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Homoclinic orbits for an almost periodically forced singular Newtonian system in $\mathbb{R}^3$

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Abstract. This work uses a variational approach to establish the existence of at least two homoclinic solutions for a family of singular Newtonian systems in $\mathbb{R}^3$ which are subjected to almost periodic forcing in time variable.

1. Introduction. This paper is motivated by [19] where the existence of homoclinic orbits was proved for a family of singular Newtonian systems in $\mathbb{R}^2$ which were subjected to almost periodic forcing in time. We extend the results of [19] to Newtonian systems in $\mathbb{R}^3$. More precisely, we will be concerned with the Newtonian system in $\mathbb{R}^3$,

$$\ddot{q} + a(t)\nabla W(q) = 0, \tag{1.1}$$

where $a(t)$ and $W(q)$ satisfy the following assumptions:

- (a1) $a : \mathbb{R} \to \mathbb{R}$ is a continuous almost periodic function such that $a(t) \geq a_0 > 0$ for all $t \in \mathbb{R}$.
- (H1) There is a line l such that $l \cap \{0\} = \emptyset$, $W \in C^2(\mathbb{R}^3 \setminus \{l\}, \mathbb{R})$ and l consists of singular points of W, i.e. $\lim_{x\to l} W(x) = -\infty$.
- (H2) $W : \mathbb{R}^3 \setminus \{l\} \to \mathbb{R}$ satisfies the Gordon condition in a neighborhood of l, i.e. there is a neighborhood $\mathcal{N}$ of l and a function $U \in C^2(\mathcal{N} \setminus \{l\}, \mathbb{R})$ such that $|U(x)| \to \infty$ as $x \to l$ and
$$|\nabla U(x)|^2 \leq -W(x) \qquad \text{for all } x \in \mathcal{N} \setminus \{l\}.$$
- (H3) $W(x) < W(0) = 0$ if $x \neq 0$ and $W''(0)$ is negative definite.
- (H4) There is a constant $W_0 < 0$ such that $\limsup_{|x|\to\infty} W(x) \leq W_0$.

Here and subsequently, $x \to l$ stands for $d(x, l) = \inf\{|x-y| : y \in l\} \to 0$ and $|\cdot| : \mathbb{R}^3 \to \mathbb{R}$ is the Euclidean norm in $\mathbb{R}^3$. Condition (H2) governs the rate at which $-W(x) \to \infty$ as $x \to l$. It was introduced by W. B. Gordon [10]. If a

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potential $W\colon \mathbb{R}^3\setminus\{l\}\to\mathbb{R}$ satisfies (H2) then its gradient $\nabla W\colon \mathbb{R}^3\setminus\{l\}\to\mathbb{R}^3$ is called a strong force. Condition (H3) implies that the origin is a nondegenerate critical point of W, and condition (H4) guarantees that W does not asymptotically converge to its global maximum 0.

When $a(t)\equiv 1$ and somewhat weaker hypotheses than (H1)–(H4) are satisfied, it was shown in [16] that (1.1) has a pair of homoclinic orbits Q^+ and Q^- which wind around l in a positive and negative sense respectively. When $\mathbb{R}^3$ is replaced by $\mathbb{R}^2$ and the line l by a fixed point ξ, it was proved in [19] that (1.1) has a pair of homoclinic orbits which wind around ξ in opposite sense. Finally, when $\mathbb{R}^3$ is replaced by $\mathbb{R}^n$ with $n>2$ and $a(t)\equiv 1$ and again weaker assumptions than the above are fulfilled, the existence of a single homoclinic orbit of (1.1) was proved by K. Tanaka [21].

There have been several other works in recent years which use variational methods to find periodic, homoclinic and heteroclinic solutions for periodically forced or conservative singular Newtonian systems with infinitely deep wells of Gordon's type: see for example [1, 2, 4]–[8, 11, 13]–[15, 18].

The goal of this paper is to obtain an analogue of the result of [19] for singular Newtonian systems in $\mathbb{R}^3$. It will be shown that (1.1) has a pair of orbits Q^+, Q^- homoclinic to 0 winding around l in opposite sense. The proof will be based on minimization and geometric arguments.

Let us introduce some basic notions and necessary notation.

Definition 1.1. A set $D\subset\mathbb{R}$ is called *relatively dense* in $\mathbb{R}$ if there exists a number $\lambda>0$ such that every interval of length λ contains at least one element of D.

Definition 1.2. Let $f\colon\mathbb{R}\to\mathbb{R}$ be a continuous function. A number τ is called an *ε-period* of f if

$$\sup_{t\in\mathbb{R}}|f(t+\tau)-f(t)|\le\varepsilon.$$

A continuous function $f\colon\mathbb{R}\to\mathbb{R}$ is called *almost periodic* if for every $\varepsilon>0$ there exists a relatively dense set $D_\varepsilon\subset\mathbb{R}$ of ε-periods of f.

When $a(t)$ is periodic of period $T>0$, and Q is a homoclinic orbit of (1.1) then so is $Q(t-kT)$ for $k\in\mathbb{Z}$. This is no longer the case when $a(t)$ is almost periodic.

E. Serra, M. Tarallo and S. Terracini [20] established the existence of at least one nonzero homoclinic solution for a Lagrangian system of the form

$$\ddot q - q + a(t)\nabla G(q)=0,$$

where $q\in\mathbb{R}^n$, $a(t)$ satisfies (a1) and $G\in C^2(\mathbb{R}^n,\mathbb{R})$ satisfies the superquadraticity growth condition due to A. Ambrosetti and P. H. Rabinowitz:

(G1) There is $\theta > 2$ such that for all $x \in \mathbb{R}^n \setminus \{0\}$,

$$0 < \theta G(x) \le \nabla G(x) \cdot x.$$

Thus G is a rather different nonlinearity than W. Nevertheless G and W have a common feature that makes proofs of several our technical results nearly identical with those of [20]. Note that (G1) implies that $G(x) = o(|x|^2)$ and $\nabla G(x) = o(|x|)$ as $x \to 0$. From this it follows that for each $M > 0$ there are $K(M) > 0$ and $L(M) > 0$ such that if $|x| < M$ then $G(x) \le K(M)|x|^2$ and $|\nabla G(x)| \le L(M)|x|$. The same inequalities hold for W by (H3). Moreover, their proof exploited the Bochner criterion.

THEOREM 1.3 ([3], Bochner's criterion). *A continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is almost periodic if and only if for every sequence $\{\sigma_n\}_{n=1}^{\infty}$ of real numbers there exists a subsequence $\{\sigma_{n_k}\}_{k=1}^{\infty}$ such that $\{f(t+\sigma_{n_k})\}_{k=1}^{\infty}$ is uniformly convergent in $\mathbb{R}$, i.e. there is a function $g \in C(\mathbb{R},\mathbb{R})$ such that*

$$\|f(\cdot+\sigma_{n_k}) - g(\cdot)\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{as } k \to \infty.$$

In particular, note that Definition 1.2 implies the existence of a sequence of real numbers $\tau_n \to \infty$ ($\tau_n \to -\infty$) as $n \to \infty$ such that

$$\|f(\cdot+\tau_n) - f(\cdot)\|_{L^\infty(\mathbb{R})} \to 0$$

as $n \to \infty$. Using this fact, it will be shown in Section 2 that (1.1) has a homoclinic solution.

DEFINITION 1.4. A solution $q\colon \mathbb{R} \to \mathbb{R}^3$ is said to be *homoclinic* (to 0) if

$$q(\pm\infty) = \lim_{t\to\pm\infty} q(t) = 0 \quad \text{and} \quad \dot{q}(\pm\infty) = 0.$$

Let E denote the Sobolev space $W^{1,2}(\mathbb{R},\mathbb{R}^3)$ with the standard norm

$$\|q\|_E^2 = \int_{\mathbb{R}} (|q(t)|^2 + |\dot{q}(t)|^2)\,dt.$$

LEMMA 1.5. *Let $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ be bounded sequences in E and assume that $\|u_n - v_n\|_E \to 0$ as $n \to \infty$. Then*

$$|u_n|^2 - |v_n|^2 \to 0 \quad \text{as } n \to \infty \text{ in } W^{1,2}(\mathbb{R},\mathbb{R}).$$

This appears in [20, proof of Proposition 3.4, Step 2]. From now on, $\|\cdot\|$ stands for the standard norm in $W^{1,2}(\mathbb{R},\mathbb{R})$.

Set

$$\mathcal{L}(q) = \tfrac{1}{2}|\dot{q}(t)|^2 - a(t)W(q(t)).$$

We will consider the family of curves that avoid the line l,

$$\Lambda = \{q \in E : q(t) \notin l \text{ for all } t \in \mathbb{R}\}.$$

For $q \in \Lambda$, set

$$I(q) = \int_{\mathbb{R}} \mathcal{L}(q)\,dt. \tag{1.2}$$

PROPOSITION 1.6. *If* (a1) *and* (H1)–(H4) *hold then* $I \in C^1(\Lambda, \mathbb{R})$.

This was proved in [9, Proposition 1.1].

PROPOSITION 1.7. *If* (a1) *and* (H1)–(H4) *are satisfied,* $q \in \Lambda$ *and* $I'(q) = 0$*, i.e.* q *is a critical point of* I *on* Λ*, then* q *is a classical solution of* (1.1) *with* $|q(t)|$, $|\dot{q}(t)| \to 0$ *as* $|t| \to \infty$.

Proof. If $q \in E$, standard embedding theorems imply $q(t) \to 0$ as $|t| \to \infty$. If q is a critical point of I, it is a weak solution of (1.1) and then standard "elliptic" arguments show it is a classical solution of (1.1). Finally, using (1.1), (a1), (H1) and (H3) arguing as in [16] it is possible to show that $\dot{q} \in E$ and therefore $|\dot{q}(t)| \to 0$ as $|t| \to \infty$. ∎

Let Π be the plane that is perpendicular to l and contains 0. We will use the cylindrical coordinate system in $\mathbb{R}^3$ with the height axis l and the reference plane Π. Let P be the intersection point of the plane Π and the line l. Then P is the pole and the half-line $P0$ is the polar axis. In this coordinate system, for all $q \in \Lambda$ we have

$$q(t) = (r(t)\cos\varphi(t), r(t)\sin\varphi(t), z(t)),$$

where $r(t)$ is the distance of $q(t)$ from l, $\varphi(t)$ is a polar angle and $z(t)$ is the distance of $q(t)$ from the plane Π. We choose an orthogonal positively oriented base $\{\vec{P0}, \vec{PR}\}$ of Π. The positive direction of l is determined by $\vec{P0} \times \vec{PR}$. The function φ is not unique. However, if q is continuous then we can assume that r, φ and z are also continuous.

DEFINITION 1.8. Following J. Janczewska and J. Maksymiuk [16], for $q \in \Lambda$ we define the *winding* (rotation) *number* as follows:

$$\mathrm{WN}(q) = \frac{\varphi(\infty) - \varphi(-\infty)}{2\pi}.$$

Set $\hat{\varepsilon} = |P|/3$. From now on, $B_\varepsilon(x)$ stands for an open ball in $\mathbb{R}^3$ of radius $\varepsilon > 0$, centered at $x \in \mathbb{R}^3$.

REMARK 1.9. Let $0 < \varepsilon \le \hat{\varepsilon}$. If $q \in \Lambda$, there is $T \in \mathbb{R}$ such that $q(T) \in B_\varepsilon(0)$. Then, by $\mathrm{WN}(q|^T_{-\infty})$ and $\mathrm{WN}(q|^\infty_T)$ we denote the winding numbers of appropriate paths in Λ that arise from $q|_{(-\infty,T]}$ and $q|_{[T,\infty)}$ resp., by connecting $q(T)$ to 0 by a line segment. By elementary homotopy arguments,

$$\mathrm{WN}(q) = \mathrm{WN}(q|^T_{-\infty}) + \mathrm{WN}(q|^\infty_T).$$

Moreover, if $q([T, \infty)) \subset B_\varepsilon(0)$ then

$$\mathrm{WN}(q) = \mathrm{WN}(q|^T_{-\infty}).$$

Let

$$\Gamma = \{q \in \Lambda : \mathrm{WN}(q) \neq 0\} = \Gamma^+ \cup \Gamma^-, \tag{1.3}$$

where

$$\Gamma^{\pm} = \{q \in \Gamma : \pm \mathrm{WN}(q) > 0\}.$$

Now we are ready to formulate our main result.

THEOREM 1.10. *If* (a1) *and* (H1)–(H4) *are satisfied then the Newtonian system* (1.1) *has at least two homoclinic solutions* $Q^{\pm} \in \Gamma^{\pm}$.

2. Proof of Theorem 1.10. For the convenience of the reader the proof of Theorem 1.10 will be divided into a sequence of lemmas, propositions and theorems.

Throughout this section we will assume that (a1) and (H1)–(H4) are satisfied.

LEMMA 2.1. *For each* $M > 0$ *there is a constant* $K > 0$ *such that if* $q \in \Lambda$ *and* $I(q) \le M$ *then*

$$d(q(t), l) \ge K \quad \text{for all } t \in \mathbb{R}.$$

The proof of Lemma 2.1 is analogous to that of [14, Proposition 2.6] and [16, Proposition 2.6] and therefore it will be omitted.

Let

$$\alpha_\varepsilon = a_0 \inf\{-W(x) : x \notin B_\varepsilon(0)\},$$

where $0 < \varepsilon \le \hat{\varepsilon}$.

LEMMA 2.2. *Suppose that* $q \in \Lambda$ *and* $q(t) \notin B_\varepsilon(0)$ *for each* $t \in \bigcup_{i=1}^{k} [r_i, s_i]$, *where* $[r_i, s_i] \cap [r_j, s_j] = \emptyset$ *for* $i \ne j$. *Then*

$$I(q) \ge \sqrt{2\alpha_\varepsilon} \sum_{i=1}^{k} |q(s_i) - q(r_i)|.$$

The proof of Lemma 2.2 is the same as that of [12, Lemma 2.1].

Let

$$c^{\pm} = \inf_{q \in \Gamma^{\pm}} I(q). \tag{2.1}$$

PROPOSITION 2.3. *There is* $c > 0$ *such that*

$$c^{\pm} \ge c. \tag{2.2}$$

Proof. We will prove the case of c^+. The second one is analogous. Let $\{q_m\}_{m=1}^{\infty}$ be a minimizing sequence for I on Γ^+. Fix $0 < \varepsilon \le \hat{\varepsilon}$. If $q_m \in \Gamma^+$ then $\mathrm{WN}(q_m) > 0$. Since $B_\varepsilon(0) \cap l = \emptyset$ there are $T_1^m, T_2^m \in \mathbb{R}$ such that q_m leaves $B_\varepsilon(0)$ at T_1^m, rotates around the line l and returns to $B_\varepsilon(0)$ at T_2^m. There is a ray starting from 0, passing through the line l, intersecting the orbit of q_m at $q_m(T_3^m)$, where $T_3^m \in (T_1^m, T_2^m)$ and $|q_m(T_3^m)| > |P|$. Thus by Lemma 2.2,

$$I(q_m) \ge \sqrt{2\alpha_\varepsilon}\, |q_m(T_1^m) - q_m(T_3^m)| \ge \sqrt{2\alpha_\varepsilon}\, \frac{|P|}{2}.$$

Finally,

$$c^+ = \lim_{m\to\infty} I(q_m) \geq \frac{|P|}{2}\sqrt{2\alpha_\varepsilon} = c. \ \blacksquare$$

LEMMA 2.4. *If $M > 0$, $q \in \Lambda$, and $I(q) \leq M$, then there exists $\omega(M) > 0$ (independent of q) such that*

$$\|q\|_E \leq \omega(M).$$

The proof of Lemma 2.4 is the same as that of [19, Proposition 1.11] and it will be omitted. Condition (H3) plays an important role here.

If $a(t)$ satisfies (a1) then it is bounded from above by some constant a_1. We set

$$\mathcal{A}_a = \{\beta \in C(\mathbb{R},\mathbb{R}) : a_0 \leq \beta(t) \leq a_1 \text{ for all } t \in \mathbb{R}\}.$$

For every $\beta \in \mathcal{A}_a$ we define a functional $I(\beta,\cdot) : \Lambda \to \mathbb{R}$ by setting

$$I(\beta,q) = \int_{\mathbb{R}} \left(\tfrac{1}{2}|\dot{q}|^2 - \beta(t)W(q)\right) dt. \tag{2.3}$$

Let

$$K_\infty = \{v \in E : v \not\equiv 0,\ \exists \beta \in \mathcal{A}_a,\ \nabla I(\beta, v) = 0\},$$
$$Q_\infty = \Big\{\varphi \in W^{1,2}(\mathbb{R},\mathbb{R}) : \varphi(t) = \sum_{\text{finite}} |\tau_{\theta_i} v_i(t)|^2,\ v_i \in K_\infty,\ \theta_i \in \mathbb{R}\Big\},$$

where $\tau_{\theta_i} v_i(t) = v_i(t - \theta_i)$.

LEMMA 2.5. *There exists $\delta > 0$ such that for all $\varphi \in Q_\infty$,*

$$0 < \varphi(t) < 2\delta \ \Rightarrow\ \varphi''(t) > 0.$$

For the proof we refer the reader to [20].

For all $\varphi \in Q_\infty$, set

$$Z(\varphi) = \{t \in \mathbb{R} : \varphi(t) = \delta\},$$

where δ is the number introduced in Lemma 2.5. Let $\mathcal{T} : Q_\infty \to \mathbb{R}$ be given by

$$\mathcal{T}(\varphi) = \max Z(\varphi).$$

In [20] it was shown that $\mathcal{T}$ is locally Lipschitz continuous on bounded subsets of Q_∞ (see [20, Proposition 3.11]).

LEMMA 2.6. *Let β_1, $\beta_2 \in L^\infty(\mathbb{R},\mathbb{R})$ and B be a bounded subset of Λ. Then there exists a constant $M = M(B)$ such that for all $q \in B$,*

$$|I(\beta_1,q) - I(\beta_2,q)| \leq M\|\beta_1 - \beta_2\|_{L^\infty(\mathbb{R},\mathbb{R})}, \tag{2.4}$$

$$\|\nabla I(\beta_1,q) - \nabla I(\beta_2,q)\|_E \leq M\|\beta_1 - \beta_2\|_{L^\infty(\mathbb{R},\mathbb{R})}. \tag{2.5}$$

Proof. Inequality (2.4) follows from

$$|I(\beta_1,q)-I(\beta_2,q)| = \Big|\int_{\mathbb{R}}(\beta_2(t)-\beta_1(t))W(q)\,dt\Big| \le \|\beta_1-\beta_2\|_{L^\infty(\mathbb{R},\mathbb{R})}\sup_{q\in B}\int_{\mathbb{R}}|W(q)|\,dt.$$

Since B is bounded, there is a constant $M_1=M_1(B)$ such that for all $q\in B$, $\|q\|_E\le M_1$. By (H3) there exists $M_2=M_2(M_1)$ such that

$$|W(x)|\le M_2|x|^2$$

for all $|x|\le M_1$. Therefore

$$\sup_{q\in B}\int_{\mathbb{R}}|W(q)|dt \le M_2\sup_{q\in B}\int_{\mathbb{R}}|q|^2\,dt \le M_2M_1^2.$$

Finally,

$$|I(\beta_1,q)-I(\beta_2,q)|\le M_2M_1^2\|\beta_1-\beta_2\|_{L^\infty(\mathbb{R},\mathbb{R})}.$$

Similarly, inequality (2.5) follows from

$$\|\nabla I(\beta_1,q)-\nabla I(\beta_2,q)\|_E \le \|\beta_1-\beta_2\|_{L^\infty(\mathbb{R},\mathbb{R})}\sup_{q\in B}\Big(\int_{\mathbb{R}}|\nabla W(q)|^2\,dt\Big)^{1/2}. \ \blacksquare$$

As a simple consequence of Lemma 2.6 we get

LEMMA 2.7. *Let $\{q_n\}_{n=1}^\infty$ be a bounded sequence in Λ and $\{\beta_n\}_{n=1}^\infty\subset L^\infty(\mathbb{R},\mathbb{R})$ such that $\beta_n\to\beta$ in $L^\infty(\mathbb{R},\mathbb{R})$. Then*

$$|I(\beta_n,q_n)-I(\beta,q_n)|\to 0 \quad \text{and} \quad \|\nabla I(\beta_n,q_n)-\nabla I(\beta,q_n)\|_E\to 0 \tag{2.6}$$

as $n\to\infty$.

To continue, it is essential to understand the behavior of Palais–Smale sequences for I. The next results provide this information.

LEMMA 2.8 (Representation lemma). *Let $\{p_m\}_{m=1}^\infty$ be a Palais–Smale sequence for I at the level $b>0$, i.e. $I(p_m)\to b$ and $\nabla I(p_m)\to 0$ as $m\to\infty$. Then there exist a number $k\in\mathbb{N}$, depending on b, k functions $\beta_i\in\mathcal{A}_a$, k functions $v_i\in E$, $v_i\not\equiv 0$, a subsequence still denoted by $\{p_m\}_{m=1}^\infty$ and k sequences $\{\theta_m^1\}_{m=1}^\infty,\dots,\{\theta_m^k\}_{m=1}^\infty\subset\mathbb{R}$ such that*

(i) $\|p_m-\sum_{i=1}^k\tau_{\theta_m^i}v_i\|_E\to 0$ *as* $m\to\infty$,
(ii) $\nabla I(\beta_i,v_i)=0$ *for all* $i=1,\dots,k$,
(iii) $b=\sum_{i=1}^k I(\beta_i,v_i)$,
(iv) $|\theta_m^i-\theta_m^j|\to\infty$ *as* $m\to\infty$ *for all* $i\ne j$.

The statement of Lemma 2.8 is the same as that of [20, Proposition 2.16] although the technical frameworks of the two results are different. Nevertheless the proof of Lemma 2.8 is nearly identical with that in [20]. Therefore

we indicate those properties for (1.1) that combined with [20] allow for the same proof.

Note that by Lemma 2.4 Palais–Smale sequences for I are bounded in E. Moreover, by (H3) there exist $\rho, \beta_0 > 0$ such that if $|x| \le \rho$ then $-W(x) \ge \beta_0 |x|^2$. Thus for $x \in \Lambda$ such that $\|x\|_E \le \rho$ we have

$$I(\beta, x) \ge \int_{\mathbb{R}} \left(\tfrac{1}{2}|\dot{x}|^2 + a_0\beta_0|x|^2\right) dt \ge \min\{1/2, a_0\beta_0\}\|x\|_E^2.$$

With the above observations and Lemma 2.2 the proof of Lemma 2.8 proceeds as in [20]. Bochner's criterion is exploited here. It is worth pointing out that analogous results can be found in many papers on homoclinic solutions (see e.g. [9], [19]).

LEMMA 2.9. *Let $\{p_m\}_{m=1}^{\infty}$ be a Palais–Smale sequence for I at the level $b > 0$ as in the representation lemma, that is, assume $p_m - \sum_{i=1}^{k} \tau_{\theta_m^i} v_i \to 0$ in E for some $v_i \in K_\infty$ and $\theta_m^i \in \mathbb{R}$. Then*

$$|p_m|^2 - \sum_{i=1}^{k} |\tau_{\theta_m^i} v_i|^2 \to 0 \quad \text{as } m \to \infty \text{ in } W^{1,2}(\mathbb{R}, \mathbb{R}). \tag{2.7}$$

For the proof we refer the reader to [20].

PROPOSITION 2.10. *Let $\{p_m\}_{m=1}^{\infty}$ be a Palais-Smale sequence for I at the level $b > 0$. Then*

$$\operatorname{dist}(|p_m|^2, Q_\infty) \to 0 \quad \text{as } m \to \infty \text{ in } W^{1,2}(\mathbb{R}, \mathbb{R}). \tag{2.8}$$

Proof. Suppose (2.8) is false. Then there is a subsequence, still denoted by $\{p_m\}_{m=1}^{\infty}$, such that

$$\lim_{m\to\infty} \operatorname{dist}(|p_m|^2, Q_\infty) > 0. \tag{2.9}$$

Taking another subsequence $\{p_m\}_{m=1}^{\infty}$, by Lemma 2.8 we have

$$\Big\| p_m - \sum_{i=1}^{k} \tau_{\theta_m^i} v_i \Big\|_E \to 0$$

as $m \to \infty$ for some k, v_i and θ_m^i. Consequently, by Lemma 2.9,

$$|p_m|^2 - \sum_{i=1}^{k} |\tau_{\theta_m^i} v_i|^2 \to 0 \quad \text{as } m \to \infty \text{ in } W^{1,2}(\mathbb{R}, \mathbb{R}),$$

contradicting (2.9). ∎

This proof is adapted from [20].

THEOREM 2.11. *Let $\{p_m\}_{m=1}^{\infty} \subset \Lambda$ be a Palais–Smale sequence for I at the level $b > 0$. Moreover,*

$$\|p_m - p_{m-1}\|_E \to 0 \quad \text{as } m \to \infty. \tag{2.10}$$

Then there is a sequence $\{\theta_m\}_{m=1}^{\infty} \subset \mathbb{R}$ *and* $r > 0$ *such that*

$$\liminf_{m\to\infty} |\tau_{\theta_m} p_m(0)| \geq r, \tag{2.11}$$

$$|\theta_m - \theta_{m-1}| \to 0 \quad \text{as } m \to \infty. \tag{2.12}$$

Proof. Since $\operatorname{dist}(|p_m|^2, Q_\infty) \to 0$ as $m \to \infty$, there exists $\{\varphi_m\}_{n=1}^{\infty} \subset Q_\infty$ such that $\||p_m|^2 - \varphi_m\| \to 0$ as $m \to \infty$ in $W^{1,2}(\mathbb{R}, \mathbb{R})$. Let $\theta_m = \mathcal{T}(\varphi_m)$. By Lemma 1.5 we have

$$\begin{aligned}\|\varphi_m - \varphi_{m-1}\| &\leq \|\varphi_m - |p_m|^2\| + \big\||p_m|^2 - |p_{m-1}|^2\big\| + \big\|\varphi_{m-1} - |p_{m-1}|^2\big\| \\ &\to 0\end{aligned}$$

as $m \to \infty$. By uniform continuity of $\mathcal{T}$ on bounded sets we have

$$|\theta_m - \theta_{m-1}| = |\mathcal{T}(\varphi_m) - \mathcal{T}(\varphi_{m-1})| \to 0$$

as $m \to \infty$. Moreover,

$$\begin{aligned}|\tau_{\theta_m} p_m(0)|^2 &= |p_m(\theta_m)|^2 - \varphi_m(\theta_m) + \varphi_m(\theta_m) = o(1) + \varphi_m(\mathcal{T}(\varphi_m)) \\ &= o(1) + \delta > 0 \qquad \text{as } m \to \infty. \ \blacksquare\end{aligned}$$

THEOREM 2.12. *Let* $q \in \Gamma$. *Then there exists a homoclinic solution* $Q \in \Lambda$ *of* (1.1) *such that* $I(Q) \in (0, I(q)]$.

Proof. If $\nabla I(q) = 0$, the result is obtained for $Q = q$.

Suppose that $\nabla I(q) \neq 0$. Let $\mathcal{V}(x)$ be a locally Lipschitz continuous pseudogradient vector field for I, i.e. $\mathcal{V}\colon \hat{E} \to E$ is locally Lipschitz continuous on $\hat{E} = \{x \in E : \nabla I(x) \neq 0\}$ and satisfies

$$\|\mathcal{V}(x)\|_E \leq 2\|\nabla I(x)\|_E, \tag{2.13}$$

$$\nabla I(x) \cdot \mathcal{V}(x) \geq \|\nabla I(x)\|_E^2. \tag{2.14}$$

Consider the Cauchy problem

$$\frac{d\eta}{ds} = -\frac{\mathcal{V}(\eta)}{1 + \|\mathcal{V}(\eta)\|_E} \equiv -\mathcal{W}(\eta) \tag{2.15}$$

with $\eta(0) = q$. Then $\mathcal{W}$ is locally Lipschitz continuous on $\hat{E}$ and $\|\mathcal{W}(x)\|_E \leq 1$ for all $x \in \hat{E}$. Thus the solution of (2.15) exists for all $s \geq 0$.

By (2.14), we have

$$\frac{dI(\eta(s))}{ds} = \nabla I(\eta(s))\frac{d\eta}{ds} = -\nabla I(\eta(s))\mathcal{W}(\eta(s)) < 0. \tag{2.16}$$

Since $\eta(0) \in \Gamma$, combining Lemma 2.1 with (2.16) we conclude that $\eta(s) \in \Gamma$ for all $s \geq 0$. Moreover, by (2.16), (2.1) and Proposition 2.3,

$$\inf_{s\geq 0} I(\eta(s)) = \lim_{s\to\infty} I(\eta(s)) \geq c^{\pm} > 0,$$

depending on whether q is in Γ^+ or Γ^-. Let $\{s_m\}_{m=1}^{\infty} \subset \mathbb{R}$ be a sequence satisfying $s_m \to \infty$ as $m \to \infty$ and

$$|s_m - s_{m-1}| \to 0$$

as $m \to \infty$. By a corollary to Ekeland's Theorem (see [17, Corollary 4.1]) there is a sequence $\{t_m\}_{m=1}^{\infty}$, $t_m \to \infty$ as $m \to \infty$ such that

$$|t_m - s_m| \to 0, \quad I(\eta(t_m)) \le I(\eta(s_m))$$

and

$$\frac{d}{ds} I(\eta(t_m)) \to 0 \tag{2.17}$$

as $m \to \infty$. Set $q_m = \eta(t_m)$. By (2.17) and (2.16) we obtain

$$\nabla I(q_m) \to 0 \tag{2.18}$$

as $m \to \infty$. Moreover, from the above it follows that

$$\begin{aligned} \|q_m - q_{m-1}\|_E = \left\| \int_{t_{m-1}}^{t_m} \frac{d\eta}{ds}\, ds \right\|_E &\le \int_{t_{m-1}}^{t_m} \left\| \frac{d\eta}{ds} \right\|_E ds \le |t_m - t_{m-1}| \\ &\le |t_m - s_m| + |s_m - s_{m-1}| + |s_{m-1} - t_{m-1}| \to 0 \end{aligned}$$

as $m \to \infty$. Hence $\{q_m\}_{m=1}^{\infty}$ satisfies the assumptions of Theorem 2.11. Thus there exists a sequence $\{\theta_m\}_{m=1}^{\infty}$ and $r > 0$ such that

$$\liminf_{m\to\infty} |\tau_{\theta_m} q_m(0)| \ge r, \tag{2.19}$$

$$|\theta_m - \theta_{m-1}| \to 0 \quad \text{as } m \to \infty. \tag{2.20}$$

We will consider two cases.

Case 1: $\{\theta_m\}_{m=1}^{\infty}$ has a bounded subsequence. Then along a subsequence $\theta_m \to \overline{\theta}$. By decreasing of I along η we have $I(q_m) < I(q)$. Therefore by Lemma 2.4, $\{q_m\}_{m=1}^{\infty}$ is bounded. Hence there is $Q \in \Lambda$ such that $q_m \rightharpoonup Q$ in E and $q_m \to Q$ in $L^{\infty}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^3)$ along a subsequence. Thus

$$\nabla I(q_m) \cdot f \to \nabla I(Q) \cdot f$$

for all $f \in C_0^{\infty}(\mathbb{R}, \mathbb{R}^3)$. Finally, by (2.18), $\nabla I(Q) \cdot f = 0$, and by (2.19), $\tau_{\overline{\theta}} Q(0) \neq 0$. Therefore Q is a nontrivial homoclinic solution of (1.1). Moreover, for all N_1, N_2 such that $N_1 < N_2$, the functional given by

$$E \ni q \mapsto \int_{N_1}^{N_2} \left(\tfrac{1}{2} |\dot{q}(t)|^2 - a(t) W(q(t)) \right) dt$$

is weakly lower semicontinuous. Hence for each $j \in \mathbb{N}$,

$$\begin{aligned}\int_{-j}^{j} \left(\tfrac{1}{2}|\dot{Q}|^2 - a(t)W(q)\right) dt &\le \liminf_{m\to\infty} \int_{-j}^{j} \left(\tfrac{1}{2}|\dot{q}_m|^2 - a(t)W(q_m)\right) dt \\ &\le \lim_{m\to\infty} I(q_m) = c^{\pm} \le I(q).\end{aligned}$$

Letting $j \to \infty$ we get

$$I(Q) \le I(q).$$

Case 2: $\{\theta_m\}_{m=1}^{\infty}$ has no bounded subsequence. Let $v_m = \tau_{\theta_m} q_m$. By the almost periodicity of the function $a(t)$, there is an unbounded sequence $\{\sigma_m\}_{m=1}^{\infty}$ (in the same direction as $\{\theta_m\}_{m=1}^{\infty}$) such that

$$\|\tau_{-\sigma_m} a - a\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{as } m \to \infty. \tag{2.21}$$

Let $\{\theta_{m_k}\}_{k=1}^{\infty}$ be a subsequence satisfying

$$|\theta_{m_k} - \sigma_k| \to 0 \quad \text{as } k \to \infty. \tag{2.22}$$

This is possible by (2.20). By boundedness of $\{v_m\}_{m=1}^{\infty}$ there is $Q \in \Lambda$ such that $v_m \rightharpoonup Q$ in E and $v_m \to Q$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^3)$ along a subsequence. Combining (2.21), (2.22) and uniform continuity of $a(t)$ we have

$$\|\tau_{-\theta_{m_k}} a - a\|_{L^\infty(\mathbb{R})} \le \|\tau_{-\theta_{m_k}} a - \tau_{-\sigma_k} a\|_{L^\infty(\mathbb{R})} + \|\tau_{-\sigma_k} a - a\|_{L^\infty(\mathbb{R})} \to 0 \tag{2.23}$$

as $k \to \infty$. For all $f \in C_0^\infty(\mathbb{R}, \mathbb{R}^3)$ we have

$$\begin{aligned}\nabla I(Q) \cdot f &= \lim_{k\to\infty} \nabla I(v_{m_k}) \cdot f = \lim_{k\to\infty} \nabla I(\tau_{\theta_{m_k}} a, v_{m_k}) \cdot f \\ &= \lim_{k\to\infty} \int_{\mathbb{R}} \left(\dot{v}_{m_k}(t) \cdot \dot{f}(t) - \tau_{\theta_{m_k}} a(t) \nabla W(v_{m_k}(t)) \cdot f(t)\right) dt \\ &= \lim_{k\to\infty} \int_{\mathbb{R}} \left(\dot{q}_{m_k}(t - \theta_{m_k}) \cdot \dot{f}(t) - a(t - \theta_{m_k}) \nabla W(q_{m_k}(t - \theta_{m_k})) \cdot f(t)\right) dt \\ &= \lim_{k\to\infty} \int_{\mathbb{R}} \left(\dot{q}_{m_k}(t) \cdot \dot{f}(t + \theta_{m_k}) - a(t) \nabla W(q_{m_k}(t)) \cdot f(t + \theta_{m_k})\right) dt \\ &= \lim_{k\to\infty} \nabla I(q_{m_k}) \tau_{-\theta_{m_k}} f = 0\end{aligned}$$

by (2.23), Lemma 2.7 and (2.18). Analogously to Case 1 we get $I(Q) \le I(q)$. ∎

Lemma 2.13. *There is a constant $\rho > 0$ such that if $w \in \Lambda \setminus \{0\}$ is a solution of (1.1) then $\|w\|_{L^\infty(\mathbb{R},\mathbb{R}^3)} > \rho$.*

Proof. From (H3), there exist $\rho, \beta > 0$ such that if $|x| \le \rho$, then

$$-\nabla W(x) \cdot x \ge \beta |x|^2.$$

Suppose that $w \in \Lambda \setminus \{0\}$ is a solution of (1.1) and $\|w\|_{L^\infty(\mathbb{R},\mathbb{R}^3)} \le \rho$. Then

$$\begin{aligned} 0 = \nabla I(w) \cdot w &= \int_{\mathbb{R}} (|\dot{w}|^2 - a(t)\nabla W(w) \cdot w)\, dt \\ &\ge \int_{\mathbb{R}} (|\dot{w}|^2 + \beta a(t)|w|^2)\, dt \\ &\ge \int_{\mathbb{R}} (|\dot{w}|^2 + \beta a_0 |w|^2)\, dt > 0, \end{aligned}$$

a contradiction. ∎

THEOREM 2.14. *There is $\varepsilon_0 > 0$ such that if $q \in \Gamma^\pm$ and*

$$I(q) < c^\pm + \varepsilon_0 \tag{2.24}$$

then the solution Q of (1.1) given by Theorem 2.12 lies in $\Gamma^\pm$ and $I(Q) \in [c^\pm, I(q)]$.

Proof. We will consider the case $q \in \Gamma^+$. Suppose that $Q \notin \Gamma^+$. Then $\mathrm{WN}(q) \le 0$. Let $\delta \in (0, \rho/2)$ with ρ given by Lemma 2.13. Since $Q \in E$ there is a time $T = T(\delta) > 0$ such that

$$Q(t) \in \overline{B}_\delta(0) \quad \text{for all } |t| \ge T.$$

We take $\delta > 0$ sufficiently small such that $\mathrm{WN}(Q) = \mathrm{WN}(Q|_{-T}^{T})$.

Let $\{q_m\}_{m=1}^\infty$ and $\{\theta_m\}_{m=1}^\infty$ be defined as in the proof of Theorem 2.12.

CASE 1: $\{\theta_m\}_{m=1}^\infty$ has a bounded subsequence. Let $Q_m(t) = q_m(t)$. Since $Q_m \rightharpoonup Q$ in E, Q_m converges to Q uniformly for $|t| \le T+1$ as $m \to \infty$. We have

$$0 < \mathrm{WN}(Q_m) = \mathrm{WN}(Q_m|_{-\infty}^{-T-1}) + \mathrm{WN}(Q_m|_{-T-1}^{T+1}) + \mathrm{WN}(Q_m|_{T+1}^{\infty}) \tag{2.25}$$

and

$$\mathrm{WN}(Q_m|_{-T-1}^{T+1}) = \mathrm{WN}(Q|_{-T-1}^{T+1}) = \mathrm{WN}(Q) < 0 \tag{2.26}$$

for m large enough. Since $\mathrm{WN}(Q) < 0$, by (2.25) and (2.26), $\mathrm{WN}(Q_m|_{-\infty}^{-T-1}) > 0$ or $\mathrm{WN}(Q_m|_{T+1}^{\infty}) > 0$. Without loss of generality suppose $\mathrm{WN}(Q_m|_{-\infty}^{-T-1}) > 0$. Define a new function by

$$\hat{q}_m(t) = \begin{cases} Q_m(t) & \text{for } t \le -T-1, \\ -(t+T)Q_m(-T-1) & \text{for } -T-1 < t \le -T, \\ 0 & \text{for } t \ge -T. \end{cases} \tag{2.27}$$

We have $\mathrm{WN}(\hat{q}_m) > 0$ and thus $\hat{q}_m \in \Gamma^+$. We obtain

$$\begin{aligned} I(\hat{q}_m) &= \int_{-\infty}^{-T-1} \mathcal{L}(\hat{q}_m)\,dt + \int_{-T-1}^{-T} \mathcal{L}(\hat{q}_m)\,dt + \int_{-T}^{\infty} \mathcal{L}(\hat{q}_m)\,dt \\ &= I(Q_m) + \int_{-T-1}^{-T} \mathcal{L}(\hat{q}_m)\,dt - \int_{-T-1}^{\infty} \mathcal{L}(Q_m)\,dt \\ &< c^+ + \varepsilon_0 + \int_{-T-1}^{-T} \mathcal{L}(\hat{q}_m)\,dt - \int_{-T-1}^{T+1} \mathcal{L}(Q_m)\,dt. \end{aligned}$$

For $t \in [-T-1, -T]$ we have

$$\begin{aligned} \mathcal{L}(\hat{q}_m) = \frac{1}{2}\left|\frac{d}{dt}\left[(t+T)Q_m(-T-1)\right]\right|^2 \\ - a(t)W(-(t+T)Q_m(-T-1)) \end{aligned}$$

and

$$-(t+T)Q_m(-T-1) \in \overline{B}_\delta(0), \tag{2.28}$$

for m large enough. Using (2.28) we obtain

$$\begin{aligned} \int_{-T-1}^{-T} \mathcal{L}(\hat{q}_m)\,dt &= \tfrac{1}{2}|Q_m(-T-1)|^2 - \int_{-T-1}^{-T} a(t)W(-(t+T)Q_m(-T-1))\,dt \\ &\le \tfrac{1}{2}\delta^2 - a_1 \int_{-T-1}^{-T} W(-(t+T)Q_m(-T-1))\,dt. \end{aligned} \tag{2.29}$$

Applying (H3) and the Maclaurin formula for W we get

$$W(x) = W(0) + W'(0)(x) + \tfrac{1}{2}W''(\xi)(x,x) = O(|x|^2), \tag{2.30}$$

where ξ is an intermediate point between 0 and x. Combining (2.28)–(2.30) we have

$$\int_{-T-1}^{-T} \mathcal{L}(\hat{q}_m)dt = O(\delta^2).$$

Thus

$$I(\hat{q}_m) < c^+ + \varepsilon_0 - \int_{-T-1}^{T+1} \mathcal{L}(Q_m)\,dt + O(\delta^2) \quad \text{as } \delta \to 0. \tag{2.31}$$

Since $Q_m \to Q \in \Lambda$ uniformly for $t \in [-T-1, T+1]$, by Lemma 2.13 we conclude that in this interval the curve Q_m passes from $\partial B_\delta(0)$ to $\partial B_\rho(0)$

and finally back to $\partial B_\delta(0)$. Therefore by Lemma 2.2,

$$\int_{-T-1}^{T+1} \mathcal{L}(Q_m)\,dt \geq \frac{\rho}{2}\sqrt{2a_0\gamma\left(\frac{\rho}{2}\right)} \equiv \varepsilon_1 \tag{2.32}$$

with $\gamma(\rho/2) := \inf_{|x|>\rho/2} -W(x)$. Combining (2.31) with (2.32) gives

$$I(\hat{q}_m) < c^+ + \varepsilon_0 - \varepsilon_1 + O(\delta^2) \quad \text{as } \delta \to 0.$$

Hence by choosing $\varepsilon_0 := \varepsilon_1/2$ and δ sufficiently small we have

$$I(\hat{q}_m) < c^+,$$

contrary to $\hat{q}_m \in \Gamma^+$. Thus $Q \in \Gamma^+$, and hence $I(Q) \geq c^+$.

Case 2: $\{\theta_m\}_{m=1}^\infty$ has no bounded subsequence. Define $Q_k = v_{m_k} = \tau_{\theta_{m_k}} q_{m_k}$, where $\{\theta_{m_k}\}_{k=1}^\infty$ is the subsequence introduced in Theorem 2.12, case 2. Then

$$\begin{aligned} I(Q_k) = I(\tau_{\theta_{m_k}} q_{m_k}) &= \int_{\mathbb{R}} \left(\tfrac{1}{2}|\tau_{\theta_{m_k}}\dot{q}_{m_k}|^2 - a(t)W(\tau_{\theta_{m_k}} q_{m_k})\right) dt \\ &= \int_{\mathbb{R}} \left(\tfrac{1}{2}|\dot{q}_{m_k}|^2 - \tau_{-\theta_{m_k}} a(t)W(q_{m_k})\right) dt \\ &= I(q_{m_k}) + \int_{\mathbb{R}} \left(a(t) - \tau_{-\theta_{m_k}} a(t)\right) W(q_{m_k}) dt < c^+ + \varepsilon_0 \end{aligned}$$

for large k, due to the uniform L^∞ bounds on $\{q_{m_k}\}_{k=1}^\infty$ given by Lemma 2.4 and the estimate (2.23). The rest of the proof is similar to the first case. ∎

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References

[1] A. Ambrosetti and V. Coti Zelati, *Periodic Solutions of Singular Lagrangian Systems*, Progr. Nonlinear Differential Equations Appl. 10, Birkhäuser Boston, Boston, MA, 1993.

[2] M. L. Bertotti and L. Jeanjean, *Multiplicity of homoclinic solutions for singular second-order conservative systems*, Proc. Roy. Soc. Edinburgh Sect. A 126 (1996), 1169–1180.

[3] A. S. Besicovitch, *Almost Periodic Functions*, Dover Publ., 1954.

[4] M. J. Borges, *Heteroclinic and homoclinic solutions for a singular Hamiltonian system*, Eur. J. Appl. Math. 17 (2006), 1–32.

[5] S. V. Bolotin, *Shadowing chains of collision orbits*, Discrete Contin. Dynam. Systems 14 (2006), 235–260.

[6] P. Caldiroli and L. Jeanjean, *Homoclinics and heteroclinics for a class of conservative singular Hamiltonian systems*, J. Differential Equations 136 (1997), 76–114.

[7] P. Caldiroli and M. Nolasco, *Multiple homoclinic solutions for a class of autonomous singular systems in* $\mathbb{R}^2$, Ann. Inst. H. Poincaré Anal. Non Linéaire 15 (1998), 113-125.

[8] A. Capozzi, C. Greco and A. Salvatore, *Lagrangian systems in the presence of singularities*, Proc. Amer. Math. Soc. 102 (1988), 125–130.

[9] V. Coti Zelati and P. H. Rabinowitz, *Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials*, J. Amer. Math. Soc. 4 (1991), 693–727.

[10] W. B. Gordon, *Conservative dynamical systems involving strong forces*, Trans. Amer. Math. Soc. 204 (1975), 113–135.

[11] C. Greco, *Periodic solutions of a class of singular Hamiltonian systems*, Nonlinear Anal. 12 (1988), 259–269.

[12] M. Izydorek and J. Janczewska, *Heteroclinic solutions for a class of the second order Hamiltonian systems*, J. Differential Equations 238 (2007), 381–393.

[13] M. Izydorek and J. Janczewska, *Connecting orbits for a periodically forced singular planar Newtonian system*, J. Fixed Point Theory Appl. 12 (2012), 59–67.

[14] M. Izydorek and J. Janczewska, *The shadowing chain lemma for singular Hamiltonian systems involving strong forces*, Cent. Eur. J. Math. 10 (2012), 1928–1939.

[15] J. Janczewska, *The existence and multiplicity of heteroclinic and homoclinic orbits for a class of singular Hamiltonian systems in* $\mathbb{R}^2$, Boll. Un. Mat. Ital. (9) 3 (2010), 471–491.

[16] J. Janczewska and J. Maksymiuk, *Homoclinic orbits for a class of singular second order Hamiltonian systems in* $\mathbb{R}^3$, Cent. Eur. J. Math. 10 (2012), 1920–1927.

[17] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer, 1989.

[18] P. H. Rabinowitz, *Homoclinic orbits for a singular Hamiltonian system*, in: Geometric Analysis and the Calculus of Variations, J. Jost (ed.), Int. Press, Cambridge, MA, 1996.

[19] P. H. Rabinowitz, *Homoclinics for an almost periodically forced singular Hamiltonian system*, Topol. Methods Nonlinear Anal. 6 (1995), 49–66.

[20] E. Serra, M. Tarallo and S. Terracini, *On the existence of homoclinic solutions for almost periodic second order systems*, Ann. Inst. H. Poincaré Anal. Non Linéaire 13 (1996), 783–812.

[21] K. Tanaka, *Homoclinic orbits for a singular second order Hamiltonian system*, Ann. Inst. H. Poincaré Anal. Non Linéaire 7 (1990), 427–438.

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