\widetilde{k}^*_{G_k}(a,z_0) \le \widetilde{k}^*_G(a,z_0) + \varepsilon$, and $\boldsymbol{p}_k(a) \ge \boldsymbol{p}(a) - \varepsilon$ for $k \ge k(a,\varepsilon)$. Hence

$$\begin{aligned}\inf_{k\in\mathbb{N}} d^{\max}_{G_k}(\boldsymbol{p}_k, z_0) &= \inf_{k\in\mathbb{N}\,:\,a\in G_k} [\widetilde{k}^*_{G_k}(a,z_0)]^{\boldsymbol{p}_k(a)} \\ &\le \inf_{a\in G} \inf\{[\widetilde{k}^*_G(a,z_0)+\varepsilon]^{\boldsymbol{p}_k(a)} : 0<\varepsilon\ll 1,\ k\ge k(a,\varepsilon)\} \\ &\le \inf_{a\in G} \inf\{[\widetilde{k}^*_G(a,z_0)+\varepsilon]^{\boldsymbol{p}(a)-\varepsilon} : 0<\varepsilon\ll 1\} = d^{\max}_G(\boldsymbol{p}, z_0). \ \blacksquare\end{aligned}$$

EXAMPLE 4.7. Let $G := \{z \in \mathbb{C}^n : |z^\alpha| < 1\}$, where $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$ with $\alpha_1,\dots,\alpha_n$ relatively prime. Then

$$d^{\min}_G(\boldsymbol{p}, z) = d^{\min}_E(\boldsymbol{p}', z^\alpha) = \prod_{\mu\in E} [m_E(\mu, z^\alpha)]^{\boldsymbol{p}'(\mu)}, \qquad z \in G,$$

where $\boldsymbol{p}'(\lambda) = \sup\{\boldsymbol{p}(a) : a^\alpha = \lambda\}$ for $\lambda \in E$, and $d^{\min}_E(\boldsymbol{p}', \cdot) := 0$ if there exists a $\lambda_0 \in E$ with $\boldsymbol{p}'(\lambda_0) = \infty$.

Indeed, it is known that any function $f \in \mathcal{O}(G,E)$ has the form $f = g\circ\Phi$, where $\Phi(z) := z^\alpha$ and $g \in \mathcal{O}(E,E)$ (cf. [Jar-Pfl 1993, §4.4]. Thus

$$d^{\min}_G(\boldsymbol{p}, z) = \sup\Big\{ \prod_{\mu\in g(\Phi(G))} [m_E(\mu, g(\Phi(z)))]^{\sup \boldsymbol{p}(\Phi^{-1}(g^{-1}(\mu)))} : g \in \mathcal{O}(E,E)\Big\}$$

$$= \sup\Big\{ \prod_{\mu\in g(E)} [m_E(\mu, g(\Phi(z)))]^{\sup \boldsymbol{p}'(g^{-1}(\mu))} : g \in \mathcal{O}(E,E)\Big\} = d^{\min}_E(\boldsymbol{p}', \Phi(z)).$$

5. Product property. Let $\underline{d} = (d_G)_G$ be a generalized holomorphically contractible family with integer-valued weights. We say that $\underline{d}$ has the *product property* if

$$\text{(P)}\quad d_{G\times D}(A\times B, (z,w)) = \max\{d_G(A,z), d_D(B,w)\}, \qquad (z,w) \in G\times D,$$

for any domains $G \subset \mathbb{C}^n$, $D \subset \mathbb{C}^m$ and for any sets $\emptyset \ne A \subset G$, $\emptyset \ne B \subset D$. Notice that the inequality "$\ge$" follows from (H) applied to the projections $G\times D \to G$, $G\times D \to D$. The definition applies to the standard holomorphically contractible families and means that

$$d_{G\times D}((a,b),(z,w)) = \max\{d_G(a,z), d_D(b,w)\}, \qquad (a,b),(z,w) \in G\times D.$$

It is well known that the families $(\widetilde{k}^*_G)_G$, $(c^*_G)_G$, $(g_G)_G$ have the product property (cf. [Jar-Pfl 1993, Ch. 9], [Edi 1997], [Edi 1999], [Edi 2001]).

Moreover, it is known that the higher order Möbius functions $(m^{(k)}_G)_G$ with $k \ge 2$ fail the product property (cf. [Jar-Pfl 1993, Ch. 9]).

Thus it is natural to ask whether the minimal and maximal families have the product property.

PROPOSITION 5.1. *The system* $(d^{\max}_G)_G$ *has the product property.*

Proof. Fix $(z_0, w_0) \in G \times D$ and $\varepsilon > 0$. Let $(a, b) \in A \times B$ be such that $\widetilde{k}_G^*(a, z_0) \le d_G^{\max}(A, z_0) + \varepsilon$, $\widetilde{k}_D^*(b, w_0) \le d_G^{\max}(B, w_0) + \varepsilon$. Then using the product property for $(\widetilde{k}_G^*)_G$, we get

$$\begin{aligned} d_{G\times D}^{\max}(A \times B, (z_0, w_0)) &\le \widetilde{k}_{G\times D}^*((a, b), (z_0, w_0)) \\ &= \max\{\widetilde{k}_G^*(a, z_0), \widetilde{k}_D^*(b, w_0)\} \\ &\le \max\{d_G^{\max}(A, z_0), d_D^{\max}(B, w_0)\} + \varepsilon. \ \blacksquare \end{aligned}$$

We do not know whether the system $(d_G^{\min})_G$ *has the product property.* So far we have been able to handle only the case where $\#B = 1$ (see Proposition 5.3). Recall that $d_G^{\min}(A, \cdot) = m_G(A, \cdot)$ (Corollary 3.1(c)).

PROPOSITION 5.2. *Assume that for any* $n \in \mathbb{N}$, *the system* $(m_G)_G$ *has the following special product property*:

$$(\mathrm{P}_0)\quad |\Psi(z, w)| \le (\max_{G\times D} |\Psi|) \max\{m_G(A, z), m_D(B, w)\}, \qquad (z, w) \in G \times D,$$

where $G, D \subset \mathbb{C}^n$ *are balls with respect to arbitrary* $\mathbb{C}$*-norms*, $A \subset D$, $B \subset G$ *are finite and non-empty*, $\Psi(z, w) := \sum_{j=1}^n z_j w_j$, *and* $\Psi|_{A\times B} = 0$. *Then the system* $(m_G)_G$ *has the product property* (P) *in full generality. Moreover*, *if* (P_0) *holds with* $\#B = 1$, *then* (P) *holds with* $\#B = 1$.

Proof (cf. [Jar-Pfl 1993, the proof of Th. 9.5]). Fix arbitrary domains $G \subset \mathbb{C}^n$, $D \subset \mathbb{C}^m$, non-empty sets $A \subset G$, $B \subset G$, and $(z_0, w_0) \in G \times D$. We have to prove that for any $F \in \mathcal{O}(G \times D, E)$ with $F|_{A\times B} = 0$,

$$|F(z_0, w_0)| \le \max\{m_G(A, z_0), m_D(B, w_0)\}.$$

By 2.12, we may assume that A, B are finite.

Let $(G_\nu)_{\nu=1}^\infty$, $(D_\nu)_{\nu=1}^\infty$ be sequences of relatively compact subdomains of G and D, respectively, such that $A \cup \{z_0\} \subset G_\nu \nearrow G$, $B \cup \{w_0\} \subset D_\nu \nearrow D$. By 2.7, it suffices to show that

$$|F(z_0, w_0)| \le \max\{m_{G_\nu}(A, z_0), m_{D_\nu}(B, w_0)\}, \qquad \nu \ge 1.$$

Fix a $\nu_0 \in \mathbb{N}$ and let $G' := G_{\nu_0}$, $D' := D_{\nu_0}$. It is well known that F may be approximated locally uniformly in $G \times D$ by functions of the form

$$(**) \qquad F_s(z, w) = \sum_{\mu=1}^{N_s} f_{s,\mu}(z) g_{s,\mu}(w), \qquad (z, w) \in G \times D,$$

where $f_{s,\mu} \in \mathcal{O}(G)$, $g_{s,\mu} \in \mathcal{O}(D)$, $s \ge 1$, $\mu = 1, \ldots, N_s$. Notice that $F_s \to 0$ uniformly on $A \times B$. Using the Lagrange interpolation formula, we find polynomials $P_s : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}$ such that $P_s|_{A\times B} = F_s|_{A\times B}$ and $P_s \to 0$ locally uniformly in $\mathbb{C}^n \times \mathbb{C}^m$. The functions $\widehat{F}_s := F_s - P_s$, $s \ge 1$, also have the form

$(**)$ and $\widehat{F}_s \to F$ locally uniformly in $G \times D$. Hence, without loss of generality, we may assume that $F_s|_{A\times B} = 0$ for $s \geq 1$. Let $m_s := \max\{1, \|F_s\|_{G'\times D'}\}$ and $\widetilde{F}_s := F_s/m_s$, $s \geq 1$. Note that $m_s \to 1$, and therefore $\widetilde{F}_s \to F$ uniformly on $G' \times D'$. Consequently, we may assume that $F_s(G' \times D') \Subset E$ for $s \geq 1$.

It is enough to prove that

$$|F_s(z_0, w_0)| \leq \max\{m_{G'}(A, z_0), m_{D'}(B, w_0)\}, \qquad s \geq 1.$$

Fix an $s = s_0 \in \mathbb{N}$ and let $N := N_{s_0}$, $f_\mu := f_{s_0,\mu}$, $g_\mu := g_{s_0,\mu}$, $\mu = 1, \ldots, N$. Let $f := (f_1, \ldots, f_N) : G \to \mathbb{C}^N$ and $g := (g_1, \ldots, g_N) : D \to \mathbb{C}^N$. Put

$$K := \{\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{C}^N : \\ |\xi_\mu| \leq \|f_\mu\|_{G'},\ \mu = 1, \ldots, N,\ |\Psi(\xi, g(w))| \leq 1,\ w \in D'\}.$$

It is clear that K is an absolutely convex compact subset of $\mathbb{C}^N$ with $f(G') \subset K$. Let

$$L := \{\eta = (\eta_1, \ldots, \eta_N) \in \mathbb{C}^N : \\ |\eta_\mu| \leq \|g_\mu\|_{D'},\ \mu = 1, \ldots, N,\ |\Psi(\xi, \eta)| \leq 1,\ \xi \in K\}.$$

Then again L is an absolutely convex compact subset of $\mathbb{C}^N$, and moreover, $g(D') \subset L$.

Let $(W_\sigma)_{\sigma=1}^\infty$ (resp. $(V_\sigma)_{\sigma=1}^\infty$) be a sequence of absolutely convex bounded domains in $\mathbb{C}^N$ such that $W_{\sigma+1} \Subset W_\sigma$ and $W_\sigma \searrow K$ (resp. $V_{\sigma+1} \Subset V_\sigma$ and $V_\sigma \searrow L$). Put $M_\sigma := \|\Psi\|_{W_\sigma \times V_\sigma}$, $\sigma \in \mathbb{N}$. By (P_0) and by the holomorphic contractibility applied to the mappings $f : G' \to W_\sigma$, $g : D' \to V_\sigma$ we have

$$\begin{aligned}
|F_{s_0}(z_0, w_0)| &= |\Psi(f(z_0), g(w_0))| \\
&\leq M_\sigma \max\{m_{W_\sigma}(f(A), f(z_0)), m_{V\sigma}(g(B), g(w_0))\} \\
&\leq M_\sigma \max\{m_{G'}(f^{-1}(f(A)), z_0), m_{D'}(g^{-1}(g(B)), w_0)\} \\
&\leq M_\sigma \max\{m_{G'}(A, z_0), m_{D'}(B, w_0)\}.
\end{aligned}$$

Letting $\sigma \to \infty$ we get the required result. ∎

PROPOSITION 5.3. *The system* $(m_G)_G$ *has the product property* (P) *whenever* $\#B = 1$, *i.e. for any domains* $G \subset \mathbb{C}^n$, $D \subset \mathbb{C}^m$, *any set* $A \subset G$, *and any* $b \in D$ *we have*

$$m_{G\times D}(A \times \{b\}, (z, w)) = \max\{m_G(A, z), m_D(b, w)\}, \qquad (z, w) \in G \times D.$$

Proof. By Proposition 5.2, we only need to check (P) in the case where D is a bounded convex domain, A is finite, and $B = \{b\}$. Fix $(z_0, w_0) \in G \times D$. Let $\varphi : E \to D$ be a holomorphic mapping such that $\varphi(0) = b$ and $\varphi(m_D(b, w_0)) = w_0$ (cf. [Jar-Pfl 1993, Ch. 8]). Consider the mapping $F : G \times E \to G \times D$, $F(z, \lambda) := (z, \varphi(\lambda))$. Then

$$m_{G\times D}(A \times \{b\}, (z_0, w_0)) \leq m_{G\times E}(A \times \{0\}, (z_0, m_G(b, w_0))).$$

Consequently, it suffices to show that

$$(\dagger)\qquad m_{G\times E}(A\times\{0\},(z_0,\lambda))\le\max\{m_G(A,z_0),|\lambda|\},\qquad \lambda\in E.$$

The case where $m_G(A,z_0)=0$ is elementary: for an $f\in\mathcal{O}(G\times E,E)$ with $f|_{A\times\{0\}}=0$ we have $f(z_0,0)=0$ and hence $|f(z_0,\lambda)|\le|\lambda|$ for $\lambda\in E$ (by the Schwarz lemma). Thus, we may assume that $r:=m_G(A,z_0)>0$. First observe that it suffices to prove (†) on the circle $|\lambda|=r$. Indeed, if the inequality holds on that circle, then by the maximum principle for subharmonic functions (applied to the function $m_{G\times E}(A\times\{0\},(z_0,\cdot))$) it holds for all $|\lambda|\le r$. In the annulus $\{r<|\lambda|<1\}$ we apply the maximum principle to the subharmonic function $\lambda\mapsto|\lambda|^{-1}m_{G\times E}(A\times\{0\},(z_0,\lambda))$.

Now fix a $\lambda_0\in E$ with $|\lambda_0|=r$. Let f be an extremal function for $m_G(A,z_0)$ with $f|_A=0$ and $f(z_0)=\lambda_0$. Consider $F:G\to G\times E$, $F(z):=(z,f(z))$. Then

$$m_G(A\times\{0\},(z_0,\lambda_0))\le m_G(A,z_0)=\max\{m_G(A,z_0),|\lambda_0|\},$$

which completes the proof. ∎

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