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Peak points for domains in $\mathbb{C}^n$

by Armen Edigarian (Kraków)

Abstract. We give a necessary and sufficient condition for the existence of a weak peak function by using Jensen type measures. We also show the existence of a weak peak function for a class of Reinhardt domains.

1. Introduction. Let X be a topological space and let $\mathscr{S}$ be a subset of the set $C(X)$ of all continuous complex-valued functions $X \to \mathbb{C}$. We say that a point $\zeta \in X$ is an *$\mathscr{S}$-peak point* if there exists a function $f \in \mathscr{S}$ such that $f(\zeta) = 1$ and $|f| < 1$ on $X \setminus \{\zeta\}$. Usually, one takes for X a compact set in $\mathbb{C}$ and for $\mathscr{S}$ a uniform algebra on X (see e.g. [5, Chapter II]). The basic problem is to determine a necessary and sufficient condition for a point to be a peak point.

Let X be any subset of $\mathbb{C}^n$. In complex analysis it is quite natural to study peak points of the family $\mathscr{A}(X) = C(X) \cap \mathcal{O}(\operatorname{int} X)$. In particular, take a bounded domain $D \subset \mathbb{C}^n$ and take $X = \overline{D}$. Then a boundary point $\zeta \in \partial D$ is a peak point if it is a peak point for the set $\mathscr{A}(X)$. However, in our paper we study the case $X = D \cup \{\zeta\}$ (here, D is not necessarily bounded), where ζ is a boundary point of D. The question is: when is ζ a peak point for $\mathscr{A}(D \cup \{\zeta\})$? To distinguish from the standard "peak point" we use the notion of "weak peak point" (see [8]).

The main result of the paper is the following characterization (cf. [5, Theorem II.11.3]).

Theorem 1.1. *Let $D \subset \mathbb{C}^n$ be any domain and let $\zeta \in \partial D$. The following conditions are equivalent:*

(1) *ζ is a weak peak point.*
(2) *There exists no Borel probability measure μ with compact support in*

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$D\cup\{\zeta\}$ such that $\mu(\{\zeta\})=0$ and

$$|f(\zeta)|\le\int_D |f(w)|\,d\mu(w)\qquad\text{for any } f\in\mathscr{A}(D\cup\{\zeta\}).$$

(3) *There exists no Borel probability measure μ with compact support in $D\cup\{\zeta\}$ such that $\mu(\{\zeta\})=0$ and*

$$f(\zeta)=\int_D f(w)\,d\mu(w)\qquad\text{for any } f\in\mathscr{A}(D\cup\{\zeta\}),$$

i.e., there exists no Borel probability representing measure μ with $\mu(\zeta)=0$.

(4) *There exists a sequence of functions $f_N\in\mathscr{A}(D\cup\{\zeta\})$ such that*
   (a) *$\|f_N\|_D\le 1$ for any $N\ge 1$;*
   (b) *$f_N(\zeta)\to 1$ when $N\to\infty$;*
   (c) *for any compact set $K\Subset D$ there exist an $\eta\in(0,1)$ and a $k\in\mathbb{N}$ such that $\|f_N\|_K\le 1-\eta$ for any $N\ge k$.*

(5) *For any bounded set $W\subset\mathbb{C}^n$ there exists a sequence of functions $f_N\in\mathscr{A}(D\cup\{\zeta\})$ such that*
   (a) *$\|f_N\|_{D\cap W}\le 1$ for any $N\ge 1$;*
   (b) *$f_N(\zeta)\to 1$ when $N\to\infty$;*
   (c) *for any compact set $K\Subset D\cap W$ there exist an $\eta\in(0,1)$ and a $k\in\mathbb{N}$ such that $\|f_N\|_K\le 1-\eta$ for any $N\ge k$.*

The implications (1)⇒(2)⇒(3) and (4)⇒(5)⇒(2) are immediate.

As a direct corollary of our result we get the following family of domains where weak peak functions exist.

COROLLARY 1.2. *Assume that a domain $D\subset\mathbb{C}^n$ can be described as*

$$D=\Big\{z\in\mathbb{C}^n:\sum_{j=1}^m\phi_j(|f_j(z)+\overline{g_j}(z)|)+\sum_{\ell=1}^k\psi_\ell\big(\Re(h_\ell(z))\big)<0\Big\},$$

where $f_j,g_j,h_\ell\in\mathcal{O}(\mathbb{C}^n)$ and ϕ_j,ψ_ℓ are convex functions. Then any boundary point of D is a weak peak point.

2. Non-compact version of Edwards' theorem. This part of the paper is motivated by [6] and we use methods developed in that paper. For the convenience of the reader, we repeat the main points from [6] without proofs, thus making our exposition self-contained.

Let X be a topological space and let $C_{\mathbb{R}}(X)$ be the set of all real-valued continuous functions on X. Note that $C_{\mathbb{R}}(X)$ is a real vector space. We say that a function $\mathscr{H}:C_{\mathbb{R}}(X)\to[-\infty,+\infty)$ is a *superlinear* operator if

(1) $\mathscr{H}(\alpha f)=\alpha\mathscr{H}(f)$ for any $f\in C_{\mathbb{R}}(X)$ and any $\alpha\ge 0$;
(2) $\mathscr{H}(f+g)\ge\mathscr{H}(f)+\mathscr{H}(g)$ for any $f,g\in C_{\mathbb{R}}(X)$.

Note that any linear operator is superlinear. Moreover, if $\mathscr{H}_1, \mathscr{H}_2$ are superlinear operators, then $\min\{\mathscr{H}_1, \mathscr{H}_2\}$ is also a superlinear operator. In this way we get superlinear operators which are not linear. As an example, for a fixed $x \in X$ the operator $\mathscr{H}_x(\phi) = \phi(x)$ is (super)linear.

We have the following version of the Hahn–Banach theorem (see e.g. [6, Theorem 2.4]).

Theorem 2.1. *Let Z be a linear space and let M be a vector subspace of Z. If $\mathscr{H} : Z \to [-\infty, +\infty)$ is a superlinear functional and $\ell : M \to \mathbb{R}$ is a linear functional such that $\ell \geq \mathscr{H}$ on M, then there is a linear functional $\mathscr{L} : Z \to \mathbb{R}$ such that $\ell = \mathscr{L}$ on M and $\mathscr{L} \geq \mathscr{H}$ on Z.*

In particular, we get a relation between positive superlinear operators and positive linear operators.

Proposition 2.2. *Let X be a topological space and let*

$$\mathscr{H} : C_{\mathbb{R}}(X) \to [-\infty, +\infty)$$

be a positive superlinear operator. Then

$$\mathscr{H}(\phi) = \inf\{\mathscr{L}(\phi) : \mathscr{L} \text{ is a positive linear operator on } C_{\mathbb{R}}(X),\ \mathscr{L} \geq \mathscr{H}\}. \tag{2.1}$$

Proof. Fix $\phi \in C_{\mathbb{R}}(X)$. Since $\mathscr{H}(\phi) + \mathscr{H}(-\phi) \leq 0$, we may find an $a \in \mathbb{R}$ such that $-\mathscr{H}(-\phi) \geq a \geq \mathscr{H}(\phi)$. Consider a real linear subspace $M = \{t\phi : t \in \mathbb{R}\} \subset C_{\mathbb{R}}(X)$ and a linear functional $\ell : M \ni t\phi \mapsto ta \in \mathbb{R}$. Note that $\ell \geq \mathscr{H}$ on M. Indeed, $\ell(t\phi) = ta \geq \mathscr{H}(t\phi)$ for any $t \in \mathbb{R}$. For if $t > 0$ then this is equivalent to $a \geq \mathscr{H}(\phi)$, and if $t < 0$ then it is equivalent to $-a \geq \mathscr{H}(-\phi)$. Hence by Theorem 2.1 there exists a linear functional $\mathscr{L} : C_{\mathbb{R}}(X) \to \mathbb{R}$ such that $\ell = \mathscr{L}$ on M and $\mathscr{L} \geq \mathscr{H}$ on $C_{\mathbb{R}}(X)$.

If $\mathscr{H}(\phi) \in \mathbb{R}$ then we take $a = \mathscr{H}(\phi)$ and get the equality (2.1). If $\mathscr{H}(\phi) = -\infty$, we may take the sequence $a_n = -n$ and get a sequence $\{\mathscr{L}_n\}_{n\geq 1}$ of linear functionals such that $\mathscr{L}_n \geq \mathscr{H}$ for any $n \geq 1$ and $\mathscr{L}_n(\phi) \to -\infty$ when $n \to \infty$. ∎

We say that a subset $\mathscr{S} \subset C_{\mathbb{R}}(X)$ is a *wedge* if $\alpha f + \beta g \in \mathscr{S}$ for any $f, g \in \mathscr{S}$ and any $\alpha, \beta \geq 0$ (see e.g. [1]). We assume moreover that any wedge $\mathscr{S}$ contains the constants.

For any $\phi \in C_{\mathbb{R}}(X)$ we consider its envelope related to a point $x \in X$ and a wedge $\mathscr{S} \subset C_{\mathbb{R}}(X)$ defined by

$$\Phi_{x,\mathscr{S}}(\phi) = \sup\{\psi(x) : \psi \in \mathscr{S},\ \psi \leq \phi\}.$$

Note that $\Phi_{x,\mathscr{S}}(\phi) \leq \phi(x)$ and that $\Phi_{x,\mathscr{S}} : C_{\mathbb{R}}(X) \to [-\infty, +\infty)$ is a positive superlinear operator. Here, *positivity* means $\Phi_{x,\mathscr{S}}(\phi) \geq 0$ for any $\phi \in C_{\mathbb{R}}(X)$ such that $\phi \geq 0$. Actually, we have $\Phi_{x,\mathscr{S}}(\phi) = \phi(x)$ for any $\phi \in \mathscr{S}$.

For a wedge $\mathscr{S}$ and a point $x \in X$ we study the set $J_{x,\mathscr{S}}(X)$ of all *Jensen measures*, i.e., the set of all Borel probability measures μ with compact support in X such that $\psi(x) \le \int \psi\, d\mu$ for any $\psi \in \mathscr{S}$.

In 1965 Edwards [3] proved the following result.

THEOREM 2.3. *Let X be a compact topological space and let $\mathscr{S} \subset C_{\mathbb{R}}(X)$ be a wedge. Assume that ϕ is a lower semicontinuous function on X. Then for any $x \in X$ we have*

$$\varPhi_{x,\mathscr{S}}(\phi) = \min\Big\{\int \phi\, d\mu : \mu \in J_{x,\mathscr{S}}(X)\Big\}.$$

Recently Gogus, Perkins, and Poletsky [6] proved the following noncompact version of Edwards' theorem.

THEOREM 2.4. *Let X be a locally compact σ-compact Hausdorff space and let $\mathscr{S} \subset C_{\mathbb{R}}(X)$ be a wedge. Assume that ϕ is a continuous function on X. Then for any $x \in X$ we have $\varPhi_{x,\mathscr{S}}(\phi) \equiv -\infty$ or*

$$\varPhi_{x,\mathscr{S}}(\phi) = \min\Big\{\int \phi\, d\mu : \mu \in J_{x,\mathscr{S}}(X)\Big\}.$$

Our main aim in this section is to extend this result to a more general class of spaces. In the next sections we show applications of our results.

There is an extensive literature on the study of positive linear functionals on $C_{\mathbb{R}}(X)$ (see e.g. [2], [7], and [10]) ([1]). Let X be a normal topological space and let $\mathscr{L} : C_{\mathbb{R}}(X) \to \mathbb{R}$ be a positive linear functional. We say that $\mathscr{L}$ has a *compact support* (in X) if there exists a compact set $K \subset X$ such that $\mathscr{L}(\phi) = 0$ for any $\phi \in C_{\mathbb{R}}(X)$ such that $\phi = 0$ on K.

We prove

THEOREM 2.5. *Let X be a normal topological space and let $\mathscr{S} \subset C_{\mathbb{R}}(X)$ be a wedge. Assume that any positive linear functional on $C_{\mathbb{R}}(X)$ has a compact support. If $x \in X$ is a fixed point, then for any $\phi \in C_{\mathbb{R}}(X)$ we have $\varPhi_{x,\mathscr{S}}(\phi) = -\infty$ or*

$$\varPhi_{x,\mathscr{S}}(\phi) = \inf\Big\{\int \phi\, d\mu : \mu \in J_{x,\mathscr{S}}\Big\}.$$

According to [6], on any locally compact σ-compact Hausdorff space any positive linear functional has a compact support. Hence, Theorem 2.5 is a generalization of the result of Gogus, Perkins, and Poletsky.

We also need the following version of the Riesz representation theorem.

PROPOSITION 2.6. *Let X be a normal topological space and let $\mathscr{L}$: $C_{\mathbb{R}}(X) \to \mathbb{R}$ be a positive linear functional with compact support. Then there*

([1]) The author is grateful to Piotr Niemiec and Jan Stochel for the references on linear functionals.

exists a Borel finite measure μ with support in X such that

$$\mathscr{L}(\phi) = \int \phi(x)\, d\mu(x) \quad \textit{for any } \phi \in C_{\mathbb{R}}(X).$$

Proof. Assume that $K \subset X$ is a compact set such that $\mathscr{L}(\phi) = 0$ whenever $\phi \in C_{\mathbb{R}}(X)$ with $\phi = 0$ on K. Let us define a positive linear operator $\widetilde{\mathscr{L}} : C_{\mathbb{R}}(K) \to \mathbb{R}$. Fix $\phi \in C_{\mathbb{R}}(K)$. From the normality of X and the Tietze extension theorem there exists $\widetilde{\phi} \in C_{\mathbb{R}}(X)$ such that $\widetilde{\phi} = \phi$ on K. We set $\widetilde{\mathscr{L}}(\phi) = \mathscr{L}(\widetilde{\phi})$. Now we use the classical Riesz representation theorem for the operator $\widetilde{\mathscr{L}}$ and get μ. ∎

Proof of Theorem 2.5. From Propositions 2.2 and 2.6 we get

$$\Phi_{x,\mathscr{S}}(\phi) = \min\Big\{\int \phi\, d\mu : \mu \text{ is a Borel finite measure with compact support in } X \text{ such that } \int \psi\, d\mu \ge \Phi_{x,\mathscr{S}}(\psi) \text{ for any } \psi \in C_{\mathbb{R}}(X)\Big\}. \tag{2.2}$$

Hence,

$$\Phi_{x,\mathscr{S}}(\phi) \ge \inf\Big\{\int \phi\, d\mu : \mu \text{ is a Borel finite measure with compact support in } X \text{ such that } \int \psi\, d\mu \ge \psi(x) \text{ for any } \psi \in \mathscr{S}\Big\}. \tag{2.3}$$

The inequality "$\le$" is immediate. ∎

Let us show an example of a set X as in Proposition 2.6.

THEOREM 2.7. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $K \subset \partial\Omega$ be a compact set. Then any positive linear functional $\mathscr{L}$ defined on $C_{\mathbb{R}}(X)$ has a compact support, where $X = \Omega \cup K$.*

Proof. Take sequences R_m, r_m such that $R_m > r_m > R_{m+1}$ and $R_m \to 0$ (e.g. $R_m = 1/3^m$ and $r_m = 2/3^{m+1}$). Consider functions $\chi_m \in C^\infty(\mathbb{R})$ such that $0 \le \chi_m \le 1$ having the following properties:

$$\chi_1(t) = \begin{cases} 1, & t \ge R_1, \\ 0, & t \le r_1, \end{cases}$$

and for any $m \ge 2$,

$$\chi_m(t) = \begin{cases} 1 - \sum_{j=1}^{m-1} \chi_j(t), & t \ge R_m, \\ 0, & t \le r_m. \end{cases}$$

Note that $\sum_{m=1}^{\infty} \chi_m(t) = 1$ for $t > 0$. Moreover, $\chi_m(t) = 0$ for $t \ge R_{m-1}$ and $t \le r_m$, $m > 1$.

We set $\widetilde{\chi}_m(x) = \chi_m(\operatorname{dist}(x, K))$, $x \in \mathbb{R}^n$. It is easy to see that $\widetilde{\chi}_m$ is in $C(\mathbb{R}^n)$. Define $A_1 = \{x \in \Omega : \operatorname{dist}(x, K) \ge r_1\}$ and $A_m = \{x \in \Omega : r_m \le \operatorname{dist}(x, K) \le R_{m-1}\}$, $m \ge 2$. Note that A_m, $m \ge 2$, are (relatively) closed sets in Ω and that $\widetilde{\chi}_m = 0$ for $x \in \mathbb{R}^n \setminus A_m$. Moreover, $\bigcup_{m=1}^{\infty} A_m = \Omega$. We want to show that for any $m \ge 1$ there exists a compact set $K_m \subset A_m$

with the following property: $\mathscr{L}(\phi)=0$ whenever $\phi\in C_{\mathbb{R}}(X)$, $\phi\geq 0$, and $\phi=0$ on $(X\setminus A_m)\cup K_m$.

Let us show that for any $m\geq 1$ there exists a $j=j(m)$ such that $\mathscr{L}(\phi)=0$ whenever $\phi\in C_{\mathbb{R}}(X)$, $\phi\geq 0$, and $\phi=0$ on $(\Omega\setminus A_m)\cup K_{mj}$, where

$$K_{mj}=\{x\in A_m:\operatorname{dist}(x,\mathbb{R}^n\setminus\Omega)\geq 1/j,\ \|x\|\leq j\}.$$

Note that K_{mj} are compact sets. We also have $\bigcup_{j=1}^{\infty}K_{mj}=A_m$ for any $m\geq 1$.

Assume for a while that for any $j\geq 1$ there exists a $\phi_j\in C_{\mathbb{R}}(X)$ such that $\phi_j\geq 0$ on X, $\phi_j=0$ on $(X\setminus A_m)\cup K_{mj}$, and $\mathscr{L}(\phi_j)\neq 0$. Without loss of generality we may assume that $\mathscr{L}(\phi_j)=1$. Consider $\phi=\sum_{j=1}^{\infty}\phi_j$. Note that $\phi\in C_{\mathbb{R}}(X)$. Moreover, $\mathscr{L}(\phi)\geq\sum_{j=1}^{N}\mathscr{L}(\phi_j)=N$ for any $N\in\mathbb{N}$. Hence, $\mathscr{L}(\phi)=+\infty$, a contradiction.

Now consider the set

$$L=\bigcup_{j=1}^{\infty}K_{mj(m)}\cup K.$$

Note that L is compact. It suffices to show that $\mathscr{L}(\phi)=0$ for any $\phi\in C_{\mathbb{R}}(X)$ such that $\phi=0$ on L. First let us prove this for $\phi\geq 0$. So, fix a $\phi\in C_{\mathbb{R}}(X)$ such that $\phi\geq 0$ and $\phi=0$ on L. Fix an $\epsilon>0$. Since $\phi=0$ on K, there exists a $\delta>0$ such that $\phi<\epsilon$ on $\{z\in\Omega:\operatorname{dist}(z,K)\leq\delta\}$. We have

$$\phi=\sum_{m=1}^{\infty}\phi\widetilde{\chi}_m=\sum_{m=1}^{m_0}\phi\widetilde{\chi}_m+\sum_{m=m_0+1}^{\infty}\phi\widetilde{\chi}_m.$$

Further, $\phi\widetilde{\chi}_m=0$ on $(X\setminus A_m)\cup K_{mj(m)}$. Hence, $\mathscr{L}(\phi\widetilde{\chi}_m)=0$. Set $\tilde{\phi}=\sum_{m=m_0+1}^{\infty}\phi\widetilde{\chi}_m$. For sufficiently large m_0 we have $\tilde{\phi}=0$ on $\{z\in\Omega:\operatorname{dist}(z,K)>\delta\}$. Therefore, $0\leq\tilde{\phi}\leq\epsilon$ on Ω. So

$$\mathscr{L}(\phi)=\sum_{m=1}^{m_0}\mathscr{L}(\phi\widetilde{\chi}_m)+\mathscr{L}(\tilde{\phi})=\mathscr{L}(\tilde{\phi})\leq\epsilon\mathscr{L}(1).$$

Since $\epsilon>0$ was arbitrary, we get $\mathscr{L}(\phi)=0$.

If ϕ is not positive, take $\phi=\phi_+-\phi_-$, where $\phi_+=\max\{\phi,0\}$ and $\phi_-=\phi_+-\phi$. Then $\mathscr{L}(\phi)=\mathscr{L}(\phi_+)-\mathscr{L}(\phi_-)=0$. ∎

Corollary 2.8. *Let $\Omega\subset\mathbb{R}^n$ be a domain and let $\zeta\in\partial\Omega$ be a boundary point. Assume that $\mathscr{S}\subset C_{\mathbb{R}}(\Omega\cup\{\zeta\})$ is a wedge. Then for any $\phi\in C_{\mathbb{R}}(\Omega\cup\{\zeta\})$ we have $\Phi_{\zeta,\mathscr{S}}(\phi)=-\infty$ or*

$$\Phi_{\zeta,\mathscr{S}}(\phi)=\inf\Big\{\int\phi\,d\mu:\mu\in J_{\zeta,\mathscr{S}}\Big\}.$$

In particular, if $J_{\zeta,\mathscr{S}}=\{\delta_\zeta\}$ then $\Phi_{\zeta,\mathscr{S}}(\phi)=\phi(\zeta)$ for any $\phi\in C_{\mathbb{R}}(\Omega\cup\{\zeta\})$.

Proof of Theorem 1.1. The implication (3)⇒(4) follows from Corollary 2.8. Indeed, consider a wedge $\mathscr{S} = \{\Re f : f \in \mathscr{A}(D \cup \{\zeta\})\} \subset C_{\mathbb{R}}(D \cup \{\zeta\})$. Now, fix a compact set $K \subset D$ and a continuous function ϕ such that $\phi \le 0$ on D, $\phi(\zeta) = 0$, and $\phi = -1$ on K. Then we get a sequence $g_N \in \mathscr{A}(D \cup \{\zeta\})$ such that $\Re g_N \le 0$ on D, $\Re g_N \le -1$ on K, and $\Re g_N(\zeta) \to 0$ when $N \to \infty$. Set $f_N = e^{g_N}$.

It remains to show (4)⇒(1). We use the method of [9].

Set $\epsilon_N = 1/4^N$. Then there is a sequence of functions $f_N \in \mathscr{A}(D \cup \{\zeta\})$, $N \ge 1$, such that $f_N(\zeta) \to 1$ as $N \to \infty$ and $|f_N| < 1$ on D, $N \ge 1$. Taking a subsequence of $\{f_N\}_{N \ge 1}$ if necessary, we may assume that $\zeta \in U_N$, where $U_N = \{z \in D \cup \{\zeta\} : |f_N(z) - 1| < \epsilon_N\}$. Set

$$h(z) = \sum_{N=1}^{\infty} \frac{1}{2^N} \cdot \frac{1 + \epsilon_N + f_N(z)}{1 + \epsilon_N - f_N(z)}. \tag{2.4}$$

Note that $\zeta \in U_N$ and that h is a well-defined holomorphic function in D. Indeed, fix a compact set $K \Subset D$. Then there exist an $\eta \in (0,1)$ and a $k \in \mathbb{N}$ such that $\|f_N\|_K \le 1 - \eta$ for any $N \ge k$. Hence, $\left|\frac{1+\epsilon_N + f_N(z)}{1+\epsilon_N - f_N(z)}\right| \le \frac{3-\eta}{\eta}$ for $z \in K$ and $N \ge k$, so the series in (2.4) is locally uniformly convergent.

It is easy to check that for any $\epsilon > 0$,

$$\Re\left(\frac{1+z}{1-z}\right) > \frac{1}{\epsilon} \ \Leftrightarrow\ \left|z - \frac{1}{1+\epsilon}\right| < \frac{\epsilon}{1+\epsilon}.$$

We have

$$\Re h(z) \ge \frac{1}{2^N} \cdot \Re\left(\frac{1 + \frac{f_N(z)}{1+\epsilon_N}}{1 - \frac{f_N(z)}{1+\epsilon_N}}\right) \ge 2^N \quad \text{for } z \in U_N.$$

Set $F = \frac{h-1}{h+1}$. Then $|F - 1| = \frac{2}{|h+1|} \le \frac{2}{|\Re h + 1|} \le \frac{2}{2^N + 1}$ on U_N. Hence F is in $\mathscr{A}(D \cup \{\zeta\})$ and $F(\zeta) = 1$. ■

Proof of Corollary 1.2. Fix a $\zeta \in \partial D$ and assume that μ is a representing measure at ζ. If $c := \mu(\{\zeta\}) \in [0,1)$, then without loss of generality we may assume that $\mu(\{\zeta\}) = 0$. Indeed, it suffices to replace μ by $\widetilde{\mu} = \frac{\mu - c\delta_\zeta}{1-c}$. Then

$$f_j(\zeta) = \int_D f_j(z)\, d\mu(z), \qquad g_j(\zeta) = \int_D g_j(z)\, d\mu(z), \qquad h_i(\zeta) = \int_D h_i(z)\, d\mu(z).$$

By Jensen's inequality we have

$$\phi_j(|f_j(\zeta) + \overline{g_j}(\zeta)|) \le \int_D \phi_j(|f_j(z) + \overline{g_j}(z)|)\, d\mu(z),$$
$$\psi_i(\Re h_i(\zeta)) \le \int_D \psi_i(\Re h_i(z))\, d\mu(z).$$

Hence,

$$(2.5)\qquad 0=\sum_{j=1}^m \phi_j(|f_j(\zeta)+\overline{g_j}(\zeta)|)+\sum_{i=1}^k \psi_i(\Re(h_i(\zeta)))$$
$$\le \int_D\Big(\sum_{j=1}^m \phi_j(|f_j(z)+\overline{g_j}(z)|)+\sum_{i=1}^k \psi_i(\Re(h_i(z)))\Big)\,d\mu(z)<0,$$

a contradiction. So, we get $c=1$. ∎

Example 2.9. Any boundary point of the domain

$$D=\{(z_1,z_2,z_3)\in\mathbb{C}^3: 2|z_3+\overline{z}_2^2|+|z_3|^2+|z_1^4-z_2^3|<1\}$$

is a weak peak point. Note that for any $(z_1,z_2,z_3)\in D$ and any $t\in[0,1]$ we have $(t^3z_1,t^4z_2,t^8z_3)\in D$. Therefore, D is indeed a domain.

3. Peak points on Reinhardt domains. Let us start with the following very special, however important case.

Proposition 3.1. *Let $\alpha_1,\dots,\alpha_n\ge 0$ be any numbers and let*

$$D=\{z\in\mathbb{C}^n: |z_1|^{\alpha_1}\cdot\ldots\cdot|z_n|^{\alpha_n}<1\}.$$

Then any boundary point of D is a weak peak point.

Proof. Without loss of generality we may assume that $\alpha_1,\dots,\alpha_n>0$. Fix a boundary point $\zeta\in\partial D$. We are going to use condition (5) of Theorem 1.1. Fix an $R>\|\zeta\|$. We want to show that there exists a sequence of polynomials f_N such that

- $\|f_N\|_{D\cap\mathbb{D}_R^n}\le 1$ for any $N\ge 1$;
- $f_N(\zeta)\to 1$ when $N\to\infty$;
- for any compact set $K\Subset D\cap\mathbb{D}_R^n$ there exist an $\eta\in(0,1)$ and a $k\in\mathbb{N}$ such that $\|f_N\|_K\le 1-\eta$ for any $N\ge k$.

Fix a compact set $K\subset D\cap\mathbb{D}_R^n$ and an $\epsilon>0$. There exist $\beta_1,\dots,\beta_n\in\mathbb{Z}$ and a $q\in\mathbb{N}$, $q\ge 2$, such that $\operatorname{sign}\beta_j=\operatorname{sign}\alpha_j$ and

$$|q\alpha_j-\beta_j|\le\epsilon\quad\text{ for any } j=1,\dots,n.$$

Set $f_\epsilon(z)=z_1^{\beta_1}\cdot\ldots\cdot z_n^{\beta_n}$, $z\in\mathbb{C}^n$. Let us estimate $\sup_{z\in D\cap\mathbb{D}^n}|f_\epsilon(z)|$.

First we want to show that there exists a $\delta\in(0,1)$ such that for any $z\in D\cap\mathbb{D}_R^n$ with $\min\{|z_1|,\dots,|z_n|\}\le\delta$ we have

$$|z_1|^{\beta_1}\cdot\ldots\cdot|z_n|^{\beta_n}\le(1/2)^q.$$

It suffices to have

$$(3.1)\qquad \delta^{\min\{\beta_1,\dots,\beta_n\}}R^{\beta_1+\dots+\beta_n-\min\{\beta_1,\dots,\beta_n\}}\le(1/2)^q.$$

We see that for small enough $\delta>0$ the inequality (3.1) holds.

If $|z_1|,\dots,|z_n|\ge\delta$ then

$$(3.2)\qquad \begin{aligned}|f_\epsilon(z)| &= (|z_1|^{\alpha_1}\cdot\ldots\cdot|z_n|^{\alpha_n})^q|z_1|^{\beta_1-q\alpha_1}\cdot\ldots\cdot|z_n|^{\beta_n-q\alpha_n}\\ &\le (|z_1|^{\alpha_1}\cdot\ldots\cdot|z_n|^{\alpha_n})^q(\max\{R,1/\delta\})^{n\epsilon}.\end{aligned}$$

We have

$$|f_\epsilon(\zeta)| = |\zeta_1|^{\beta_1-\alpha_1 q}\cdot\ldots\cdot|\zeta_n|^{\beta_n-\alpha_n q}.$$

Hence, $|f_\epsilon(\zeta)|\to 1$ when $\epsilon\to 0$. We see that condition (5) of Theorem 1.1 is fulfilled. So, ζ is a weak peak point of D. ∎

A domain $D\subset\mathbb{C}^n$ is called *Reinhardt* if $(\lambda_1 z_1,\dots,\lambda_n z_n)\in D$ for all points $z=(z_1,\dots,z_n)\in D$ and any $|\lambda_1|=\dots=|\lambda_n|=1$. Let

$$V_j=\{z\in\mathbb{C}^n : z_j=0\},\qquad j=1,\dots,n,$$

and $V=\bigcup_{j=1}^n V_j$. The following result is well-known (see e.g. [11]).

Theorem 3.2. *Let $D\subset\mathbb{C}^n$ be a Reinhardt domain. Then D is pseudoconvex if and only if the set*

$$\log D := \{x\in\mathbb{R}^n : (e^{x_1},\dots,e^{x_n})\in D\}$$

is convex and for any $j\in\{1,\dots,n\}$,

$$(3.3)\qquad \text{if } D\cap V_j\ne\emptyset \text{ and } (z',z_j,z'')\in D \text{ then } (z',\lambda z_j,z'')\in D \text{ for any } \lambda\in\overline{\mathbb{D}}.$$

In particular, if D is a pseudoconvex domain and $D\cap V_j\ne\emptyset$ for some $j\in\{1,\dots,n\}$, then $\pi_j(D)=D\cap V_j$ is a pseudoconvex domain in $\mathbb{C}^{n-1}$ (after natural identification), where

$$\pi_j:\mathbb{C}^n\ni z\mapsto(z_1,\dots,z_{j-1},0,z_{j+1},\dots,z_n)\in V_j.$$

For Reinhardt domains, which satisfy Fu's condition (see (3.4) below), we have the following result (cf. [4], [12]).

Theorem 3.3. *Let $D\subset\mathbb{C}^n$ be a pseudoconvex Reinhardt domain such that for any $j\in\{1,\dots,n\}$,*

$$(3.4)\qquad \text{if } \overline{D}\cap V_j\ne\emptyset \text{ then } D\cap V_j\ne\emptyset.$$

Then any $\zeta\in\partial D$ is a weak peak point.

Proof. Note that $0\notin\partial D$. For if $0\in\partial D$ then from (3.3) and (3.4) we deduce that D is a complete Reinhardt domain, a contradiction.

Fix a boundary point $\zeta\in\partial D$. First assume that $\zeta_1\cdot\ldots\cdot\zeta_n\ne 0$. Without loss of generality, we may assume that for some $m\in\{0,1,\dots,n\}$ we have

- $D\cap V_j\ne\emptyset$, $j=1,\dots,m$;
- $\overline{D}\cap V_j=\emptyset$, $j=m+1,\dots,n$.

These conditions imply that

- if $z\in D$ then $(\lambda_1 z_1,\dots,\lambda_m z_m,z_{m+1},\dots,z_n)\in D$ for any $\lambda_1,\dots,\lambda_m\in\overline{\mathbb{D}}$;

- there exists a $\delta_0 \in (0,1)$ such that for any $z \in D$ we have $|z_j| > \delta_0$ for $j = m+1, \dots, n$.

Since $\log D$ is convex, there exist $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that

$$\alpha_1 x_1 + \dots + \alpha_n x_n < \alpha_1 \log|\zeta_1| + \dots + \alpha_n \log|\zeta_n| \quad \text{for any } x \in \log D.$$

Note that $\alpha_1, \dots, \alpha_m \geq 0$. We have $D \subset G$ and $\zeta \in \partial G$, where

$$G := \{z \in \mathbb{C}^n : |z_1|^{\alpha_1} \cdot \ldots \cdot |z_n|^{\alpha_n} < |\zeta_1|^{\alpha_1} \cdot \ldots \cdot |\zeta_n|^{\alpha_n}\} \cap \{z \in \mathbb{C}^n : |z_j| > \delta_0,\ j = m+1, \dots, n\}. \tag{3.5}$$

Hence, it suffices to show that ζ is a weak peak point of any domain $\widetilde{G}$ such that $D \subset \widetilde{G}$ and $\zeta \in \partial\widetilde{G}$. Without loss of generality we may assume that $\alpha_j \neq 0$ for any $j = 1, \dots, n$. Moreover, we may assume that $|\zeta_1|^{\alpha_1} \cdot \ldots \cdot |\zeta_n|^{\alpha_n} = 1$. Indeed, consider

$$\widetilde{G} := \{z \in \mathbb{C}^n : |z_1|^{\alpha_1} \cdot \ldots \cdot |z_n|^{\alpha_n} < |\zeta_1|^{\alpha_1} \cdot \ldots \cdot |\zeta_n|^{\alpha_n}\} \cap \{z \in \mathbb{C}^n : |z_j/\zeta_j| > \delta_1,\ j = m+1, \dots, n\}, \tag{3.6}$$

where $\delta_1 > 0$ is sufficiently small, such that $G \subset \widetilde{G}$. Note that $\zeta \in \partial\widetilde{G}$. Now, we use the linear mapping

$$L : \mathbb{C}^n \ni (z_1, \dots, z_n) \mapsto (z_1/\zeta_1, \dots, z_n/\zeta_n) \in \mathbb{C}^n$$

and proceed to show the existence of a peak function for the domain $L(\widetilde{G})$.

Let us show that we may assume $\alpha_1, \dots, \alpha_n > 0$, and hence the result follows from Proposition 3.1. Indeed, consider the domain

$$H = \{z \in \mathbb{C}^n : |z_1|^{|\alpha_1|} \cdot \ldots \cdot |z_n|^{|\alpha_n|} < 1\}.$$

Note that $(\zeta_1^{\epsilon_1}, \dots, \zeta_n^{\epsilon_n})$, where $\epsilon_j = \operatorname{sign}\alpha_j$, $j = 1, \dots, n$, is a boundary point of H. If f is a peak function for H, then $f \circ \psi$ is a peak function for G, where

$$\psi : \mathbb{C}^m \times (\mathbb{C}_*)^{n-m} \ni z \mapsto (z_1, \dots, z_m, z_{m+1}^{\epsilon_{m+1}}, \dots, z_n^{\epsilon_n}) \in \mathbb{C}^n.$$

Now assume that $\zeta \in \partial D$ and that $\zeta_1 \cdot \ldots \cdot \zeta_n = 0$. Then there are $1 \leq j_1 < \dots < j_k \leq m$ such that $\zeta_{j_s} = 0$ for $s = 1, \dots, k$, and $z_\ell \neq 0$ for $\ell \notin \{j_1, \dots, j_k\}$. Consider the projection

$$\pi : \mathbb{C}^n \ni (z_1, \dots, z_n) \mapsto (z_1, \dots, z_{j_1-1}, z_{j_1+1}, \dots, z_{j_k-1}, z_{j_k+1}, \dots, z_n) \in \mathbb{C}^{n-k}$$

and the domain $\widetilde{D} = \pi(D)$. Note that $\widetilde{D}$ is pseudoconvex and $\pi(\zeta) \in \partial\widetilde{D}$. Hence, $\pi(\zeta)$ is a weak peak point for $\partial\widetilde{D}$, and therefore ζ is a weak peak point for ∂D. ∎

Example 3.4. Using Sibony's ideas we show that there exist a domain $D \subset \mathbb{C}^2$ and a boundary point such that there exists a weak peak function, but peak functions do not exist.

Let $D \Subset \mathbb{C}^N$ be a bounded domain and let $\phi_n : \mathbb{D} \to D$ be a sequence of holomorphic mappings such that $\phi_n(0) \to p \in \partial D$. Assume that $f \in A(\overline{D})$

is such that $f(p) = 1$ and $|f| < 1$ on D. Then there exists a subsequence $\{n_k\}$ such that $f(\phi_{n_k}) \to 1$ locally uniformly on $\mathbb{D}$. In particular, if $\phi_n \to \phi$ locally uniformly on $\mathbb{D}$ then $f(\phi) \equiv 1$. So, $f(w) = 1$ for any $w \in \overline{\phi(\mathbb{D})}$.

Let us now consider a special case. Fix an irrational number $\alpha > 0$. Let $D \subset \mathbb{C}^2$ be a domain and let $(z_0, w_0) \in \partial D$. Assume that there exists a bounded neighborhood $U \subset \mathbb{C}^2$ of (z_0, w_0) such that

$$D \cap U = \{(z, w) \in \mathbb{C}^2 : |z| \cdot |w|^\alpha < 1\} \cap U.$$

We want to show that there does not exist a holomorphic function f in $\mathcal{O}(D) \cap C(\overline{D} \cap U)$ such that $|f| < 1$ on D and $f(z_0, w_0) = 1$.

Indeed, assume that such a function exists. Then there also exists a neighborhood $V \subset \mathbb{C}$ of the origin such that $(z_0 e^{-\alpha\lambda}, w_0 e^\lambda) \in U$ whenever $\lambda \in V$. For sufficiently large $n \in \mathbb{N}$ consider the functions

$$\psi_n(\lambda) = f(z_0 e^{-\alpha\lambda}, (1 - 1/n) w_0 e^\lambda).$$

Note that $\psi_n : V \to D \cap U$ is a holomorphic mapping. Hence, a subsequence ψ_{n_k} tends locally uniformly on V to a holomorphic mapping $\psi : V \to \mathbb{C}^2$. It is easy to see (use continuity of f) that

$$\psi(\lambda) = f(z_0 e^{-\alpha\lambda}, w_0 e^\lambda).$$

So, $\psi : V \to \overline{D \cap U}$ is a holomorphic mapping such that $|\psi| \le 1$ and $\psi(0) = 1$. Hence, $\psi \equiv 1$. Since α is irrational, we deduce that $\{(z_0 e^{-\alpha\lambda}, w_0 e^\lambda) : \lambda \in V\}$ is dense in a neighborhood of (z_0, w_0). From the continuity of f we get $f = 1$ on a relatively open subset ∂D containing (z_0, w_0). Then $f(z_0, \lambda w_0) = 1$ on the open subset of the unit circle containing 1. Hence, $f(z_0, \lambda w_0) = 1$ everywhere, a contradiction.

QUESTION 3.5. *Do there exist a bounded pseudoconvex domain $D \subset \mathbb{C}^n$ with smooth boundary and a boundary point $\zeta \in \partial D$ such that ζ is not a weak peak point?*

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References

[1] C. D. Aliprantis and R. Tourky, *Cones and Duality*, Grad. Stud. Math. 84, Amer. Math. Soc., Providence, RI, 2007.

[2] P. Berti and P. Rigo, *Integral representation of linear functionals on spaces of unbounded functions*, Proc. Amer. Math. Soc. 128 (2000), 3251–3258.

[3] D. A. Edwards, *Choquet boundary theory for certain spaces of lower semicontinuous functions*, in: Function Algebras (New Orleans, LA, 1965), Scott-Foresman, 1966, 300–309.

[4] S. Fu, *On completeness of invariant metrics of Reinhardt domains*, Arch. Math. (Basel) 63 (1994), 166–172.

[5] T. W. Gamelin, *Uniform Algebras*, Prentice-Hall, 1969.
[6] N. G. Gogus, T. L. Perkins and E. A. Poletsky, *Non-compact versions of Edwards' Theorem*, Positivity 17 (2013), 459–473.
[7] E. Hewitt, *Linear functionals on spaces of continuous functions*, Fund. Math. 37 (1950), 161–189.
[8] Ł. Kosiński and W. Zwonek, *Proper holomorphic mappings vs. peak points and Shilov boundary*, Ann. Polon. Math. 107 (2013), 97–108.
[9] P. Pflug, *About the Carathéodory completeness of all Reinhardt domains*, in: Functional Analysis, Holomorphy and Approximation Theory II, G. I. Zapata (ed.), North-Holland Math. Stud. 86, North-Holland, 1984, 331–337.
[10] P. Ressel, *Integral representations on convex semigroups*, Math. Scand. 61 (1987), 93–111.
[11] V. S. Vladimirov, *Methods of the Theory of Complex Variables*, MIT Press, Cambridge, MA, 1966.
[12] W. Zwonek, *On Carathéodory completeness of pseudoconvex Reinhardt domains*, Proc. Amer. Math. Soc. 128 (2000), 857–864.

Armen Edigarian
Faculty of Mathematics and Computer Science
Jagiellonian University
Łojasiewicza 6
30-348 Kraków, Poland
E-mail: armen.edigarian@uj.edu.pl

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