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The set of probability distribution solutions of a linear functional equation

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Abstract. Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\tau\colon \mathbb{R}\times\Omega \to \mathbb{R}$ be a function which is strictly increasing and continuous with respect to the first variable, measurable with respect to the second variable. Given the set of all continuous probability distribution solutions of the equation

$$F(x) = \int_{\Omega} F(\tau(x,\omega))\, dP(\omega)$$

we determine the set of all its probability distribution solutions.

1. Introduction. Fix a probability space $(\Omega, \mathcal{A}, P)$, a function $\tau\colon \mathbb{R}\times\Omega \to \mathbb{R}$ and consider the equation

$$F(x) = \int_{\Omega} F(\tau(x,\omega))\, dP(\omega). \tag{1}$$

This equation as well as its particular cases appear in many branches of mathematics and their solutions are extensively studied in various classes of functions (see [2], [12]). Measurable and nonnegative solutions of (1) have been investigated in [7], [8]. Continuous and bounded solutions of (1) have been studied in [3], [5], [9], [10]. Existence and uniqueness of solutions of (1) in a class of bounded functions have been proved in [13]. Properties of this unique solution have been examined in [14]. Absolutely continuous probability distribution (p.d.) solutions of (1) are interesting in connection with the refinement equation. Such solutions have been considered in [4], [6], [16], [17]. Equation (1) contains the integral Cauchy functional equation which appears in probability theory (see [18], [19]). Interesting and very general results on solutions of (1) in the class of p.d. functions can be found in [1]. P.d. solutions of a very special case of (1) have been studied in [15]

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in connection with a problem coming from game theory. Motivated by Theorem 6 from that paper we are interested in answering the following main question: Is it possible to determine all p.d. solutions of (1) if one knows all solutions which are continuous p.d. functions? Another motivation to pose the above question comes from [1], [9] and [11]. For example, we know from [1] that equation (1) may possess exactly one solution in the class of all p.d. functions (see Theorem A in Section 7). But the problem is: When is the unique solution continuous? According to [9] in some cases equation (1) has no continuous p.d. solutions (see Theorem B in Section 7) and the problem reads: Are there discontinuous p.d. solutions of (1)? Further, the paper [11] brings conditions under which equation (1) has exactly one continuous p.d. solution (see Theorem C in Section 7). Now the problem is: Is the set of p.d. solutions of (1) nonvoid and how large is it? It turns out that we can solve all these problems if we know a positive answer to our main question.

2. Notation. From now on we assume that $\tau\colon \mathbb{R}\times\Omega\to\mathbb{R}$ is a given function such that for every $x\in\mathbb{R}$ the function $\tau(x,\cdot)$ is measurable, and for every $\omega\in\Omega$ the function $\tau(\cdot,\omega)$ is strictly increasing and continuous.

Denote by $\mathcal{C}$ the class of all continuous p.d. solutions of (1). In this paper we determine the class

$$\mathcal{I}=\{F\colon\mathbb{R}\to[0,1]\mid F \text{ is a nondecreasing solution of (1) such that } F^{+}(-\infty)=\lim_{x\to-\infty}F(x)=0 \text{ and } F_{-}(+\infty)=\lim_{x\to+\infty}F(x)=1\},$$

assuming that the class $\mathcal{C}$ is given. Since $\mathcal{I}$ contains the classes

$$\mathcal{R}=\{F\in\mathcal{I}\mid F \text{ is right continuous}\}$$

and

$$\mathcal{L}=\{F\in\mathcal{I}\mid F \text{ is left continuous}\},$$

it follows that the results on $\mathcal{I}$ can be easily reformulated to results concerning $\mathcal{R}$ and $\mathcal{L}$. The details are left to the reader.

The following remark can be proved by a simple verification.

REMARK 1. *Let $F_1,\ldots,F_n\colon\mathbb{R}\to\mathbb{R}$ be solutions of* (1) *and let $\alpha_1,\ldots,\alpha_n$ be real numbers. Then $F=\alpha_1F_1+\cdots+\alpha_nF_n$ is a solution of* (1), *and if moreover $F_1,\ldots,F_n\in\mathcal{I}$ $[\mathcal{L},\mathcal{R},\mathcal{C}]$ and $\alpha_1,\ldots,\alpha_n$ are nonnegative and sum to one*, *then $F\in\mathcal{I}$ $[\mathcal{L},\mathcal{R},\mathcal{C}]$.*

We now define two sets which will play the key role in the description of the class $\mathcal{I}$. Put

$$\mathbf{E}=\{x\in\mathbb{R}\mid \tau(x,\omega)=x \text{ for almost all } \omega\in\Omega\},\qquad E_0=\mathbf{E}\cup\{-\infty,+\infty\}.$$

REMARK 2. *Let $a,b\in E_0$ and let J be an interval with endpoints a and b. If $F\colon\mathbb{R}\to\mathbb{R}$ is a solution of* (1), *then so is $F|_J$.*

In particular, any constant function defined on J is a solution of (1).

Proof. It is enough to observe that $\{\tau(x,\omega) \mid x \in J\} \subset J$ for almost all $\omega \in \Omega$, by the strict monotonicity of τ as a function of the first variable. ∎

Given a nondecreasing solution $F\colon \mathbb{R} \to \mathbb{R}$ of (1) we define functions $F_-, F^+, \Delta F\colon \mathbb{R} \to \mathbb{R}$ by

$$F_-(x) = \lim_{y\to x^-} F(y), \qquad F^+(x) = \lim_{y\to x^+} F(y), \qquad \Delta F(x) = F^+(x) - F_-(x),$$

and if $a, b \in E_0$ with $a < b$, $F^+(a) < F_-(b)$, we also define a function $F_{a,b}\colon \mathbb{R} \to \mathbb{R}$ by

$$F_{a,b}(x) = \begin{cases} 0 & \text{if } x \in (-\infty, a], \\ \dfrac{F(x) - F^+(a)}{F_-(b) - F^+(a)} & \text{if } x \in (a,b), \\ 1 & \text{if } x \in [b, +\infty). \end{cases}$$

By simple computations we get the following remark.

Remark 3. *Let $F \in \mathcal{I}$. Then*:

(i) $F_- \in \mathcal{L}$, $F^+ \in \mathcal{R}$.
(ii) *ΔF is a nonnegative solution of* (1) *and* $\Delta F(x_1) + \cdots + \Delta F(x_n) \le 1$ *for all* $x_1 < \cdots < x_n$ *and* $n \in \mathbb{N}$.
(iii) *F is continuous at x if and only if* $\Delta F(x) = 0$.
(iv) $F_{a,b} \in \mathcal{I}$ *for all* $a, b \in E_0$.

3. Basic properties of solutions of (1). The following lemma is the key observation for our next considerations.

Lemma 1. *Let $F \in \mathcal{I}$. If $x \notin \mathbf{E}$, then* $\Delta F(x) = 0$.

Proof. Suppose, on the contrary, that there is an $x_0 \notin \mathbf{E}$ such that $\Delta F(x_0) > 0$. Put

$$a = \begin{cases} \sup\{x \in \mathbf{E} \mid x < x_0\} & \text{if } \{x \in \mathbf{E} \mid x < x_0\} \ne \emptyset, \\ -\infty & \text{otherwise}, \end{cases}$$

$$b = \begin{cases} \inf\{x \in \mathbf{E} \mid x_0 < x\} & \text{if } \{x \in \mathbf{E} \mid x_0 < x\} \ne \emptyset, \\ +\infty & \text{otherwise}. \end{cases}$$

Clearly,

$$(a,b) \cap \mathbf{E} = \emptyset. \tag{2}$$

Since $\mathbf{E}$ is closed, we have $x_0 \in (a,b)$. Hence $F^+(a) < F_-(b)$, and by Remark 3(iv) we get $F_{a,b} \in \mathcal{I}$. Moreover, from the definition of $F_{a,b}$ we see that

$$\Delta F_{a,b}(x_0) > 0, \tag{3}$$

$$\Delta F_{a,b}(x) = 0 \quad \text{for all } x \notin (a,b). \tag{4}$$

Let $G = F_{a,b}$. We see from (3), Remark 3(ii) and (4) that the set $\{x \in (a,b) \mid \Delta G(x) \geq \Delta G(x_0)\}$ is nonempty, finite and contained in (a,b). Put

$$L = \sup\{\Delta G(x) \mid x \in (a,b)\}.$$

Clearly, $L > 0$ and the set $\{x \in (a,b) \mid \Delta G(x) = L\}$ is also nonempty, finite and contained in (a,b). Hence there are $a < x_1 < \cdots < x_n < b$ such that

$$\Delta G(x_1) = \cdots = \Delta G(x_n) = L, \tag{5}$$

$$\Delta G(x) < L \qquad \text{for all } x \notin \{x_1, \ldots, x_n\}. \tag{6}$$

Remarks 3(i), 3(iii) and (5) imply

$$0 < G^+(x_1) < \cdots < G^+(x_n) \leq 1. \tag{7}$$

Put

$$A_i = \{\omega \in \Omega \mid \tau(x_1, \omega) = x_i\}$$

for all $i \in \{1, \ldots, n\}$. Notice that the sets A_i are pairwise disjoint and

$$P(A_1) < 1, \tag{8}$$

by (2) and the fact that $x_1 \in (a,b)$.

Suppose that $P\left(\bigcup_{i=1}^n A_i\right) < 1$. Then using Remark 3(ii), (5) and (6) we obtain

$$L = \Delta G(x_1) = \sum_{i=1}^n \int_{A_i} \Delta G(\tau(x_1,\omega))\, dP(\omega) + \int_{\Omega \setminus \bigcup_{i=1}^n A_i} \Delta G(\tau(x_1,\omega))\, dP(\omega)$$

$$< L \sum_{i=1}^n P(A_i) + L\Big[1 - P\Big(\bigcup_{i=1}^n A_i\Big)\Big] = L,$$

which is impossible. Hence

$$P\Big(\bigcup_{i=1}^n A_i\Big) = 1 \tag{9}$$

and $n \geq 2$, by (8). Using Remark 3(i), (1) and (9) we obtain

$$G^+(x_1) = \sum_{i=1}^n G^+(x_i) P(A_i).$$

This jointly with (7) and (9) gives

$$\begin{aligned}
[1 - P(A_1)]G^+(x_1) &= \sum_{i=2}^n G^+(x_i) P(A_i) \\
&> \sum_{i=2}^n G^+(x_1) P(A_i) \\
&= [1 - P(A_1)]G^+(x_1),
\end{aligned}$$

a contradiction. ∎

An obvious consequence of Lemma 1 and Remark 3(iii) is the following corollary.

Corollary 1. *If* $\mathbf{E} = \emptyset$, *then* $\mathcal{C} = \mathcal{R} = \mathcal{L} = \mathcal{I}$.

4. The case $\mathcal{C} = \emptyset$**.** Throughout this section we assume that $\mathcal{C} = \emptyset$. (This case occurs in [9].)

Lemma 2. *Let* $F \in \mathcal{I}$ *and let* $a, b \in E_0$ *with* $a < b$. *If* $(a, b) \cap \mathbf{E} = \emptyset$, *then* F *is constant on* (a, b).

Proof. Suppose, contrary to our claim, that $F^+(a) < F_-(b)$. From Remark 3(iv) we conclude that $F_{a,b} \in \mathcal{I}$. This jointly with Lemma 1 and Remark 3(iii) shows that $F_{a,b} \in \mathcal{C}$, a contradiction. ∎

From Corollary 1, Lemma 2 and Remark 2 we get the following description of the class $\mathcal{I}$.

Theorem 1.

(i) *If* $\mathbf{E} = \emptyset$, *then* $\mathcal{I} = \emptyset$.
(ii) *If* $\mathbf{E} \neq \emptyset$, *then* $\mathcal{I} \neq \emptyset$. *Moreover, if* $F\colon \mathbb{R} \to [0, 1]$, *then* $F \in \mathcal{I}$ *if and only if* $F^+(-\infty) = 0$, $F_-(+\infty) = 1$, F *is nondecreasing and constant on any interval* (a, b), *where* $a, b \in E_0$ *with* $\mathbf{E} \cap (a, b) = \emptyset$.

5. The case $\mathcal{C} \neq \emptyset$**.** Until the end of the paper we assume that $\mathcal{C} \neq \emptyset$. (This case occurs in [13].) We begin with a general observation concerning the members of $\mathcal{C}$.

Lemma 3. *Either* $|\mathcal{C}| = 1$ *or* $|\mathcal{C}| = \mathfrak{c}$.

Proof. Fix $F_1, F_2 \in \mathcal{C}$. For every $\alpha \in [0, 1]$ define $G_\alpha\colon \mathbb{R} \to \mathbb{R}$ by

$$G_\alpha = \alpha F_1 + (1 - \alpha) F_2.$$

From Remark 1 we conclude that $G_\alpha \in \mathcal{C}$ for all $\alpha \in [0, 1]$.

Since the set of all continuous real functions on $\mathbb{R}$ has the cardinality of the continuum, we have $|\mathcal{C}| \leq \mathfrak{c}$. If $|\mathcal{C}| < \mathfrak{c}$, then there are $\alpha_1, \alpha_2 \in [0, 1]$, $\alpha_1 \neq \alpha_2$, such that $G_{\alpha_1} = G_{\alpha_2}$. Consequently, $F_1 = F_2$. ∎

The main result of this section reads as follows.

Theorem 2.

(i) *If* $\mathbf{E} = \emptyset$, *then* $\mathcal{I} = \mathcal{C}$.
(ii) *If* $\mathbf{E} \neq \emptyset$, *then* $\mathcal{I} \supsetneq \mathcal{C}$. *Moreover, if* $F\colon \mathbb{R} \to [0, 1]$, *then* $F \in \mathcal{I}$ *if and only if* F *is nondecreasing*, $F^+(-\infty) = 0$, $F_-(+\infty) = 1$ *and for all* $a, b \in E_0$ *with* $\mathbf{E} \cap (a, b) = \emptyset$ *there are* $G \in \mathcal{C}$ *and* $\alpha, \beta \in [0, 1]$ *with* $\alpha + \beta \leq 1$ *such that*

$$F|_{(a,b)} = \alpha + \beta G|_{(a,b)}. \tag{10}$$

Proof. By Corollary 1 we only need to prove assertion (ii). Fix $F \in \mathcal{I}$ and $a, b \in E_0$ such that $\mathbf{E} \cap (a,b) = \emptyset$.

If $F^+(a) = F_-(b)$, then (10) holds with $\alpha = F^+(a)$, $\beta = 0$ and arbitrary $G \in \mathcal{C}$.

If $F^+(a) < F_-(b)$, then from Remark 3(iv), 3(iii) and Lemma 1 we conclude that $F_{a,b} \in \mathcal{C}$. Then (10) holds with $\alpha = F^+(a)$, $\beta = F_-(b) - F^+(a)$ and $G = F_{a,b}$.

The rest of the proof follows from Remarks 1, 2 and the fact that each function defined on $\mathbf{E}$ satisfies (1). ■

6. The case $|\mathcal{C}| = 1$. Throughout this section we assume that $\mathcal{C} = \{\mathbf{F}\}$. (This case occurs in [11].) In this particular case Theorem 2 is still valid. To improve it we need the following observation.

LEMMA 4. $\mathbf{E} \subset \{x \in \mathbb{R} \mid \mathbf{F}(x) \in \{0,1\}\}$.

Proof. Fix an $x_0 \in \mathbf{E}$ such that $\mathbf{F}(x_0) > 0$. Remark 3(iv) leads to $\mathbf{F}_{-\infty,x_0} = \mathbf{F}$. Hence $\mathbf{F}(x_0) = 1$. ■

Put

$$A = \begin{cases} \sup\{x \in \mathbf{E} \mid \mathbf{F}(x) = 0\} & \text{if } \{x \in \mathbf{E} \mid \mathbf{F}(x) = 0\} \neq \emptyset, \\ -\infty & \text{otherwise,} \end{cases}$$

$$B = \begin{cases} \inf\{x \in \mathbf{E} \mid \mathbf{F}(x) = 1\} & \text{if } \{x \in \mathbf{E} \mid \mathbf{F}(x) = 1\} \neq \emptyset, \\ +\infty & \text{otherwise.} \end{cases}$$

Clearly, $A, B \in E_0$, $-\infty \leq A < B \leq +\infty$ and, by Lemma 4, $(A,B) \cap \mathbf{E} = \emptyset$ and if $\mathbf{E} \neq \emptyset$, then $A \in \mathbb{R}$ or $B \in \mathbb{R}$. We now have the following description of the class $\mathcal{I}$.

THEOREM 3.

(i) *If* $\mathbf{E} = \emptyset$, *then* $\mathcal{I} = \{\mathbf{F}\}$.
(ii) *If* $\mathbf{E} \neq \emptyset$, *then* $\mathcal{I} \supsetneq \{\mathbf{F}\}$. *Moreover, if* $F \colon \mathbb{R} \to [0,1]$, *then* $F \in \mathcal{I}$ *if and only if* F *is nondecreasing, constant on any interval* $(a,b) \neq (A,B)$, *where* $a, b \in E_0$ *with* $\mathbf{E} \cap (a,b) = \emptyset$, *and*

$$F|_{(A,B)} = \alpha + \beta \mathbf{F}|_{(A,B)} \tag{11}$$

with $\alpha, \beta \in [0,1]$, $\alpha + \beta \leq 1$, *and*

(a) *if* $A \in \mathbb{R}$ *and* $B \in \mathbb{R}$, *then* $F^+(-\infty) = 0$ *and* $F_-(+\infty) = 1$,
(b) *if* $A = -\infty$ *and* $B \in \mathbb{R}$, *then* $F_-(+\infty) = 1$ *and* $\alpha = 0$,
(c) *if* $A \in \mathbb{R}$ *and* $B = +\infty$, *then* $F^+(-\infty) = 0$ *and* $\beta = 1 - \alpha$.

Proof. The representation (11) follows from (10) with $G = \mathbf{F}$. The other assertions follow from Theorem 2 and Lemma 4. ■

7. Applications. We now give three examples of the possible applications of the results obtained. The following theorem on the existence and uniqueness of solutions of (1) in the class of p.d. functions is proved in [1].

THEOREM A. *Assume that for every $\omega \in \Omega$ the function $\tau(\cdot,\omega)$ is a strictly increasing transformation of $\mathbb{R}$ onto $\mathbb{R}$ and for every $x \in \mathbb{R}$ the function $\tau(x,\cdot)$ is measurable. Let $L\colon \Omega \to (0,+\infty)$ be a measurable function such that*

$$|\tau(x,\omega) - \tau(y,\omega)| \ge L(\omega)|x-y| \quad \textit{for } x, y \in \mathbb{R},\ \omega \in \Omega, \tag{12}$$

$$0 < \int_{\Omega} \log L(\omega)\, dP(\omega) < +\infty. \tag{13}$$

If there is $x_0 \in \mathbb{R}$ such that

$$\int_{\{\omega \mid L(\omega) < |\tau(x_0,\omega) - x_0|\}} \log \frac{|\tau(x_0,\omega) - x_0|}{L(\omega)}\, dP(\omega) < +\infty,$$

then (1) *has exactly one solution F in the class of all p.d. functions.*

Conditions (12) and (13) imply that the set $\mathbf{E}$ consists of at most one element. Now it follows from Corollary 1 that the unique solution F of (1) from Theorem A is continuous if and only if $\mathbf{E} = \emptyset$; moreover $F = \chi_{[y,+\infty)}$ if and only if $\mathbf{E} = \{y\}$.

The next two results concern a special but very interesting case of (1), namely, equations of the form

$$F(x) = \int_{\Omega} F(L(\omega)x - M(\omega))\, dP(\omega). \tag{14}$$

The following result follows from [9].

THEOREM B. *Let $L\colon \Omega \to (0,+\infty)$, $M\colon \Omega \to \mathbb{R}$ be measurable functions such that $-\infty < \int_{\Omega} \log L(\omega)\, dP(\omega) < 0$ and $\int_{\Omega} \log \max\{|M(\omega)|, 1\}\, dP(\omega) < +\infty$. Then* (14) *has no continuous p.d. solutions.*

It is clear that if $\tau(x,\omega) = L(\omega)x - M(\omega)$ for $x \in \mathbb{R}$ and $\omega \in \Omega$, then the set $\mathbf{E}$ consists of at most one element. Now from Theorem 1 we conclude that under the assumptions of Theorem B equation (14) has no p.d. solution if and only if $\mathbf{E} = \emptyset$; moreover $\chi_{[y,+\infty)}$ is the unique p.d. solution of (14) if and only if $\mathbf{E} = \{y\}$.

The next theorem can be found in [11]; cf. [9].

THEOREM C. *Let $L\colon \Omega \to (0,+\infty)$, $M\colon \Omega \to \mathbb{R}$ be measurable functions such that $\int_{\Omega} \log \max\{|M(\omega)|/L(\omega), 1\}\, dP(\omega) < +\infty$ and* (13) *holds. If*

$$P(\{\omega \mid L(\omega)x - M(\omega) = x\}) < 1 \quad \textit{for all } x \in \mathbb{R}, \tag{15}$$

then (14) *has exactly one continuous p.d. solution F.*

Condition (15) yields $\mathbf{E} = \emptyset$. Hence the word "continuous" can be omitted from the assertion of Theorem C, by Corollary 1.

8. The set E. It is clear that for a given function τ (strictly increasing and continuous with respect to the first variable, and measurable with respect to the second) the set $\mathbf{E}$ is closed. One can ask if for a given closed set $E \subset \mathbb{R}$ there is a function τ, having all required properties, such that $\mathbf{E} = E$. The next example answers this question.

EXAMPLE. Fix a closed set $E \subset \mathbb{R}$ and let $\{(a_i, b_i) \mid i \in I\}$ be a (countable) set of pairwise disjoint intervals such that $\mathbb{R} \setminus E = \bigcup_{i \in I}(a_i, b_i)$. For every $i \in I$ fix a function $g_i\colon (a_i, b_i) \times \Omega \to (a_i, b_i)$ such that for every $x \in (a_i, b_i)$ the function $g_i(x, \cdot)$ is measurable, for every $\omega \in \Omega$ the function $g_i(\cdot, \omega)$ is strictly increasing without fixed points, continuous, $\lim_{x \to a_i} g_i(x, \omega) = a_i$ and $\lim_{x \to b_i} g_i(x, \omega) = b_i$. Next define the function $\tau\colon \mathbb{R} \times \Omega \to \mathbb{R}$ by

$$\tau(x, \omega) = \begin{cases} x & \text{if } x \in E, \\ g_i(x, \omega) & \text{if } x \in (a_i, b_i). \end{cases}$$

It is clear that τ is strictly increasing and continuous with respect to the first variable, measurable with respect to the second variable, and $\mathbf{E} = E$.

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