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Alpha-invariant of toric line bundles

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Abstract. We generalize the work of Jian Song by computing the α-invariant of any (nef and big) toric line bundle in terms of the associated polytope. We use the analytic version of the computation of the log canonical threshold of monomial ideals to give the log canonical threshold of any non-negatively curved singular hermitian metric on the line bundle, and deduce the α-invariant from this.

Introduction. The α-invariant of a line bundle L on a complex manifold X is an invariant measuring the singularities of the non-negatively curved singular hermitian metrics on L. It was introduced by Tian in the case of the anticanonical bundle on a Fano manifold. Tian [Tia87] showed that if the α-invariant of the anticanonical bundle is strictly greater than $n/(n+1)$, then the Fano manifold admits a Kähler–Einstein metric.

The Yau–Tian–Donaldson conjecture asserts in general that X admits an extremal metric in $c_1(L)$ if and only if the line bundle L is K-stable. It was proved in [CDS15a, CDS15b, CDS15c, Tia12] that it holds when L is the anticanonical bundle. In particular (as also shown in [OS12]), if the α-invariant of the anticanonical bundle is greater than $n/(n+1)$, then the anticanonical bundle is K-stable. Dervan [Der13] gave a similar condition of K-stability for a general line bundle, involving again its α-invariant. This is one motivation to compute explicitly the α-invariants of line bundles when possible.

In [CS08], Chel'tsov and Shramov computed for example the α-invariant of the anticanonical bundle for many Fano manifolds of dimension three. In higher dimensions, Song [Son05] proved a formula giving the α-invariant of the anticanonical bundle on a toric Fano manifold in terms of its polytope. The only toric manifolds satisfying Tian's criterion are the symmetric toric manifolds. Batyrev and Selivanova [BS99] proved first that the α-invariant of

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those manifolds was one, so that they admit a Kähler–Einstein metric. Wang and Zhu [WZ04] fully settled the question of the existence of Kähler–Einstein metrics on toric Fano manifolds, and an illustration that Tian's criterion is only a sufficient condition can be found in the toric world [NP11].

The α-invariant of a line bundle L is strongly related to the log canonical thresholds (lct) of metrics on L. The log canonical threshold was initially an algebraic invariant defined for ideal sheaves, but it was shown to coincide with the complex singularity exponent, and Demailly defines the log canonical threshold of any non-negatively curved singular hermitian metric on a line bundle (see [CS08] for example).

One of the main examples of computation of log canonical threshold is in the case of monomial ideals. Howald [How01] carried out the computation of the lct of such an ideal in terms of its Newton polygon. One can find in Guenancia [Gue12] an analytic proof of this result, generalized to compute the lct of an ideal generated by a "toric" psh function on a neighborhood of $0 \in \mathbb{C}^n$, i.e. a function invariant under rotation in each coordinate.

Since the only smooth affine toric manifolds without torus factor are isomorphic to $\mathbb{C}^n$, the computation of Guenancia in fact gives the log canonical threshold of any invariant metric on an affine smooth toric manifold, as we explain in Section 2.

In this note, we give a formula for the α-invariant of any line bundle L on a compact smooth toric manifold in terms of its polytope. We also compute the log canonical threshold of any invariant non-negatively curved singular metric on L.

After this article was accepted, the author was informed that other authors computed the same α-invariants using different methods ([LSY15], [Amb14]).

1. Line bundles on smooth toric manifolds

1.1. Toric manifolds. We recall some basic facts about toric varieties (see [Ful93], [Oda88], [CLS11]).

Let $T = (\mathbb{C}^*)^n$ be an algebraic torus. Denote its group of characters by M, which is isomorphic to $\mathbb{Z}^n$ through the choice of a basis, and let $M_{\mathbb{R}} := M \otimes \mathbb{R} \simeq \mathbb{R}^n$. The dual N of M consists of the one-parameter subgroups of T, and we also let $N_{\mathbb{R}} := N \otimes \mathbb{R} \simeq \mathbb{R}^n$.

We denote by $T_c \simeq (S^1)^n$ the compact torus in T.

Considering only cones for the toric setting, we will call $\sigma \subset N_{\mathbb{R}}$ a *cone* if σ is a convex cone generated by a finite set of elements of N. The *dual cone* $\sigma^\vee$ is defined as

$$\sigma^\vee = \{x \in M_{\mathbb{R}} \mid \langle x, y\rangle \geq 0, \forall y \in \sigma\}.$$

A *fan* Σ consists of a finite collection of cones $\sigma \subset N_{\mathbb{R}}$ such that every cone is *strongly convex* (i.e. $\{0\}$ is a face of σ), the faces of cones in Σ are in Σ and the intersection of two cones in Σ is a union of faces of both. The *support* of Σ is $|\Sigma| := \bigcup_{\sigma\in\Sigma} \sigma \subset N_{\mathbb{R}}$.

Recall that a fan Σ in $N_{\mathbb{R}}$ determines a toric variety X_Σ, that is, a normal T-variety with an open and dense orbit isomorphic to T, and every toric variety is obtained this way.

By the orbit-cone correspondence [CLS11, Theorem 3.2.6], a maximal cone σ of Σ corresponds to a fixed point z_σ in X_Σ. Also, a one-dimensional cone ρ in Σ corresponds to a prime invariant divisor D_ρ of X_Σ, and these divisors generate the group of Weyl divisors of X_Σ. Let ρ be such a cone; then we denote by u_ρ the primitive vector in N generating this ray. We will denote by $\Sigma(r)$ the set of r-dimensional cones in Σ.

Many properties of X_Σ can be read off from the fan. For example, X_Σ is smooth if and only if every cone in the fan Σ is generated by part of a basis of N. We will call a cone *smooth* if it satisfies this condition. The variety X_Σ is complete if and only if $|\Sigma| = N_{\mathbb{R}}$.

In the following we will assume in general that either $|\Sigma| = N_{\mathbb{R}}$ or Σ is given by a strongly convex, full-dimensional cone σ and its faces, in which case we will denote by X_σ the corresponding (affine) toric variety.

1.2. Line bundles. Recall that a line bundle L on a G-variety X is called *linearized* if there is an action of G on L such that for any $g \in G$ and $x \in X$, g sends the fiber L_x to the fiber $L_{g\cdot x}$ and the map defined this way between L_x and $L_{g\cdot x}$ is linear.

To a T-linearized line bundle L on X_Σ is associated a set of characters v_σ for $\sigma \in \Sigma(n)$. We define v_σ as the opposite of the character of the action of T on the fiber over the fixed point z_σ.

This defines the *support function* g_L of L, which is a function on the support $|\Sigma|$ of Σ, linear on each cone, which takes integral values at points of N, by $x \mapsto \langle v_\sigma, x\rangle$ for $x \in \sigma$.

Another equivalent data is the Weyl divisor D_L associated to L, which is related to g_L by $D_L = -\sum_\rho g_L(u_\rho)D_\rho$.

If L is effective, then to L is associated a polytope P_L in $M_{\mathbb{R}}$. This polytope can be defined as

$$P_L = \{m \in M_{\mathbb{R}} \mid g_L(x) \le \langle m, x\rangle, \forall x \in |\Sigma|\}.$$

The properties of the line bundle can be read off from the polytope or the support function. In particular, we can associate to each point of $P_L \cap M$ a global section of L, and the collection of these sections form a basis of the space of algebraic sections of L. Recall also the following, where we assume that $|\Sigma| = N_{\mathbb{R}}$.

Proposition 1.1 ([CLS11, Theorem 6.1.7]). *The following are equivalent:*

- *L is nef.*
- *L is generated by global sections.*
- *$\{v_\sigma\}$ is the set of vertices of P_L.*
- *g_L is concave.*

Proposition 1.2 ([CLS11, Lemma 9.3.9]). *L is big iff P_L has non-empty interior.*

Proposition 1.3 ([CLS11, Lemma 6.1.13]). *The line bundle L is ample iff g_L is concave and $v_\sigma \neq v_{\sigma'}$ whenever $\sigma \neq \sigma' \in \Sigma(n)$.*

Example 1.4. The anticanonical divisor $-K_{X_\Sigma}$ on a toric manifold is given by $-K_{X_\Sigma} = \sum_\rho D_\rho$. It is always big on a toric manifold.

1.3. Non-negatively curved singular metrics on line bundles

1.3.1. *Potential on the torus.* Let L be a T-linearized line bundle on X_Σ.

Recall that any linearized line bundle on $T \simeq (\mathbb{C}^*)^n$ is trivial. Fix an invariant trivialization s of L on T.

Given a hermitian metric h on the line bundle L, we denote by ϕ_h the local potential of h on T, which is the function on T defined by

$$\phi_h(z) := -\ln \|s(z)\|_h.$$

The local potentials of a smooth hermitian metric are smooth. We will work here with singular metrics, whose local potential are *a priori* only in L^1_{loc}. A singular hermitian metric h is said to have *non-negative curvature* (in the sense of currents) if and only if every local potential of h is a psh function.

A T_c-invariant function ϕ on T is determined by a function f on $N_\mathbb{R}$, identified with the Lie algebra of T_c, through the equivariant isomorphism

$$T_c \times N_\mathbb{R} \to T, \qquad ((e^{i\theta_j})_j, (x_j)_j) \mapsto (e^{x_j + i\theta_j})_j.$$

Furthermore, ϕ is psh if and only if f is convex.

So to a non-negatively curved, T_c-invariant metric h on L is associated a convex function f_h, which is the function on $N_\mathbb{R}$ determined by ϕ_h.

1.3.2. *Behavior at infinity of potentials*

Definition 1.5. Let L be a nef line bundle on X_Σ. The function $f_L : x \mapsto -g_L(-x)$ is a convex function on $N_\mathbb{R}$, and it is the potential of a continuous, T_c-invariant, non-negatively curved metric on L called the *Batyrev–Tschinkel metric* (see [Mai00]), which we denote by h_L.

Proposition 1.6. *The map $h \mapsto f_h$ defines a bijection between the singular hermitian T_c-invariant metrics on L with non-negative curvature, and the convex functions on $N_{\mathbb{R}}$, such that there exists a constant C with $f_h \leq f_L + C$ on $N_{\mathbb{R}}$.*

Proof. See also [BB13, Proposition 3.3]. Let h be a singular hermitian T_c-invariant metric on L with non-negative curvature. Write $h = e^{-v}h_L$, and let ω_L be the curvature current of h_L. Then v is a ω_L-psh function on X. In particular, v is bounded from above on X. Denote by u the convex function on $\mathbb{R}^n$ associated to the T_c-invariant function $v|_T$. Then we see that $f_h(x) - f_L(x) = u(x)$ is bounded above on $N_{\mathbb{R}}$.

Conversely, the standard fact that a psh function, which is bounded from above, extends uniquely over an analytic set, allows one to extend $u := f - f_L$ to an ω_L-psh function on the whole of X if f satisfies the condition of the proposition. ∎

2. Log canonical thresholds

2.1. Definition. Let X be a compact complex manifold, and L a line bundle on X. Let h be a singular hermitian metric on L. We recall the definition of the log canonical threshold of h (see the appendix of [CS08]).

Definition 2.1. Let $z \in X$. The complex singularity exponent $c_z(h)$ of h at z is the supremum of the real $c > 0$ such that $e^{-2c\phi}$ is integrable in a neighborhood of z, where ϕ is a local potential of h near z.

Definition 2.2. The *log canonical threshold* $\mathrm{lct}(h)$ of h is defined as

$$\mathrm{lct}(h) = \inf_{z \in X} c_z(h).$$

2.2. Newton body of a function

Definition 2.3. Let σ be a cone. Let f be a function defined on $N_{\mathbb{R}}$. Define the *Newton body* of f on σ as

$$N_\sigma(f) = \{m \in M_{\mathbb{R}} \mid f(x) - \langle m, x\rangle \geq O(1),\ \forall x \in \sigma\}.$$

If $\sigma = N_{\mathbb{R}}$, we will write $N(f)$.

The following properties of the Newton body will be useful.

Proposition 2.4. *For any function f, $N_\sigma(f)$ is convex and $N_\sigma(f) = N_\sigma(f) - \sigma^\vee$. If f is convex then for any $y \in N_{\mathbb{R}}$,*

$$N_\sigma(f) = \{m \in M_{\mathbb{R}} \mid f(t) - \langle m, t\rangle \geq O(1),\ \forall t \in y + \sigma\}.$$

Proof. The first two properties are trivial. Let us briefly prove the last statement.

Let m be in the right-hand set, i.e. $f(t) - \langle m, t\rangle \geq O(1)$, $\forall t \in y + \sigma$. Let $x = t - y \in \sigma$ for $t \in y + \sigma$. By convexity, $f(x + y) \leq \frac{1}{2}(f(2x) + f(2y))$, so we get

$$f(2x) \geq 2f(x+y) - f(2y) = 2f(t) - f(2y).$$

Subtracting $\langle m, 2x\rangle$ gives

$$f(2x) - \langle m, 2x\rangle \geq 2(f(t) - \langle m, t\rangle) + (2\langle m, y\rangle - f(2y)).$$

The right-hand side is the sum of a lower-bounded function of $t \in y + \sigma$ and a constant, so the left-hand side is a lower-bounded function of $x \in \sigma$.

This shows one inclusion; the other is proved by a similar argument. ∎

Given a non-negatively curved T_c-invariant metric h on L, we define the associated convex subset P_h of $M_{\mathbb{R}}$ to be the Newton body of f_h.

PROPOSITION 2.5.

- *For the Batyrev–Tschinkel metric h_L, we recover the polytope P_L.*
- *For any T_c-invariant, non-negatively curved metric h on L, $P_h \subset P_L$.*
- *If h is smooth, then also $P_h = P_L$.*

Proof. For the first statement, observe that $m \in P_L$ if and only if for any cone $\sigma \in \Sigma$, for all $x \in \sigma$, we have $g_L(x) = \langle v_\sigma, x\rangle \leq \langle m, x\rangle$. This inequality is equivalent to $-\langle v_\sigma, x\rangle + \langle m, x\rangle \geq 0$, and since the functions involved are linear, it is satisfied for all $x \in \sigma$ if and only if $-\langle v_\sigma, x\rangle + \langle m, x\rangle$ is bounded below on σ. Since $f_L(-x) = -g_L(x) = -\langle v_\sigma, x\rangle$ for $x \in \sigma$, we see that $m \in P_L$ if and only if for every cone $\sigma \in \Sigma$, the function $f_L(-x) - \langle m, -x\rangle$ is bounded below on σ. Finally, this can be translated as: for every cone $\sigma \in \Sigma$, the function $f_L(y) - \langle m, y\rangle$ is bounded below on $-\sigma$. To conclude the proof, we note that $N(f_L) = \bigcap_\sigma N_{-\sigma}(f_L)$.

The second statement is an easy consequence of the first and Proposition 1.6, since whenever two functions f and g satisfy $f \leq g + C$ for a constant C, we trivially have $N_\sigma(f) \subset N_\sigma(g)$.

For the last statement, note that in this case, $f_h - f_L$ extends to a continuous function on X_Σ, so $f_L - C \leq f_h \leq f_L + C$ for some constant C. The same property of Newton bodies allows one to conclude the proof. ∎

2.3. Integrability condition. The first result on log canonical thresholds on toric varieties was the computation by Howald [How01] in the case of monomial ideals. Guenancia gave an analytic proof of this result, extending the computation to the case of non-algebraic psh functions. The key ingredient in this analytic version is the following integrability condition.

PROPOSITION 2.6 (see [Gue12]). *Let σ be a smooth cone of maximum dimension. Let f be a convex function on $N_{\mathbb{R}}$. Then e^{-f} is integrable on all translates of σ if and only if $0 \in \mathrm{Int}(N_\sigma(f))$.*

This is essentially the result in Guenancia [Gue12] because any smooth affine toric manifold with no torus factor is isomorphic to $\mathbb{C}^n$. However we precisely describe the change of variables used, to apply it later in the compact case.

Proof. Choose a basis of N formed by the generators of the extremal rays of σ, then define S_σ to be the isomorphism from N to $\mathbb{Z}^n$ sending the chosen basis to the canonical basis of $\mathbb{Z}^n$.

Let f be a function on $N_\mathbb{R}$, and g the function on $\mathbb{R}^n$ such that $f = g \circ S_\sigma$. Then from the definition of the Newton body we have $N_\sigma(f) = S_\sigma^*(N_D(g))$, where S_σ^* is the dual isomorphism from $\mathbb{Z}^n$ to M and D is the cone generated by the canonical basis of $\mathbb{Z}^n$.

By a change of variables, e^{-f} is integrable on all translates of σ if and only if $e^{-f\circ S_\sigma^{-1}}$ is integrable on all translates of D. Applying [Gue12, Proposition 1.9] to the concave function $-f \circ S_\sigma^{-1}$ proves that we have integrability if and only if $0 \in \mathrm{Int}(N_D(f \circ S_\sigma^{-1}))$. Using S_σ^*, which is linear, this indeed translates to $0 \in \mathrm{Int}(N_\sigma(f))$.

Note that the statement in [Gue12, Proposition 1.9] only mentions integrability on D, but the equivalence with integrability on all translates is easily derived from Proposition 2.4. ∎

2.4. Lct on an affine smooth toric manifold

PROPOSITION 2.7. *Let σ be a smooth cone of maximum dimension, and X_σ the corresponding smooth affine toric manifold. Let L be a linearized line bundle on X_σ, and h a T_c-invariant metric with non-negative curvature. Then*

$$\mathrm{lct}(h) = \sup\{c > 0 \mid cv_\sigma \in \mathrm{Int}(N_{-\sigma}(cf_h)) - S_\sigma^*(1, \dots, 1)\}.$$

Proof. The change of variables for cones S_σ in the proof of Proposition 2.6 gives (by [CLS11, Theorem 3.3.4]) an equivariant isomorphism between X_σ and $\mathbb{C}^n$, which we again denote by S_σ.

Any linearized line bundle on $\mathbb{C}^n$ is trivial, so L admits a global equivariant trivialization t on X_σ. Note that, at the fixed point z_σ, we have $g \cdot t(z_\sigma) = -v_\sigma(t(z_\sigma))$ by definition of v_σ. Restricting to T and remembering that s is an invariant trivialization of L on T, we deduce that up to renormalization by a constant, $t(z) = v_\sigma(z)s(z)$ on T.

We can now look at the potential ψ of h with respect to the trivialization t, and observe that, on T, if ϕ denotes the potential of h with respect to s on T, we have $\psi(z) = \langle -v_\sigma, \ln|z|\rangle + \phi(z)$.

Let $y \in N_\mathbb{R}$. Using again the isomorphism $T_c \times N_\mathbb{R} \simeq T$, we consider $T_c \times (y - \sigma)$ as a subset of T, and denote by C_y the closure of this set in X_σ. Each set C_y is a neighborhood of z_σ in X_σ, and they form a basis of neighborhoods. Observe that the collection of translates of $-\sigma$ cover $N_\mathbb{R}$, and so the corresponding sets cover X_σ. More precisely, for any point z in X_σ, there is a translate of $-\sigma$ which corresponds to a neighborhood of z.

We first consider the complex singularity exponent of h at z_σ. Suppose $c > 0$ is such that $e^{-2c\psi}$ is integrable in a neighborhood of z_σ. Then it is integrable in a neighborhood C_y. We have

$$\int_{C_y} e^{-2c\psi(z)}\, dz \wedge d\overline{z} = \int_{T_c\times(y-\sigma)} e^{-2c\psi(z)}\, dz \wedge d\overline{z}.$$

Recall that $\psi(z) = \langle -v_\sigma, \ln|z|\rangle + \phi(z)$, and that f is the function on $N_\mathbb{R}$ such that $f(x) = \phi(e^x)$.

Say we have chosen a basis of N or equivalently of M, and we denote by $(x_i)_{i=1,\dots,n}$ the coordinates of $x \in N_\mathbb{R}$ in this basis. This determines local holomorphic coordinates $z_i = e^{x_i+i\theta_i}$ on $T \simeq N_\mathbb{R} \times T_c$. Using the fact that $\frac{dz_i}{z_i} \wedge \frac{d\overline{z_i}}{\overline{z_i}} = dx_i \wedge d\theta_i$, and T_c-invariance, we find that, up to a constant,

$$\int_{C_y} e^{-2c\psi(z)}\, dz \wedge d\overline{z} = \int_{y-\sigma} e^{-2c(f(x)+\langle -v_\sigma,x\rangle)} e^{2\sum_i x_i}\, dx.$$

Since $\sum_i x_i$ is equal to $\langle S^*_\sigma(1,\dots,1), x\rangle$, we conclude by using Proposition 2.6 that the complex singularity exponent $c_{z_\sigma}(h)$ is the supremum of the $c > 0$ such that $0 \in \operatorname{Int}(N_{-\sigma}(2c(f + \langle -v_\sigma,\cdot\rangle) - 2\langle S^*_\sigma(1,\dots,1),\cdot\rangle))$.

To obtain a simpler condition, note that for any function g and positive scalar λ, we have $N_{-\sigma}(\lambda g) = \lambda N_{-\sigma}(g)$, and that if g_2 is a linear function, then $N_{-\sigma}(g + g_2)$ is the Minkowski sum of $N_{-\sigma}(g)$ and $N_{-\sigma}(g_2)$.

So we get $c_{z_\sigma}(h) = \sup\{c > 0 \mid cv_\sigma \in \operatorname{Int}(N_{-\sigma}(cf)) - S^*_\sigma(1,\dots,1)\}$.

Furthermore, for any $c < c_{z_\sigma}(h)$, Proposition 2.6 shows that $e^{-2c\psi}$ is integrable on every C_y for $y \in N_\mathbb{R}$. Observe now that for any point $z \in X_\sigma$, there exists a C_y containing z. So $c_z(h) \geq c_{z_\sigma}(h)$ for any $z \in X_\sigma$. This concludes the proof of the proposition. ∎

2.5. Lct on a compact smooth toric manifold

THEOREM 2.8. *Let X_Σ be a smooth compact toric manifold, L a linearized line bundle on X_Σ and h a T_c-invariant non-negatively curved metric on L. Then*

$$\operatorname{lct}(h) = \sup\{c > 0 \mid cP_L \subset \operatorname{Int}(cP_h + P_{-K_{X_\Sigma}})\}.$$

Proof. The compact manifold X_Σ is covered by the affine toric manifolds X_σ for $\sigma \in \Sigma(n)$. By the definition of log canonical threshold,

$$\operatorname{lct}(h) = \min_{\sigma\in\Sigma(n)} \operatorname{lct}(h|_{Z_\sigma}).$$

Another way to say this is that $\operatorname{lct}(h)$ is the sup of the $c > 0$ such that $c \leq \operatorname{lct}(h|_{X_\sigma})$ for all $\sigma \in \Sigma(n)$.

Now this condition means, by Proposition 2.7, that for all $\sigma \in \Sigma(n)$,

$$cv_\sigma \in \operatorname{Int}\big(N_{-\sigma}(cf_h + \langle -S^*_\sigma(1,\dots,1),\cdot\rangle)\big).$$

By Proposition 2.4, this is equivalent to the condition that for all $\sigma \in \Sigma(n)$,

$$cv_\sigma + \sigma^\vee \subset \mathrm{Int}\big(N_{-\sigma}(cf_h + \langle -S_\sigma^*(1,\dots,1),\cdot\rangle)\big).$$

This is further equivalent to the condition that for all $\sigma \in \Sigma(n)$,

$$\bigcap_{\sigma\in\Sigma(n)} (cv_\sigma + \sigma^\vee) \subset \mathrm{Int}\big(N_{-\sigma}(cf_h + \langle -S_\sigma^*(1,\dots,1),\cdot\rangle)\big).$$

Recall from Proposition 2.5 that $\bigcap_{\sigma\in\Sigma(n)}(v_\sigma + \sigma^\vee) = N(f_L) = P_L$, so that the condition can be written as

$$N(cf_L) \subset \bigcap_{\sigma\in\Sigma(n)} \mathrm{Int}\big(N_{-\sigma}(cf_h + \langle -S_\sigma^*(1,\dots,1),\cdot\rangle)\big) = \mathrm{Int}(N(cf_h + f_{-K_{X_\Sigma}})).$$

Indeed, the support function of the anticanonical bundle is, from Example 1.4,

$$f_{-K_{X_\Sigma}}(x) = \langle -S_\sigma^*(1,\dots,1), x\rangle. \ \blacksquare$$

3. Alpha-invariant

3.1. Log canonical threshold and α-invariant. Let X be a compact Kähler manifold, and L a big and nef line bundle on X.

Definition 3.1. Assume that a compact group K acts on X, and that L is K-linearized. The α-invariant $\alpha_K(L)$ of L with respect to the group K is defined as the infimum of the log canonical thresholds of all K-invariant, non-negatively curved singular hermitian metrics on L.

The linear systems in a multiple of L give singular metrics on L, which we will call *algebraic metrics*, in the following way. Let $\delta_1,\dots,\delta_r \in H^0(X, mL)$ be linearly independent sections, and denote by Δ the linear system generated by these. Then it defines an algebraic metric $h_{\Delta/m}$ on L by setting, in any trivialization,

$$\|\xi\|^2_{h_{\Delta/m}} = \frac{|\xi|^2}{(\sum |\delta_j(z)|^2)^{1/m}}$$

for any $\xi \in L_z$. The local potential $\phi_{\Delta/m}(z) = \frac{1}{2m}\ln \sum |\delta_j(z)|^2$ is psh.

If Δ is one-dimensional, generated by δ, we denote by $h_{\delta/m}$ the corresponding metric.

Recall the following result of Demailly, relating the α-invariant with log canonical thresholds of algebraic metrics:

Theorem 3.2 ([CS08, Appendix A]). *Let K be a compact group, let X be a compact complex K-variety and L a big and nef K-linearized line bundle on X. Then*

$$\alpha_K(L) = \inf_{m\in\mathbb{N}^*} \inf_{\Delta\subset H^0(X,mL),\, \Delta^K=\Delta} \mathrm{lct}(h_{\Delta/m}).$$

One can slightly improve this result and give the following statement, which is only given in the case of a trivial group K by Demailly.

Corollary 3.3. *Let K be a compact group, let X be a compact complex K-variety and L a big and nef K-linearized line bundle on X. Then*

$$\alpha_K(L) = \inf_{m\in\mathbb{N}^*} \inf_{\Delta\in\mathrm{Irr}(H^0(X,mL))} \mathrm{lct}(h_{\Delta/m}),$$

where $\mathrm{Irr}(H^0(X,mL))$ *is the set of all irreducible K-subrepresentations of* $H^0(X,mL)$.

Proof. Let Δ be a K-invariant subspace of $H^0(X,mL)$. Then we have $\Delta = \Delta_1 \oplus \cdots \oplus \Delta_s$ with Δ_i irreducible subspaces. For all i, one can choose a basis δ_j^i of Δ_i. Together they form a basis of Δ and we can obtain the metric h_Δ this way.

In particular, $\phi_{\Delta/m}(z) = \frac{1}{2m}\ln\sum_i\sum_j|\delta_j^i(z)|^2$. Since the logarithm is increasing, we can write

$$\phi_{\Delta/m}(z) \geq \frac{1}{2m}\ln\sum|\delta_j^1(z)|^2 = \phi_{\Delta_1/m}(z).$$

This implies, by elementary properties of the complex singularity exponent [DK01, 1.4], that $\mathrm{lct}(h_{\Delta/m}) \geq \mathrm{lct}(h_{\Delta_1/m})$.

We conclude that the log canonical threshold of a metric associated to a K-invariant linear system is greater than the log canonical threshold of at least one metric associated to an irreducible linear system, so it is enough to consider only these. ∎

3.2. General formula. Let X_Σ be a smooth compact toric manifold. Let $N(T)$ be the normalizer of T in $\mathrm{Aut}(X_\Sigma)$, and denote by $W = N(T)/T$ the Weyl group obtained from T.

The group $N(T)$ naturally acts on M, and since T acts trivially on M, this induces an action of W on M. By duality one also gets an action on N.

From the description of morphisms between toric varieties [CLS11, Theorem 3.3.4], we can see that W is isomorphic to the subgroup of $\mathrm{GL}(N)$ composed of the γ such that $\gamma(\Sigma) = \Sigma$.

Given a subgroup G of W, we denote by K_G the compact subgroup of $\mathrm{Aut}(X_\Sigma)$ generated by T_c and G, and by T_G the subgroup generated by T and G. If P is a polytope in $M_\mathbb{R}$, we let P^G be the set of G-invariant points of P.

Finally, if P is a polytope in $M_\mathbb{R}$, we denote by $P(\mathbb{Q})$ the set of *rational points* in P, i.e. points p such that there exists $m \in \mathbb{N}^*$ with $mp \in M$.

Theorem 3.4. *Let L be a T_G-linearized line bundle on X_Σ. Then*

$$\alpha_{K_G}(L) = \inf_{p\in P_L^G(\mathbb{Q})} \sup\{c > 0 \mid cP_L \subset \mathrm{Int}(cp + P_{-K_{X_\Sigma}})\}.$$

Proof. Corollary 3.3 shows that it is enough to consider algebraic metrics on L associated to K_G-irreducible linear systems in a multiple of L.

The T_c-irreducible subrepresentations of $H^0(X_\Sigma, mL)$ are the dimension one subspaces corresponding to integral points of the polytope P_{mL} associated to mL. Recall that $P_{mL} = mP_L$.

Now a K_G-irreducible subrepresentation of $H^0(X_\Sigma, mL)$ is the union of the images by G of a T_c-irreducible representation.

Let p be an integral point in mP_L, and denote by Δ the K_G-irreducible linear system generated by the G-orbit of p.

The potential of $h_{\Delta/m}$ is

$$\phi_{\Delta/m}(z) = \frac{1}{2m} \ln \sum_{g\in G} |(g\cdot p)(z)|^2.$$

By the arithmetic-geometric mean inequality,

$$\phi_{\Delta/m}(z) \geq \frac{1}{2m} \ln \left| \left(\frac{\sum_{g\in G}(g\cdot p)}{|G|} \right)(z) \right|^2.$$

The right-hand side is the potential of the algebraic metric $h_{\sum_{g\in G}(g\cdot p)/(m|G|)}$ corresponding to the linear system of $H^0(X_\Sigma, m|G|L)$ generated by the section $\sum_{g\in G}(g\cdot p)$.

Using again the fact that the complex singularity exponent is increasing [DK01, 1.4], we get

$$\operatorname{lct}(h_{\Delta/m}) \geq \operatorname{lct}(h_{\sum_{g\in G}(g\cdot p)/(m|G|)}).$$

We have thus shown that it is enough to compute the log canonical thresholds of algebraic metrics associated to one-dimensional G-invariant sublinear systems of multiples of L.

We use Theorem 2.8 to conclude the proof. Indeed, if $p \in mP_L$ generates a one-dimensional G-invariant sublinear system in $H^0(X_\Sigma, mL)$, and $f_{p/m}$ denotes the convex function associated to the potential of the corresponding algebraic metric $h_{p/m}$, we have $N(f_{p/m}) = \{p/m\}$.

Applying Theorem 2.8 gives

$$\operatorname{lct}(h_{p/m}) = \sup\{c > 0 \mid cP_L \subset \operatorname{Int}(cp/m + P_{-K_{X_\Sigma}})\}.$$

Finally, observe that as p and m vary, they describe the set $P_L^G(\mathbb{Q})$ of G-invariant points of P_L with rational coordinates. ∎

Remark 3.5. One can also prove, without the use of Corollary 3.3, that we can consider only metrics corresponding to points of P_L (not necessarily with rational coordinates), by considering the expression of the log canonical threshold of any metric.

Indeed, if f is a convex function on $N_\mathbb{R}$, corresponding to a metric h on L, and p is a point in $N(f)$, then the metric h_p associated to the convex function $x \mapsto \langle p, x\rangle$ is also a non-negatively curved metric on L, and $\operatorname{lct}(h_p) \leq \operatorname{lct}(h)$.

3.3. Case of the anticanonical line bundle. We assume in this section that $L = -K_{X_\Sigma}$.

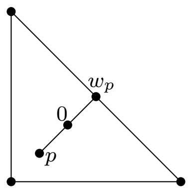
This line bundle admits a natural T_W-linearization, and the polytope associated to this linearization contains the origin in its interior.

For any subgroup G of W, let $S_G := \{p \in \partial P_L \mid g \cdot p = p, \forall g \in G\}$. If $0 \neq p \in P_L$, let w_p be the point $\partial P_L \cap \{-tp \mid t \geq 0\}$.

Remark 3.6. Note that S_G is empty if and only if $\{0\}$ is the only point fixed by G in P. Moreover, if S_W is empty then X_Σ is called *symmetric.*

Proposition 3.7. *Assume that $P_h = \{p\}$ with $0 \neq p \in P_L$. Then*

$$\operatorname{lct}(h) = \frac{|w_p|}{|w_p| + |p|}.$$



Proof. By Theorem 2.8 we have

$$\operatorname{lct}(h) = \sup\{c > 0 \mid cP \subset \operatorname{Int}(cp + P)\}.$$

Consider the half-line starting from p and containing the origin. It intersects ∂P at w_p. Denote by r its intersection with $\partial(p + P)$.

Then it is easy to see that the log canonical threshold of h_p is equal to the quotient of the distance between p and r by the distance between p and w_p. The translation sending 0 to p also sends w_p to r, so $|r - p| = |w_p|$. The result follows. ■

Remark 3.8. If $P_h = \{0\}$ then $\operatorname{lct}(h) = 1$.

Example 3.9. Consider the case $P_h = \{b\}$, where b is the barycenter of the polytope P_L. Then $\operatorname{lct}(h)$ is equal to the greatest lower bound for the Ricci curvature $R(X)$, introduced by Székelyhidi [Szé11], and computed for toric manifolds by Li [Li11].

From this formula we recover the previous results of Song and Chel'tsov–Shramov.

Theorem 3.10 ([Son05], [CS08, Lemma 6.1]). *Let X be a smooth Fano toric manifold, and G be a subgroup of W. Then*

- *if S_G is empty, we have $\alpha_{K_G}(X) = 1$;*
- *else,*

$$\alpha_{K_G}(X) = \frac{1}{1 + \max_{p \in S_G} |p|/|w_p|} \leq \frac{1}{2}.$$

Proof. By Theorem 3.4, it is enough to consider only the (rational) G-invariant points of P.

The first case follows immediately by using Remark 3.8.

In the second case, we obtain the formula using Proposition 3.7. Indeed, it is enough to consider points p in S_G because if $q \neq 0$ is not in ∂P, and p is the intersection of ∂P with the half-line starting from the origin and going through q, then $\mathrm{lct}(h_q) \geq \mathrm{lct}(h_p)$.

Furthermore, $\max_{p\in S_G} |p|/|w_p| \geq 1$ because otherwise if p were a point at which this maximum was attained and the maximum were < 1, then we would have $|w_p|/|p| > 1$ with $w_p \in S_G$, a contradiction. ∎

3.4. Example. We compute the α-invariant of any linearized line bundle on the blow up X of $\mathbb{P}^2$ at one point.

Identify N with $\mathbb{Z}^2$. The fan of X has four rays, with generators $u_1 = (1,0)$, $u_2 = (1,1)$, $u_3 = (0,1)$ and $u_4 = (-1,-1)$.

The group W is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and acts on $M_{\mathbb{R}}$ by exchanging the coordinates, $(x,y) \mapsto (y,x)$.

We define $P(k,l)$ to be the polytope whose vertices are $(0,k)$, $(0,l)$, $(k,0)$ and $(l,0)$, for $k,l \in \mathbb{N}$ with $l > k$. It is easy to see that the polytopes of nef and big divisors are the $P(k,l)$, up to translation by a character. For example, the polytope of the anticanonical bundle is $Q := (-1,-1) + P(1,3)$.

Proposition 3.11. *The α-invariant with respect to K_W of the nef and big line bundle corresponding to $P(k,l)$ is equal to* $\inf\left(\frac{1}{l-k}, \frac{2}{l}\right)$.

Proof. By Theorem 3.4, it is enough to consider points (with rational coordinates) in the intersection of $P(k,l)$ with the first diagonal. However, one easily observes that it is enough to consider only the point $(l/2, l/2)$, similarly to the proof of Theorem 3.10.

We want to compute

$$\sup\{c > 0 \mid cP(k,l) \subset \mathrm{Int}(c(l/2,l/2) + Q)\}.$$

This is of course equal to

$$\sup\left\{c > 0 \,\middle|\, P(k,l) \subset \mathrm{Int}\left((l/2,l/2) + \frac{1}{c}Q\right)\right\}.$$

Observe that $l/2$ is the least positive constant b such that

$$\{(0,l),(l,0)\} \subset (l/2,l/2) + bQ.$$

If $k \geq l/2$, then we also have $\{(0,k),(k,0)\} \subset (l/2,l/2) + (l/2)Q$, so

$$P(k,l) \subset (l/2,l/2) + (l/2)Q.$$

Thus $\alpha_{K_W}(P(k,l)) = 2/l$ when $k \geq l/2$.

For the other case, observe that $l-k$ is the least positive constant b such that $(k/2,k/2) \in (l/2,l/2)+bQ$. If $k \leq l/2$, then we also have

$$P(k,l) \subset (l/2,l/2)+(l-k)Q.$$

Thus $\alpha_{K_W}(P(k,l)) = \frac{1}{l-k}$ when $k \geq l/2$. ■

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