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Gradient estimates for the $p(x)$-Laplacian equation in $\mathbb{R}^N$

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Abstract. Under some assumptions on the function $p(x)$, we obtain global gradient estimates for weak solutions of the $p(x)$-Laplacian type equation in $\mathbb{R}^N$.

1. Introduction. In this paper we study the gradient estimates of weak solutions for the following $p(x)$-Laplacian type equation:

$$\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \operatorname{div}(|\mathbf{f}|^{p(x)-2}\mathbf{f}) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N \geq 2$, the variable exponent $p : \mathbb{R}^N \to (1, N)$ is a continuous function, and $\mathbf{f} = (f^1, \ldots, f^N)$ with $|\mathbf{f}|^{p(x)}$ belonging to $L^q(\mathbb{R}^N)$ $(q \geq 1)$.

Differential equations and variational problems with nonstandard growth conditions arouse much interest with the development of elastic mechanics, image processing, electro-rheological fluid dynamics, etc. We refer the readers to [AM1, CLR, RR, R] and the references therein. Elliptic equations of the type (1.1) are simplified versions of equations which arise naturally in the mathematical modeling of electro-rheological fluids developed by Rajagopal and Růžička [RR]. These are particular non-Newtonian fluids, characterized by their ability of changing their mechanical properties when interacting with an electromagnetic field $\mathbf{E}(x)$. Their viscosity strongly depends on external electromagnetic fields and therefore varies in space and time.

To introduce the main result we need some notation and assumptions on the variable exponent $p(x)$. We set

$$B_\rho(y) = \{x \in \mathbb{R}^N : |x-y| < \rho\}.$$

If $y = 0$, we write $B_\rho = B_\rho(0)$ for simplicity. The integral average of $f \in L^1(E)$ on a bounded subset E of $\mathbb{R}^N$ is defined by

$$f_E = \fint_E f(x)\,dx = \frac{1}{|E|}\int_E f(x)\,dx.$$

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Throughout this paper, we assume that $p(x)$ satisfies the *strong log-Hölder* condition

$$|p(x)-p(y)| \le w(|x-y|), \qquad \lim_{R\to 0} w(R)\log(1/R) = 0, \tag{1.2}$$

where $w : \mathbb{R}^+ \to \mathbb{R}^+$ denotes the modulus of continuity of $p(x)$.

Denote by

$$p^*(\cdot) = \frac{Np(\cdot)}{N-p(\cdot)}$$

the Sobolev conjugate of $p(\cdot)$ with $\sup_{x\in\mathbb{R}^N} p(x) < N$. As usual, solutions of (1.1) are taken in a weak sense. We use the following classical definition of a weak solution.

DEFINITION 1.1. We say that $u \in D^{p(\cdot)}(\mathbb{R}^N)$ is a weak solution of (1.1) in $\mathbb{R}^N$ if for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} |\nabla u|^{p(x)-2}\nabla u\cdot\nabla\varphi\,dx = \int_{\mathbb{R}^N} |\mathbf{f}|^{p(x)-2}\mathbf{f}\cdot\nabla\varphi\,dx,$$

where

$$D^{p(\cdot)}(\mathbb{R}^N) = \{u \in L^{p^*(\cdot)}(\mathbb{R}^N) \mid \nabla u \in L^{p(\cdot)}(\mathbb{R}^N)\}.$$

In this paper we are interested in studying how the regularity of $|\mathbf{f}|^{p(x)}$ reflects in the solutions. When $p(x)$ is a constant function, DiBenedetto and Manfredi [DM] have obtained the estimate

$$\int_{\mathbb{R}^N} |\nabla u|^q\,dx \le C\int_{\mathbb{R}^N} |\mathbf{f}|^q\,dx$$

if $\mathbf{f} \in L^q(\mathbb{R}^N,\mathbb{R}^m)$ for any $q \ge p$, where C is a constant independent of u and $\mathbf{f}$. In fact, there have been a large number of results on local and global L^q estimates for the gradients of solutions of general quasilinear elliptic systems of p-Laplacian type with variable coefficients where the domain is a bounded domain in $\mathbb{R}^N$; see [BW, BWZ, BYZ, CP, KZ1, KZ2], etc. The main approaches are based either on the method of approximation developed by Caffarelli and Peral [CP] within the maximal function technique, or the so-called maximal function free technique which was first introduced by Acerbi and Mingione [AM3]. The latter is a purely PDE method and it is suitable for situations in which scaling in time and space differs, as is the case for p-Laplacian parabolic equations and systems. However, in most results, this approach is very technical and delicate and it cannot be applied to equation (1.1) due to the existence of variable exponents and the anisotropy of $p(x)$-Laplacian operators. Moreover, recently similar local gradient estimates for (1.1) have been obtained by Acerbi and Mingione [AM2]. Under the condition that $p(x)$ satisfies the strong log-Hölder condition (1.2), they proved that

$$\Big(\fint_{Q_R} |\nabla u|^{p(x)q}\, dx \Big)^{1/q} \le CK^{\varepsilon} \fint_{Q_{4R}} |\nabla u|^{p(x)}\, dx + CK^{\varepsilon} \Big(\fint_{Q_{4R}} |\mathbf{f}|^{p(x)q}\, dx + 1 \Big)^{1/q},$$

where Q_R is a cube with side length $2R$, $\varepsilon \in (0, q-1)$ and

$$K := \int_{Q_{4R}} (|\nabla u|^{p(x)} + |\mathbf{f}|^{p(x)(1+\varepsilon)})\, dx + 1.$$

The methods rely on Calderón–Zygmund type covering arguments and iteration of level sets combined with a careful localization technique, fine estimates in $L \log^{\beta} L$ spaces and the use of certain restricted maximal operators.

In this paper we revisit the maximal function technique introduced in [DM] to find a version of L^q regularity results for (1.1) in the whole space $\mathbb{R}^N$. The main difficulties are that (1.1) has more complicated nonlinearities than the usual p-Laplacian equation. An essential difference is that the p-Laplacian operator is $(p-1)$-homogeneous, that is, $\Delta_p(\lambda u) = \lambda^{p-1} \Delta_p u$ for every $\lambda > 0$, but the $p(x)$-Laplacian operator, when $p(x)$ is not a constant, is not homogeneous. To overcome this, we need to make strong assumptions on $p(x)$ and make use of appropriate localization techniques and estimates in $L \log^{\beta} L$ spaces, as in [AM2]. Here we employ the reverse Hölder inequality which gives a better regularity of solutions, which in fact can compensate the lack of compactness of weak solutions.

Now we state our main result.

THEOREM 1.2. *Let $q > 1$ be a real number and let $p(x)$ satisfy the strong log-Hölder condition* (1.2). *For all $\mathbf{f}$ with $|\mathbf{f}|^{p(x)} \in L^q(\mathbb{R}^N)$ and such that there exist positive numbers r_0 and c_0 such that*

$$|\mathbf{f}(x)| \ge c_0, \qquad \forall x \in B_{r_0}(0), \tag{1.3}$$

if $u \in D^{p(\cdot)}(\mathbb{R}^N)$ is a weak solution of (1.1) *and $|\nabla u|^{p(x)}$ belongs to $L^q(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} |\nabla u|^{p(x)q}\, dx \le CK^{\sigma q} \int_{\mathbb{R}^N} |\mathbf{f}|^{p(x)q}\, dx, \tag{1.4}$$

where $C = C(N, p(\cdot), q, c_0)$, $\sigma \in (0, q-1)$ is some fixed constant and

$$K = \int_{\mathbb{R}^N} (|\nabla u|^{p(x)} + |\mathbf{f}|^{p(x)q})\, dx + 1.$$

This paper is organized as follows. In Section 2, we first recall some properties of generalized Lebesgue–Sobolev spaces and then state some preliminary tools and known results to be used later. We prove Theorem 1.2 in Section 3.

2. Preliminaries

2.1. Generalized Lebesgue–Sobolev spaces. We first recall some definitions and basic properties of the generalized Lebesgue spaces $L^{p(\cdot)}(\Omega)$ and generalized Lebesgue–Sobolev spaces $W^{k,p(\cdot)}(\Omega)$. We refer to [DHHR, FZ, KR] for more details.

Set $C_+(\bar{\Omega}) = \{h \in C(\bar{\Omega}) : \min_{x\in\bar{\Omega}} h(x) > 1\}$. For any $h \in C_+(\bar{\Omega})$ we define

$$h_+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h_- = \inf_{x\in\Omega} h(x).$$

For any $p \in C_+(\bar{\Omega})$, we introduce the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions such that

$$\int_\Omega |u(x)|^{p(x)}\,dx < \infty,$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\Big\{\lambda > 0 : \int_\Omega |u(x)/\lambda|^{p(x)}\,dx \le 1\Big\};$$

this is a separable and reflexive Banach space. The dual space of $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. If $p(x)$ is a constant function, then the variable exponent Lebesgue space coincides with the classical Lebesgue space. Variable exponent Lebesgue spaces are special cases of Orlicz–Musielak spaces [M].

For any positive integer k, denote

$$W^{k,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : D^\alpha u \in L^{p(\cdot)}(\Omega),\ |\alpha| \le k\},$$

where the norm is defined as

$$\|u\|_{W^{k,p(\cdot)}(\Omega)} = \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^{p(\cdot)}(\Omega)}.$$

$W^{k,p(\cdot)}(\Omega)$ is called a *generalized Lebesgue–Sobolev space*, which is a special generalized Orlicz–Sobolev space. An interesting feature of this space is that smooth functions are not dense in it without additional assumptions on the exponent $p(x)$. This was observed by Zhikov [Z] in connection with the Lavrent'ev phenomenon. However, when the exponent $p(x)$ satisfies the log-Hölder continuity condition, i.e., there is a constant C such that

$$|p(x) - p(y)| \le \frac{C}{-\log|x-y|} \tag{2.1}$$

for all $x, y \in \Omega$ with $|x - y| \le 1/2$, then smooth functions are dense in variable exponent Sobolev spaces, and the Sobolev space with zero boundary values, $W_0^{1,p(\cdot)}(\Omega)$, can be defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{1,p(\cdot)}(\Omega)}$ (see [Har]). It is obvious that the strong log-Hölder continuity (1.2) of $p(x)$ implies that the log-Hölder continuity (2.1) holds.

Lemma 2.1 ([FZ]). *Denote*

$$\rho(u) = \int_{\Omega} |u|^{p(x)}\, dx, \qquad \forall u \in L^{p(\cdot)}(\Omega).$$

Then

$$\min\{\|u\|^{p_-}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p_+}_{L^{p(\cdot)}(\Omega)}\} \le \rho(u) \le \max\{\|u\|^{p_-}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p_+}_{L^{p(\cdot)}(\Omega)}\}.$$

Lemma 2.2 ([Die, FZZ]). *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary and $p \in C_+(\bar{\Omega})$ with $1 < p_- \le p_+ < N$ satisfy the log-Hölder continuity condition* (2.1). *If $r \in L^\infty(\Omega)$ with $r_- > 1$ satisfies*

$$r(x) \le p^*(x) := \frac{Np(x)}{N - p(x)} \quad \text{for all } x \in \Omega,$$

then

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega),$$

and the imbedding is compact if $\inf_{x\in\Omega}(p^*(x) - r(x)) > 0$.

Lemma 2.3 ([Has, Theorem 3.4]). *Suppose that $p(x)$ satisfies the log-Hölder continuity condition* (2.1) *with $1 \le p(x) \le c < N$ in $\mathbb{R}^N$. Then the following continuous embedding holds:*

$$W^{1,p(\cdot)}(\mathbb{R}^N) \hookrightarrow L^{p^*(\cdot)}(\mathbb{R}^N).$$

2.2. Maximal function. For a locally integrable function f defined on $\mathbb{R}^N$, we define its *maximal function* $(\mathcal{M}f)(x)$ as

$$\mathcal{M}(f)(x) = \sup_{r>0} \fint_{B_r(x)} |f(y)|\, dy.$$

Let also $f^{\#}(\cdot)$ denote the *sharp maximal function* defined by

$$f^{\#}(x) = \sup_{r>0} \fint_{B_r(x)} |f(y) - f_{B_r}|\, dy.$$

From the above definitions, we can see that $|f| \le \mathcal{M}(f)$ and $f^{\#} \le 2\mathcal{M}(f)$. The basic properties of the maximal operators are the following. The first inequality below is the maximal function theorem of Hardy, Littlewood and Wiener. The second inequality is due to Fefferman and Stein.

Lemma 2.4 ([FS, S]).

(1) *If $f \in L^p(\mathbb{R}^N)$ with $1 < p \le \infty$, then $\mathcal{M}f \in L^p(\mathbb{R}^N)$ and*

$$\frac{1}{C(N,p)}\|f\|_{L^p} \le \|\mathcal{M}f\|_{L^p} \le C(N,p)\|f\|_{L^p}.$$

If $f \in L^1(\mathbb{R}^N)$, then

$$|\{x \in \mathbb{R}^N : (\mathcal{M}f)(x) > t\}| \le \frac{C(N)}{t}\int |f(x)|\, dx.$$

(2) *If* $f \in L^p(\mathbb{R}^N)$ *with* $1 < p \le \infty$, *then* $f^{\#}(x) \in L^p(\mathbb{R}^N)$ *and*

$$\frac{1}{C(N,p)}\|f\|_{L^p} \le \|f^{\#}\|_{L^p} \le C(N,p)\|f\|_{L^p}.$$

REMARK 2.5. In the same way, if $s \ge 1$ we define

$$\mathcal{M}_s(f)(x) = \sup_{r>0}\Big(\fint_{B_r(x)} |f(y)|^s \, dy\Big)^{1/s}$$

whenever $f \in L^s(\mathbb{R}^N)$. From [I, Theorem 7.1], we have the estimate

$$\int_{\mathbb{R}^N} |\mathcal{M}_s(f)(y)|^q \, dy \le \frac{C(N)q^2}{s(q-s)} \int_{\mathbb{R}^N} |f(y)|^q \, dy, \qquad \forall q > s. \tag{2.2}$$

2.3. The spaces $L\log^\beta L(\Omega)$. In this subsection we state some properties and inequalities in the space $L\log^\beta L(\Omega)$, $\beta \ge 1$, which are taken from [AIKM, I, IV].

The Orlicz space

$$L\log^\beta L(\Omega) := \Big\{ f \in L^1(\Omega) : \int_\Omega |f|\log^\beta(e+|f|)\, dx < \infty \Big\}$$

is a Banach space with the Luxemburg norm

$$\|f\|_{L\log^\beta L(\Omega)} := \inf\Big\{ \lambda > 0 : \fint_\Omega \Big|\frac{f}{\lambda}\Big| \log^\beta\Big(e + \Big|\frac{f}{\lambda}\Big|\Big)\, dx \le 1 \Big\}.$$

This space embeds in any $L^p(\Omega)$, that is, for any $p > 1$,

$$\|f\|_{L\log^\beta L(\Omega)} \le C\Big(\fint_\Omega |f|^p \, dx\Big)^{1/p}, \qquad \forall f \in L^p(\Omega), \tag{2.3}$$

where the constant C only depends on p, and blows up as $p \to 1$. Set

$$[f]_{L\log^\beta L(\Omega)} := \fint_\Omega |f|\log^\beta\Big(e + \frac{|f|}{|f|_\Omega}\Big)\, dx,$$

where

$$|f|_\Omega := \fint_\Omega |f| \, dx.$$

Here we recall a fact, basically due to T. Iwaniec [AIKM, I, IV]: There exists a constant $C = C(\beta) \ge 1$ such that

$$C^{-1}\|f\|_{L\log^\beta L(\Omega)} \le [f]_{L\log^\beta L(\Omega)} \le C\|f\|_{L\log^\beta L(\Omega)} \tag{2.4}$$

for all $f \in L\log^\beta L(\Omega)$. We shall need these inequalities for the range

$$\frac{p_+}{p_+ - 1} \le \beta \le \frac{p_-}{p_- - 1}. \tag{2.5}$$

Therefore, since the constant appearing in (2.4) is continuous with respect to $\beta > 0$ (see [AIKM, IV]), we may assume that the constant C in (2.4) only depends on p_- and p_+, and is valid for the full range in (2.5). Moreover, we know that, for every $f \in L\log^\beta L(\Omega)$ and β as in (2.5),

$$\fint_\Omega |f|\log^\beta\Big(e + \frac{|f|}{|f|_\Omega}\Big)\,dx \le C(p,\beta)\Big(\fint_\Omega |f|^p\,dx\Big)^{1/p}, \tag{2.6}$$

where

$$C(p,\beta) \approx \Big(\frac{1}{p-1}\Big)^\beta.$$

In particular,

$$(e+t)\log^\beta(e+t) \le C(p_-,p_+)\sigma^{-\beta}(e+t)^{1+\sigma/8}, \qquad \forall t \ge 0, \tag{2.7}$$

for every β satisfying (2.5) and every $0 < \sigma < 1$. Finally, it is obvious that

$$\log(e+ab) \le \log(e+a) + \log(e+b),$$

where a and b are positive real numbers. Therefore

$$\log^\beta(e+ab) \le 2^{\frac{p_-}{p_--1}-1}[\log^\beta(e+a) + \log^\beta(e+b)] \tag{2.8}$$

whenever β satisfies the right inequality in (2.5).

2.4. The reference problem with constant exponent. Consider the reference boundary value problem

$$\begin{cases} -\operatorname{div}(|\nabla v|^{p_*-2}\nabla v) = 0 & \text{in } B_R, \\ v = u & \text{on } \partial B_R, \end{cases} \tag{2.9}$$

where $p_* > 1$ is some fixed constant and the boundary value $u \in W^{1,p_*}(B_R)$ is some known function.

The next lemma can be found in [DM] and [L].

LEMMA 2.6. *Let v be the unique solution of problem* (2.9). *Then*

(1) $v \in C^{1,\gamma}_{\mathrm{loc}}(B_R)$ *with* $\gamma \in (0,1)$ *and*

$$\|\nabla v\|_{L^\infty(B_\rho)} \le C\Big(\fint_{B_R} |\nabla v|^{p_*}\,dx\Big)^{1/p_*}, \qquad \forall \rho \in (0, R/2), \tag{2.10}$$

$$\fint_{B_\rho} |\nabla v - (\nabla v)_{B_\rho}|^{p_*}\,dx \le C\Big(\frac{\rho}{R}\Big)^\gamma \fint_{B_R} |\nabla v - (\nabla v)_{B_R}|^{p_*}\,dx, \qquad \forall \rho \in (0,R), \tag{2.11}$$

$$\int_{B_R} |\nabla v|^{p_*}\,dx \le C\int_{B_R} |\nabla u|^{p_*}\,dx, \tag{2.12}$$

$$\sup_{B_R} |u - v| \le \operatorname{osc}_{B_R} u, \tag{2.13}$$

where $(\nabla v)_{B_\rho} = \fint_{B_\rho} \nabla v\,dx$ *and* C *are positive constants depending only on* p_* *and* N.

(2) *For any* $0 < \rho < R$,

$$\fint_{B_\rho} |\nabla v - (\nabla v)_{B_\rho}|^2\,dx \le C\left(\frac{\rho}{R}\right)^{2\gamma} \fint_{B_R} |\nabla v - (\nabla v)_{B_R}|^2\,dx. \tag{2.14}$$

Remark 2.7. The proof of the oscillation estimate (2.14) can be found in [DM, §7, part II].

2.5. Existence and uniqueness of the weak solution. We end this section by proving the existence and uniqueness of weak solutions for (1.1) when $|\mathbf{f}|^{p(x)}$ belongs to $L^1(\mathbb{R}^N)$. Some of the ideas are based on personal communications with J. Manfredi.

Proposition 2.8. *If* $|\mathbf{f}|^{p(x)} \in L^1(\mathbb{R}^N)$, *then there is a unique weak solution* $u \in D^{p(\cdot)}(\mathbb{R}^N)$ *of* (1.1) *such that*

$$\int_{\mathbb{R}^N} |\nabla u|^{p(x)}\,dx \le C \int_{\mathbb{R}^N} |\mathbf{f}|^{p(x)}\,dx, \tag{2.15}$$

where the constant C *depends only on* p_- *and* p_+.

Proof. We will divide the proof into the following steps.

Step 1. Fix a positive integer $k \in \mathbb{N}$. We consider the variational functional

$$I[w] = \int_{B_k} \frac{1}{p(x)} |\nabla w|^{p(x)}\,dx - \int_{B_k} |\mathbf{f}|^{p(x)-2}\mathbf{f} \cdot \nabla w\,dx$$

in the variable exponent Sobolev spaces $W_0^{1,p(\cdot)}(B_k)$. Then by the classical calculus of variations, one can show the existence and uniqueness of a minimizer of $I[\cdot]$ over $W_0^{1,p(\cdot)}(B_k)$, and the minimizer u_k is a weak solution of (1.1) in B_k with the estimate

$$\int_{B_k} |\nabla u_k|^{p(x)}\,dx \le C(p_-,p_+) \int_{B_k} |\mathbf{f}|^{p(x)}\,dx \le C(p_-,p_+) \int_{\mathbb{R}^N} |\mathbf{f}|^{p(x)}\,dx. \tag{2.16}$$

Step 2. Using Lemmas 2.2, 2.1 and (2.16), we have

$$\begin{aligned}
\|u_k\|_{L^{p^*(\cdot)}(B_k)} &\le C(N,p_-,p_+)\|\nabla u_k\|_{L^{p(\cdot)}(B_k)} \\
&\le C(N,p_-,p_+) \max\Big\{\Big(\int_{B_k} |\nabla u_k|^{p(x)}\,dx\Big)^{1/p_-}, \Big(\int_{B_k} |\nabla u_k|^{p(x)}\,dx\Big)^{1/p_+}\Big\} \\
&\le C(N,p_-,p_+) \max\Big\{\Big(\int_{\mathbb{R}^N} |\mathbf{f}|^{p(x)}\,dx\Big)^{1/p_-}, \Big(\int_{\mathbb{R}^N} |\mathbf{f}|^{p(x)}\,dx\Big)^{1/p_+}\Big\}.
\end{aligned}$$

Step 3. Extending u_k to $\mathbb{R}^N$ by setting it to be 0 outside B_k for each $k \in \mathbb{N}$, we obtain a sequence $\{u_k\} \subset W^{1,p(\cdot)}(\mathbb{R}^N)$ of approximate solutions such that

$$\int_{\mathbb{R}^N} |\nabla u_k|^{p(x)}\,dx \le C(p_-,p_+) \int_{\mathbb{R}^N} |\mathbf{f}|^{p(x)}\,dx$$

and

$$\|u_k\|_{L^{p^*(\cdot)}(\mathbb{R}^N)} \le C(N,p_-,p_+) \max\Big\{\Big(\int_{\mathbb{R}^N} |\mathbf{f}|^{p(x)}\,dx\Big)^{1/p_-}, \Big(\int_{\mathbb{R}^N} |\mathbf{f}|^{p(x)}\,dx\Big)^{1/p_+}\Big\}.$$

Step 4. Fixing a positive integer $j \in \mathbb{N}$, in a similar way we can obtain

$$\|u_k\|_{W^{1,p(\cdot)}(B_j)} \le C(j,N,p_-,p_+) \max\Big\{\Big(\int_{\mathbb{R}^N} |\mathbf{f}|^{p(x)}\,dx\Big)^{1/p_-}, \Big(\int_{\mathbb{R}^N} |\mathbf{f}|^{p(x)}\,dx\Big)^{1/p_+}\Big\}.$$

Step 5. Using a diagonal argument and Lemma 2.2, we find a function $u \in W^{1,p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^N)$ and a subsequence of $\{u_k\}$ (still denoted by $\{u_k\}$) such that

$$\begin{aligned} u_k &\to u && \text{strongly in } L^{p(\cdot)}(B_j),\\ u_k &\rightharpoonup u && \text{weakly in } L^{p^*(\cdot)}(B_j),\\ \nabla u_k &\rightharpoonup \nabla u && \text{weakly in } L^{p(\cdot)}(B_j), \end{aligned}$$

for each fixed $j \in \mathbb{N}$.

Step 6. Invoking the lower semicontinuity of the $L^{p(\cdot)}$-norm, we conclude that

$$\int_{\mathbb{R}^N} |\nabla u|^{p(x)}\,dx \le C(p_-,p_+) \int_{\mathbb{R}^N} |\mathbf{f}|^{p(x)}\,dx$$

and

$$\|u\|_{L^{p^*(\cdot)}(\mathbb{R}^N)} \le C(N,p_-,p_+) \max\Big\{\Big(\int_{\mathbb{R}^N} |\mathbf{f}|^{p(x)}\,dx\Big)^{1/p_-}, \Big(\int_{\mathbb{R}^N} |\mathbf{f}|^{p(x)}\,dx\Big)^{1/p_+}\Big\}.$$

Step 7. Applying the convergence in Step 5 and recalling u_k is a solution of an approximate equation, we easily prove that $u \in D^{p(\cdot)}(\mathbb{R}^N)$ is a weak solution of (1.1). It is obviously unique since it is an admissible test function in the weak form. ∎

3. Global gradient estimates in $\mathbb{R}^N$. We start with the following reverse Hölder inequality which rests on an application of Gehring's lemma as in [AM2, Theorem 5].

Lemma 3.1 (Reverse Hölder inequality). *Let $u \in W^{1,p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^N)$ be a weak solution of* (1.1) *and $|\mathbf{f}|^{p(x)} \in L^q_{\mathrm{loc}}(\mathbb{R}^N)$ with $q > 1$. There exist constants $c \equiv c(N,p_-,p_+)$ and $\tilde{c} \equiv \tilde{c}(N,p_-,p_+)$ such that the following is true: Assume R_0 satisfies*

$$w(4R_0) \le \sqrt{\frac{N+1}{N}} - 1, \qquad 0 < w(R)\log\Big(\frac{1}{R}\Big) \le L, \qquad \forall R \le 4R_0.$$

Set

$$K_0 = \int_{B_{4R_0}} |\nabla u|^{p(x)}\,dx + 1$$

and let $\sigma > 0$ *be any number such that*

$$\sigma \le \min\left\{\frac{\tilde{c}}{K_0^{2qw(4R_0)/p_-}}, q-1, 1\right\} =: \sigma_0. \tag{3.1}$$

Then for every $B_R \subset B_{4R_0}$,

$$\left(\fint_{B_{R/2}} |\nabla u|^{p(x)(1+\sigma)}\,dx\right)^{\frac{1}{1+\sigma}} \le c\fint_{B_R} |\nabla u|^{p(x)}\,dx + c\left(\fint_{B_R} |\mathbf{f}|^{p(x)(1+\sigma)}\,dx + 1\right)^{\frac{1}{1+\sigma}}. \tag{3.2}$$

Remark 3.2 (Further restrictions on σ and R_0). We remark that since $K_0 \ge 1$ we have, for every $\tilde{K} \ge K_0$,

$$\sigma_0 \ge \min\{1, q-1, \tilde{c}\}\tilde{K}^{-2qw(4R_0)/p_-}.$$

Set

$$K := \int_{\mathbb{R}^N} (|\mathbf{f}|^{p(x)q} + |\nabla u|^{p(x)})\,dx + 1$$

(this will be larger than all the different versions of K) and

$$\sigma_m := \min\left\{\frac{\tilde{c}}{K^{2q(p_+-p_-)/p_-}}, \frac{q-1}{2}, 1\right\} > 0, \qquad \sigma_M := \tilde{c} + q.$$

Clearly, with $\tilde{K} \le K$, we have

$$\sigma_m \le \sigma_0 \le \sigma_M. \tag{3.3}$$

Now we are going to bound the maximal size of a quantity, $\sigma > 0$, that we shall later use as a higher integrability exponent. We shall pick σ of the form

$$\sigma := \tilde{\sigma}\sigma_0, \qquad 0 < \tilde{\sigma} < \min\{p_- - 1, 1/2\}, \tag{3.4}$$

where σ_0 appears in (3.1). In particular by (3.3) for all β satisfying (2.5) and all $\tilde{K} \ge K_0$,

$$\sigma^{-\beta} \le C\tilde{\sigma}^{-\beta}\tilde{K}^{\beta 2qw(4R_0)/p_-} \le C(p_-, p_+, N, q)\tilde{\sigma}^{-\beta}\tilde{K}^{2qw(4R_0)/(p_--1)}. \tag{3.5}$$

We also remark that by (3.1) and (3.4),

$$\sigma \le (q-1)/2.$$

With the size of σ initially bounded by (3.4), let us come back to the "large" ball B_{4R_0}, making further restrictions on the size of R_0, in addition to those already considered in Lemma 3.1. We shall require that

$$\max\left\{2qw(4R_0), \frac{2qw(4R_0)}{p_- - 1}\right\} \leq \frac{\tilde{\sigma}\sigma_m}{8}.$$

From (3.4) and (3.2) and the definition of σ_m it immediately follows that

$$w(4R_0) \leq \max\left\{2qw(4R_0), \frac{2qw(4R_0)}{p_- - 1}\right\} \leq \frac{\tilde{\sigma}\sigma_m}{8} \leq \frac{\tilde{\sigma}\sigma_0}{8} = \frac{\sigma}{8}. \tag{3.6}$$

We will use the following approximation lemma which plays an important role in proving our main result.

LEMMA 3.3. *Let $p(x)$ satisfy the strong log-Hölder continuity condition (1.2). For any $\eta \in (0,1)$, there exists a small $\delta = \delta(\eta)$ such that if $u \in W^{1,p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^N)$ is a weak solution of (1.1), then there exists a weak solution $v \in W^{1,p_2}(B_{R/2})$ of*

$$\begin{cases} -\operatorname{div}(|\nabla v|^{p_2-2}\nabla v) = 0 & \text{in } B_{R/2}, \\ v = u & \text{on } \partial B_{R/2} \end{cases}$$

with $p_2 = \sup_{x \in B_{R/2}} p(x)$ and $R \leq 4R_0$ satisfying

$$w(R) \leq \min\{\sigma/8, \delta\}, \qquad w(R)\log(1/R) \leq \delta, \tag{3.7}$$

where σ is a fixed constant defined as in Lemma 3.1, such that for any $0 < \rho \leq R/2$,

$$\fint_{B_\rho} |\nabla u - \nabla v|^{p_2}\, dx$$
$$\leq \eta\left(\frac{R}{\rho}\right)^N K_1^{\sigma/2} \fint_{B_R} |\nabla u|^{p(x)}\, dx + C(\eta)\left(\frac{R}{\rho}\right)^N K_1^{\sigma/2}\left[\fint_{B_R} (|\mathbf{f}|^{p(x)}+1)^{1+\sigma}\right]^{\frac{1}{1+\sigma}},$$

where

$$K_1 = \int_{B_R} (|\nabla u|^{p(x)} + |f|^{p(x)q})\, dx + 1.$$

Proof. Using the definition of weak solutions, we have

$$\int_{B_{R/2}} |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi\, dx = \int_{B_{R/2}} |\mathbf{f}|^{p(x)-2}\mathbf{f} \cdot \nabla\varphi\, dx \tag{3.8}$$

and

$$\int_{B_{R/2}} |\nabla v|^{p_2-2}\nabla v \cdot \nabla\varphi\, dx = 0 \tag{3.9}$$

for all $\varphi \in W^{1,p_2}_0(B_{R/2}) \subset W^{1,p(\cdot)}_0(B_{R/2})$.

Since $u - v \in W^{1,p(\cdot)}_0(B_{R/2})$, we substitute $\varphi = u - v$ into the identities (3.8) and (3.9), and write the resulting expression after simple computations as

$$I_1 = I_2 + I_3,$$

where

$$I_1 = \int_{B_{R/2}} (|\nabla u|^{p_2-2}\nabla u - |\nabla v|^{p_2-2}\nabla v) \cdot (\nabla u - \nabla v)\, dx,$$

$$I_2 = \int_{B_{R/2}} (|\nabla u|^{p_2-2}\nabla u - |\nabla u|^{p(x)-2}\nabla u) \cdot (\nabla u - \nabla v)\, dx,$$

$$I_3 = \int_{B_{R/2}} |\mathbf{f}|^{p(x)-2}\mathbf{f} \cdot \nabla(u - v)\, dx.$$

Next we estimate I_1–I_3 one by one.

Estimate of I_1. We consider two cases.

CASE 1: $p_2 \geq 2$. Using the elementary inequality (see [DiB, p. 13])

$$(|s|^{p_2-2}s - |t|^{p_2-2}t) \cdot (s - t) \geq C|s - t|^{p_2}$$

for all $s, t \in \mathbb{R}^N$, we have

$$I_1 \geq C \int_{B_{R/2}} |\nabla(u - v)|^{p_2}\, dx.$$

CASE 2: $1 < p_2 < 2$. Using the elementary inequality (see [KZ1, (3.23)])

$$|s - t|^{p_2} \leq C(p_2)\theta^{(p_2-2)/2}(|s|^{p_2-2}s - |t|^{p_2-2}t) \cdot (s - t) + \theta|t|^{p_2}$$

for all $s, t \in \mathbb{R}^N$ and $\theta \in (0, 1]$, we have

$$C\theta^{(p_2-2)/p_2} I_1 + \theta \int_{B_{R/2}} |\nabla v|^{p_2}\, dx \geq \int_{B_{R/2}} |\nabla(u - v)|^{p_2}\, dx,$$

that is,

$$I_1 + C\theta^{2/p_2} \int_{B_{R/2}} |\nabla v|^{p_2}\, dx \geq C\theta^{(2-p_2)/p_2} \int_{B_{R/2}} |\nabla(u - v)|^{p_2}\, dx.$$

Selecting $\tau = \theta^{2/p_2}/C$, we observe that

$$I_1 + \tau \int_{B_{R/2}} |\nabla v|^{p_2}\, dx \geq C(\tau) \int_{B_{R/2}} |\nabla(u - v)|^{p_2}\, dx.$$

Estimate of I_2. Now we will estimate I_2 based on [AM2, Step 3 of Lemma 2]. To make the paper self-contained, we repeat the proof with necessary modifications.

Observe that

$$I_2 \leq |B_{R/2}| \fint_{B_{R/2}} \big||\nabla u|^{p_2-1} - |\nabla u|^{p(x)-1}\big| \cdot |\nabla u - \nabla v|\, dx.$$

Denote

$$p_1 = \inf_{x\in B_{R/2}} p(x) \quad \text{and} \quad p_2 = \sup_{x\in B_{R/2}} p(x).$$

By the mean-value theorem and an inequality in [BB], we have, for all $x \in B_{R/2}$ and $b \geq 0$,

$$\begin{aligned}|b^{p_2-1} - b^{p(x)-1}| &\leq |p_2 - p(x)| \sup_{\alpha\in[p_1-1,p_2-1]} b^{\alpha}|\log b| \\ &\leq w(R)\left[b^{p_2-1}\log(e+b^{p_2}) + \frac{1}{e(p_- - 1)}\right],\end{aligned}$$

where we have used

$$b^{\alpha}|\log b| \leq \begin{cases} \dfrac{1}{e(p_- - 1)} & \text{for } b\in[0,1],\ \alpha\in[p_1-1,p_2-1], \\ b^{p_2-1}\log(e+b^{p_2-1}) & \text{for } b>1,\ \alpha\in[p_1-1,p_2-1]. \end{cases}$$

It follows from Hölder's inequality that

$$\begin{aligned} I_2 &\leq C|B_{R/2}|w(R) \fint_{B_{R/2}} [|\nabla u|^{p_2-1}\log(e+|\nabla u|^{p_2})+1]\cdot|\nabla u-\nabla v|\,dx \\ &\leq C|B_{R/2}|w(R)\Big(\fint_{B_{R/2}} |\nabla u|^{p_2}\log^{\frac{p_2}{p_2-1}}(e+|\nabla u|^{p_2})\,dx+1\Big)^{\frac{p_2-1}{p_2}} \\ &\quad\times\Big(\fint_{B_{R/2}} |\nabla u-\nabla v|^{p_2}\,dx\Big)^{1/p_2} \\ &\leq C|B_{R/2}|w(R)\Big(\fint_{B_{R/2}} |\nabla u|^{p_2}\log^{\frac{p_2}{p_2-1}}(e+|\nabla u|^{p_2})\,dx+1\Big)^{\frac{p_2-1}{p_2}} \\ &\quad\times\Big(\fint_{B_{R/2}} |\nabla u|^{p_2}\,dx+1\Big)^{1/p_2}. \end{aligned}$$

Since (3.4) implies that $\sigma \leq p_1 - 1$, for all $x \in B_{R/2}$ we have

$$\begin{aligned} p_2(1+\sigma/8) &\leq (p_1+w(R))(1+\sigma/8) \leq p_1(1+w(R)+\sigma/8) \\ &\leq p(x)(1+w(R)+\sigma/8) \leq p(x)(1+\sigma), \end{aligned} \tag{3.10}$$

where $w(R) \leq \sigma/8$ and σ is defined as in Lemma 3.1. Also for all $x \in B_{R/2}$,

$$\begin{aligned} p_2 &= (p_2-p_1)+p_1 \leq w(R)+p_1 \leq p_1(1+w(R)) \\ &\leq p(x)(1+w(R)) \leq p(x)(1+w(R)+\sigma/4) \\ &\leq p(x)(1+\sigma). \end{aligned} \tag{3.11}$$

Then from Lemma 3.1 we have

$$
\begin{aligned}
(3.12)\qquad \fint_{B_{R/2}} |\nabla u|^{p_2}\,dx &\le \fint_{B_{R/2}} (|\nabla u|^{p_2}+1)\,dx \\
&\overset{(3.11)}{\le} 2 \fint_{B_{R/2}} (|\nabla u|^{p(x)(1+w(R))}+1)\,dx \\
&\overset{(3.2),(3.6)}{\le} C\Big(\fint_{B_R} (|\nabla u|^{p(x)}+1)\,dx\Big)^{1+w(R)} + C\fint_{B_R} (|\mathbf{f}|^{p(x)}+1)^{1+w(R)}\,dx \\
&\le C\Big(\int_{B_R} (|\nabla u|^{p(x)}+1)\,dx\Big)^{w(R)} \cdot R^{-Nw(R)} \cdot \fint_{B_R} (|\nabla u|^{p(x)}+1)\,dx \\
&\quad + C\Big(\int_{B_R} (|\mathbf{f}|^{p(x)(1+w(R))}+1)\,dx\Big)^{\frac{w(R)}{1+w(R)}} \cdot R^{\frac{-Nw(R)}{1+w(R)}} \\
&\quad \cdot \Big(\fint_{B_R} (|\mathbf{f}|^{p(x)(1+w(R))}+1)\,dx\Big)^{\frac{1}{1+w(R)}} \\
&\overset{(3.6)}{\le} CK_1^{\sigma/8} \fint_{B_R} |\nabla u|^{p(x)}\,dx + CK_1^{\sigma/8}\Big(\fint_{B_R} (|\mathbf{f}|^{p(x)(1+\sigma)}+1)\,dx\Big)^{\frac{1}{1+\sigma}},
\end{aligned}
$$

where we have used the fact that $R^{-Nw(R)}$ stays bounded as $0 < R \le 4R_0$, $\delta < 1$, $w(R) \le \sigma/8$ and Hölder's inequality:

$$
\Big(\fint_{B_R} |\mathbf{f}|^{p(x)(1+w(R))}\,dx\Big)^{\frac{1}{1+w(R)}} \le \Big(\fint_{B_R} |\mathbf{f}|^{p(x)(1+\sigma)}\,dx\Big)^{\frac{1}{1+\sigma}}.
$$

Similarly, by rewriting the previous estimates we get

$$
\begin{aligned}
(3.13)\qquad \int_{B_{R/2}} |\nabla u|^{p_2}\,dx
&\le C\Big(\int_{B_R} (|\nabla u|^{p(x)}+1)\,dx\Big)^{w(R)} R^{-Nw(R)} \int_{B_R} (|\nabla u|^{p(x)}+1)\,dx \\
&\quad + C\int_{B_R} (|\mathbf{f}|^{p(x)(1+w(R))}+1)\,dx \\
&\le CK_1^{\sigma/8} \int_{B_R} (|\nabla u|^{p(x)} + |\mathbf{f}|^{p(x)(1+\sigma)}+1)\,dx \le CK_1^{1+\sigma/8}.
\end{aligned}
$$

Setting $\beta = p_2/(p_2-1)$, we now estimate the term

$$
I_{21} = \fint_{B_{R/2}} |\nabla u|^{p_2} \log^{\beta}(e+|\nabla u|^{p_2})\,dx + 1
$$

using the above estimates, the properties of the space $L\log^{\beta} L(\Omega)$ and

Remark 3.2, as follows:

$$
\begin{aligned}
I_{21} &= \fint_{B_{R/2}} |\nabla u|^{p_2} \log^{\beta}(e + |\nabla u|^{p_2})\, dx + 1 \\
&\overset{(2.8)}{\leq} C \fint_{B_{R/2}} |\nabla u|^{p_2} \log^{\beta}\Big(e + \frac{|\nabla u|^{p_2}}{(|\nabla u|^{p_2})_{B_{R/2}}}\Big)\, dx \\
&\quad + C \fint_{B_{R/2}} |\nabla u|^{p_2} \log^{\beta}(e + (|\nabla u|^{p_2})_{B_{R/2}})\, dx + 1 \\
&\overset{(2.6)}{\leq} C\sigma^{-\beta}\Big(\fint_{B_{R/2}} |\nabla u|^{p_2(1+\sigma/8)}\, dx\Big)^{\frac{1}{1+\sigma/8}} \\
&\quad + C\log^{\beta}\Big(eR^{-N} + R^{-N}\int_{B_{R/2}} |\nabla u|^{p_2}\, dx\Big) \fint_{B_{R/2}} |\nabla u|^{p_2}\, dx + 1 \\
&\overset{(3.10)}{\leq} C\sigma^{-\beta}\Big(1 + \fint_{B_{R/2}} |\nabla u|^{p(x)(1+\sigma/8+w(R))}\, dx\Big)^{\frac{1}{1+\sigma/8}} \\
&\quad + C\log^{\beta}\Big(\frac{1}{R}\Big) \fint_{B_{R/2}} |\nabla u|^{p_2}\, dx \\
&\quad + \frac{C}{|B_R|}\Big(e + \int_{B_{R/2}} |\nabla u|^{p_2}\, dx\Big)\log^{\beta}\Big(e + \int_{B_{R/2}} |\nabla u|^{p_2}\, dx\Big) + 1 \\
&\overset{(2.7),(3.2),(3.5),(3.11)}{\leq} C(q)\tilde{\sigma}^{-\beta} K_1^{\frac{2qw(R_0)}{p_- - 1}} \Big(\fint_{B_R} |\nabla u|^{p(x)}\, dx\Big)^{\frac{1+\sigma/8+w(R)}{1+\sigma/8}} \\
&\quad + C(q)\tilde{\sigma}^{-\beta} K_1^{\frac{2qw(R_0)}{p_- - 1}} \Big(\fint_{B_R} |\mathbf{f}|^{p(x)(1+\sigma/8+w(R))}\, dx\Big)^{\frac{1}{1+\sigma/8}} \\
&\quad + C\log^{\beta}\Big(\frac{1}{R}\Big) \fint_{B_{R/2}} |\nabla u|^{p_2}\, dx \\
&\quad + C(q)\tilde{\sigma}^{-\beta} K_1^{\frac{2qw(R_0)}{p_- - 1}} \Big(1 + \int_{B_{R/2}} |\nabla u|^{p_2}\, dx\Big)^{\sigma/8} \fint_{B_{R/2}} |\nabla u|^{p_2}\, dx \\
&\quad + C(q)\tilde{\sigma}^{-\beta} K_1^{\frac{2qw(R_0)}{p_- - 1}} \\
&\overset{(3.6)}{\leq} C\tilde{\sigma}^{-\beta} K_1^{\sigma/8} R^{-Nw(R)} \Big(\int_{B_R} |\nabla u|^{p(x)}\, dx\Big)^{\frac{w(R)}{1+\sigma/8}} \cdot \fint_{B_R} |\nabla u|^{p(x)}\, dx \\
&\quad + C\tilde{\sigma}^{-\beta} K_1^{\sigma/8} \Big(\fint_{B_R} |\mathbf{f}|^{p(x)(1+\sigma)}\, dx\Big)^{\frac{1}{1+\sigma}} + C\log^{\beta}\Big(\frac{1}{R}\Big) \fint_{B_{R/2}} |\nabla u|^{p_2}\, dx \\
&\quad + C\tilde{\sigma}^{-\beta} K_1^{\sigma/8} \Big(1 + \int_{B_{R/2}} |\nabla u|^{p_2}\, dx\Big)^{\sigma/8} \fint_{B_{R/2}} |\nabla u|^{p_2}\, dx + C\tilde{\sigma}^{-\beta} K_1^{\sigma/8}
\end{aligned}
$$

$$\overset{(3.6),\atop (3.12),(3.13)}{\leq} C\tilde{\sigma}^{-\beta} K_1^{\sigma/2}\Big[\fint_{B_R} (|\nabla u|^{p(x)}+1)\,dx + \Big(\fint_{B_R} (|\mathbf{f}|^{p(x)(1+\sigma)}+1)\,dx\Big)^{\frac{1}{1+\sigma}}\Big]$$
$$+ C\log^{\beta}\Big(\frac{1}{R}\Big) K_1^{\sigma/2}\Big[\fint_{B_R} (|\nabla u|^{p(x)}+1)\,dx + \Big(\fint_{B_R} (|\mathbf{f}|^{p(x)(1+\sigma)}+1)\,dx\Big)^{\frac{1}{1+\sigma}}\Big].$$

Thus from (3.7) we obtain

$$\begin{aligned} I_2 &\leq Cw(R)\log\Big(\frac{1}{R}\Big)|B_{R/2}|K_1^{\sigma/2} \\ &\quad\times\Big[\fint_{B_R} (|\nabla u|^{p(x)}+1)\,dx + \Big(\fint_{B_R} (|\mathbf{f}|^{p(x)(1+\sigma)}+1)\,dx\Big)^{\frac{1}{1+\sigma}}\Big] \\ &\quad + C\tilde{\sigma}^{-1}w(R)|B_{R/2}|K_1^{\sigma/2} \\ &\quad\times\Big[\fint_{B_R} (|\nabla u|^{p(x)}+1)\,dx + \Big(\fint_{B_R} (|\mathbf{f}|^{p(x)(1+\sigma)}+1)\,dx\Big)^{\frac{1}{1+\sigma}}\Big] \\ &\leq C\delta|B_{R/2}|K_1^{\sigma/2}\Big[\fint_{B_R} |\nabla u|^{p(x)}\,dx + \Big(\fint_{B_R} (|\mathbf{f}|^{p(x)(1+\sigma)}+1)\,dx\Big)^{\frac{1}{1+\sigma}}\Big]. \end{aligned}$$

Estimate of I_3. Note that $p_2 \geq p(x)$ in $B_{R/2}$, so

$$\frac{p_2(p(x)-1)}{p_2-1} \leq p(x) \quad \text{for all } x \in B_{R/2}.$$

Therefore, from Hölder's and Young's inequalities and (3.12) we have

$$\begin{aligned} I_3 &\leq \int_{B_{R/2}} |\mathbf{f}|^{p(x)-1}|\nabla u - \nabla v|\,dx \\ &\leq |B_{R/2}|\Big(\fint_{B_{R/2}} |\nabla u - \nabla v|^{p_2}\,dx\Big)^{1/p_2}\cdot\Big(\fint_{B_{R/2}} |\mathbf{f}|^{\frac{p_2(p(x)-1)}{p_2-1}}\,dx\Big)^{\frac{p_2-1}{p_2}} \\ &\leq C|B_{R/2}|\Big(\fint_{B_{R/2}} (|\nabla u|^{p_2}+1)\,dx\Big)^{1/p_2}\cdot\Big(\fint_{B_{R/2}} (|\mathbf{f}|^{p(x)}+1)\,dx\Big)^{\frac{p_2-1}{p_2}} \\ &\leq C|B_{R/2}|\Big(\fint_{B_{R/2}} (|\nabla u|^{p_2}+1)\,dx\Big)^{1/p_2}\cdot\Big(\fint_{B_{R/2}} (|\mathbf{f}|^{p(x)}+1)^{1+\sigma}\,dx\Big)^{\frac{p_2-1}{p_2(1+\sigma)}} \\ &\leq C|B_{R/2}|\Big[\delta\fint_{B_{R/2}} (|\nabla u|^{p_2}+1)\,dx + C(\delta)\Big(\fint_{B_{R/2}} (|\mathbf{f}|^{p(x)}+1)^{1+\sigma}\,dx\Big)^{\frac{1}{1+\sigma}}\Big] \\ &\leq C|B_{R/2}|K_1^{\sigma/2}\Big[\delta\fint_{B_R} |\nabla u|^{p(x)}\,dx + C(\delta)\Big(\fint_{B_R} (|\mathbf{f}|^{p(x)}+1)^{1+\sigma}\,dx\Big)^{\frac{1}{1+\sigma}}\Big]. \end{aligned}$$

Combining all the estimates for I_1–I_3, by selecting the constant τ so small that $0<\tau\ll\delta$, we get

$$\fint_{B_{R/2}} |\nabla(u-v)|^{p_2}\,dx \le C\delta K_1^{\sigma/2}\fint_{B_R}(|\nabla u|^{p(x)}+1)\,dx + C(\delta)K_1^{\sigma/2}\Big[\fint_{B_R}(|\mathbf{f}|^{p(x)}+1)^{1+\sigma}\,dx\Big]^{\frac{1}{1+\sigma}},$$

where

$$K_1=\int_{B_R}(|\nabla u|^{p(x)}+|\mathbf{f}|^{p(x)q})\,dx+1.$$

This further implies that for any $0<\rho\le R/2$,

$$\begin{aligned}\fint_{B_\rho}|\nabla u-\nabla v|^{p_2}\,dx &\le \frac{1}{2^N}\Big(\frac{R}{\rho}\Big)^N\fint_{B_{R/2}}|\nabla(u-v)|^{p_2}\,dx\\ &\le C\delta\Big(\frac{R}{\rho}\Big)^N K_1^{\sigma/2}\fint_{B_R}|\nabla u|^{p(x)}\,dx + C(\delta)\Big(\frac{R}{\rho}\Big)^N K_1^{\sigma/2}\Big[\fint_{B_R}(|\mathbf{f}|^{p(x)}+1)^{1+\sigma}\,dx\Big]^{\frac{1}{1+\sigma}}\\ &\le \eta\Big(\frac{R}{\rho}\Big)^N K_1^{\sigma/2}\fint_{B_R}|\nabla u|^{p(x)}\,dx + C(\eta)\Big(\frac{R}{\rho}\Big)^N K_1^{\sigma/2}\Big[\fint_{B_R}(|\mathbf{f}|^{p(x)}+1)^{1+\sigma}\,dx\Big]^{\frac{1}{1+\sigma}}\end{aligned}$$

by choosing δ to satisfy the last inequality above. ∎

LEMMA 3.4. *For any $\varepsilon\in(0,1)$, there exist constants $C=C(N,p(\cdot),\varepsilon)$ and $h=h(N,p(\cdot),\varepsilon)\in(0,1)$ such that for all $x_0\in\mathbb{R}^N$ and $0<\rho\le hR$,*

$$\begin{aligned}\fint_{B_\rho(x_0)}\big||\nabla u|^{p(x)}-(|\nabla u|^{p(x)})_{B_\rho(x_0)}\big|\,dx \\ \le CK^\sigma\Big(\fint_{B_{2R}(x_0)}[|\mathbf{f}|^{p(x)}+1]^{1+\sigma}\,dx\Big)^{\frac{1}{1+\sigma}}+\varepsilon\fint_{B_{2R}(x_0)}|\nabla u|^{p(x)}\,dx,\end{aligned}\tag{3.14}$$

where

$$K=\int_{\mathbb{R}^N}(|\nabla u|^{p(x)}+|f|^{p(x)q})\,dx+1.$$

Proof. After a translation we may assume $x_0=0$. For all $\rho\in(0,R/2)$ and $\eta\in(0,1)$, by Lemma 3.3 we have

$$\begin{aligned}\fint_{B_\rho}\big||\nabla u|^{p(x)}-(|\nabla u|^{p(x)})_{B_\rho}\big|\,dx &\le 2\fint_{B_\rho}\big||\nabla u|^{p(x)}-(|\nabla v|^{p(x)})_{B_\rho}\big|\,dx\\ &\le C\fint_{B_\rho}|\nabla u-\nabla v|^{p(x)}\,dx + C\fint_{B_\rho}|\nabla v-(\nabla v)_{B_\rho}|^{p(x)}\,dx\\ &\le C\fint_{B_\rho}(|\nabla u-\nabla v|^{p_2}+1)\,dx + C\fint_{B_\rho}(|\nabla v-(\nabla v)_{B_\rho}|^{p_2}+1)\,dx\end{aligned}$$

$$\leq \eta\Big(\frac{R}{\rho}\Big)^N K^{\sigma/2} \fint_{B_R} |\nabla u|^{p(x)}\,dx + C\Big(\frac{R}{\rho}\Big)^N K^{\sigma/2}\Big[\fint_{B_R} (|\mathbf{f}|^{p(x)(1+\sigma)}+1)\,dx\Big]^{\frac{1}{1+\sigma}}$$
$$+ C\fint_{B_\rho} (|\nabla v - (\nabla v)_{B_\rho}|^{p_2}+1)\,dx.$$

If $p_2 > 2$, by Lemma 2.6 and similar arguments to those in (3.12) we have

$$\fint_{B_\rho} |\nabla v - (\nabla v)_{B_\rho}|^{p_2}\,dx \leq 2\|\nabla v\|^{p_2-2}_{L^\infty(B_{R/2})} \fint_{B_\rho} |\nabla v - (\nabla v)_{B_\rho}|^2\,dx$$
$$\leq C\Big(\frac{\rho}{R}\Big)^{2\gamma} \fint_{B_R} |\nabla u|^{p_2}\,dx$$
$$\leq C\Big(\frac{\rho}{R}\Big)^{2\gamma} K^{\sigma/8}\Big[\fint_{B_{2R}} |\nabla u|^{p(x)}\,dx + \Big(\fint_{B_{2R}} (|\mathbf{f}|^{p(x)(1+\sigma)}+1)\,dx\Big)^{\frac{1}{1+\sigma}}\Big].$$

If $1 < p_2 < 2$, Hölder's inequality, Lemma 2.6 and similar arguments to those in (3.12) again yield

$$\fint_{B_\rho} |\nabla v - (\nabla v)_{B_\rho}|^{p_2}\,dx \leq \Big(\fint_{B_\rho} |\nabla v - (\nabla v)_{B_\rho}|^2\,dx\Big)^{p_2/2}$$
$$\leq C\Big(\frac{\rho}{R}\Big)^{\gamma p_2} \|\nabla v\|^{p_2}_{L^\infty(B_{R/2})} \leq C\Big(\frac{\rho}{R}\Big)^{\gamma p_2} \fint_{B_R} |\nabla u|^{p_2}\,dx$$
$$\leq C\Big(\frac{\rho}{R}\Big)^{\gamma p_2} K^{\sigma/8}\Big[\fint_{B_{2R}} |\nabla u|^{p(x)}\,dx + \Big(\fint_{B_{2R}} (|\mathbf{f}|^{p(x)(1+\sigma)}+1)\,dx\Big)^{\frac{1}{1+\sigma}}\Big].$$

Therefore, we conclude that for every $\eta \in (0,1)$ fixed, there exist constants C such that for all $\rho \in (0, R/2)$,

$$\fint_{B_\rho} \big||\nabla u|^{p(x)} - (|\nabla u|^{p(x)})_{B_\rho}\big|\,dx$$
$$\leq CK^{\sigma/2}\Big[\Big(\frac{R}{\rho}\Big)^N + \Big(\frac{\rho}{R}\Big)^\gamma\Big]\Big(\fint_{B_{2R}} (|\mathbf{f}|^{p(x)(1+\sigma)}+1)\,dx\Big)^{\frac{1}{1+\sigma}}$$
$$+ K^{\sigma/2}\Big\{\eta\Big(\frac{R}{\rho}\Big)^N + C\Big(\frac{\rho}{R}\Big)^\gamma\Big\} \fint_{B_{2R}} (|\nabla u|^{p(x)}+1)\,dx$$
$$\leq C(N, p(\cdot), \varepsilon)K^\sigma\Big(\fint_{B_{2R}} [|\mathbf{f}|^{p(x)}+1]^{1+\sigma}\,dx\Big)^{\frac{1}{1+\sigma}} + \varepsilon \fint_{B_{2R}} |\nabla u|^{p(x)}\,dx$$

by choosing η and ρ to satisfy the last inequality above. ■

Proof of Theorem 1.2. From Lemma 3.4, assuming as before that $x_0 = 0$ we find that for every $\varepsilon \in (0,1)$ fixed, there exists a constant $C = C(N, p(\cdot), \varepsilon)$

such that

$$\fint_{B_\rho} \big||\nabla u|^{p(x)} - (|\nabla u|^{p(x)})_{B_\rho}\big|\,dx \le CK^\sigma \Big(\fint_{B_{2R}} [|\mathbf{f}|^{p(x)} + 1]^{1+\sigma}\,dx\Big)^{\frac{1}{1+\sigma}} + \varepsilon \fint_{B_{2R}} |\nabla u|^{p(x)}\,dx.$$

Recalling the assumption (1.3) on $\mathbf{f}$ that there exist $r_0, c_0 > 0$ such that

$$|\mathbf{f}(x)| \ge c_0, \qquad \forall x \in B_{r_0}(0),$$

and taking $2R \le r_0$, we can conclude that

$$\fint_{B_\rho} \big||\nabla u|^{p(x)} - (|\nabla u|^{p(x)})_{B_\rho}\big|\,dx \le C(N, p(\cdot), \varepsilon, c_0) K^\sigma \Big(\fint_{B_{2R}} |\mathbf{f}|^{p(x)(1+\sigma)}\,dx\Big)^{\frac{1}{1+\sigma}} + \varepsilon \fint_{B_{2R}} |\nabla u|^{p(x)}\,dx.$$

Furthermore, after a translation we know from the definitions of maximal functions in Section 2 that for every $\varepsilon \in (0,1)$ fixed, there exists a constant $C = C(N, p(\cdot), \varepsilon, c_0)$ such that

$$(|\nabla u|^{p(x)})^{\#}(x_0) \le CK^\sigma M_{1+\sigma}[|\mathbf{f}|^{p(x)}](x_0) + \varepsilon M[|\nabla u|^{p(x)}](x_0), \quad \text{a.e. } x_0 \in \mathbb{R}^N.$$

Fix $q > 1$. Since $|\nabla u|^{p(x)} \in L^q(\mathbb{R}^N)$, by Lemma 2.4 there is a constant C depending only on $N, p(\cdot)$ and q such that

$$\int_{\mathbb{R}^N} |\nabla u|^{p(x)q}\,dx \le C \int_{\mathbb{R}^N} \{(|\nabla u|^{p(x)})^{\#}(x)\}^q\,dx \le CK^{\sigma q} \int_{\mathbb{R}^N} \{M_{1+\sigma}[|\mathbf{f}|^{p(x)}](x)\}^q\,dx + \varepsilon^q \int_{\mathbb{R}^N} \{M[|\nabla u|^{p(x)}](x)\}^q\,dx.$$

In addition, from Remark 2.5 and Lemma 2.4 we have

$$\int_{\mathbb{R}^N} \{M_{1+\sigma}[|\mathbf{f}|^{p(x)}](x)\}^q\,dx \le C \int_{\mathbb{R}^N} |\mathbf{f}|^{p(x)q}\,dx,$$

and analogously

$$\int_{\mathbb{R}^N} \{M[|\nabla u|^{p(x)}](x)\}^q\,dx \le C \int_{\mathbb{R}^N} |\nabla u|^{p(x)q}\,dx.$$

This yields

$$\int_{\mathbb{R}^N} |\nabla u|^{p(x)q}\,dx \le CK^{\sigma q} \int_{\mathbb{R}^N} |\mathbf{f}|^{p(x)q}\,dx$$

by choosing ε small enough. ∎

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