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Generalized P-reducible (α, β)-metrics with vanishing S-curvature

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Abstract. We study one of the open problems in Finsler geometry presented by Matsumoto–Shimada in 1977, about the existence of a concrete P-reducible metric, i.e. one which is not C-reducible. In order to do this, we study a class of Finsler metrics, called generalized P-reducible metrics, which contains the class of P-reducible metrics. We prove that every generalized P-reducible (α, β)-metric with vanishing S-curvature reduces to a Berwald metric or a C-reducible metric. It follows that there is no concrete P-reducible (α, β)-metric with vanishing S-curvature.

1. Introduction. In 1975, the well-known the physicist Y. Takano published a paper which considered the field equation in a Finsler space and proposed certain geometrical problems in Finsler geometry [15]. He requested mathematicians to find some special forms of hv-curvature, interseting from the standpoint of physics. In 1978, Matsumoto introduced the notion of P-reducible Finsler metrics as an answer to Takano's request which was a generalization of C-reducible Finsler metrics [7]. For a Finsler metric of dimension $n \geq 3$, he found some conditions under which the Finsler metric was P-reducible.

Since the study of hv-curvature became necessary for Finsler geometry as well as for theoretical physics, Matsumoto–Shimada [10] studied the curvature properties of P-reducible metrics. They posed the following problem:

Is there any concrete P-reducible metric, i.e. one which is not C-reducible?

In [9], Matsumoto–Hōjō proved that F is C-reducible if and only if it is a Randers metric or a Kropina metric. These metrics are defined by $F = \alpha + \beta$ and $F = \alpha^2/\beta$, respectively, where $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta := b_i(x)y^i$ is a 1-form on a manifold M. The Randers metrics were introduced by G. Randers in the context of general relativity, and have been widely applied in many areas of natural sciences, including biology, ecol-

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ogy, physics and psychology [3], [12]. The Kropina metric was introduced by L. Berwald in connection with a two-dimensional Finsler space with rectilinear extremals [1].

In [11], Numata introduced an interesting family of Finsler metrics which were called Numata-type metrics. They are defined by $F := \bar{F} + \eta$, where $\bar{F}(y) = \sqrt{g_{ij}(y)y^iy^j}$ is a locally Minkowskian metric and $\eta = \eta_i(x)y^i$ a closed one-form on a manifold M; F is called a *Randers change* of $\bar{F}$. By a simple calculation, we get

$$C_{ijk} = \bar{C}_{ijk} + \frac{1}{2\bar{F}}\{h_{ij}D_m + h_{jk}D_i + h_{ki}D_j\},$$

where $D_i := \eta_i - \eta y_i/(\bar{F})^2$ and $h_{ij} := FF_{ij}$ is the angular metric. Define $\eta_{i|j}$ by $\eta_{i|j}\gamma^j := d\eta_i - \eta_j\gamma_i^j$, where $\gamma^i := dx^i$ and $\gamma_i^j := \Gamma^j_{ik}dx^k$ denote the coefficients of the linear connection form of $\bar{F}$. Set

$$\mathfrak{D}_{ij} := \tfrac{1}{2}(\eta_{i|j} + \eta_{j|i}).$$

Then the *Landsberg curvature* of F is given by

$$L_{ijk} = \lambda C_{ijk} + a_i h_{jk} + a_j h_{ki} + a_k h_{ij}, \tag{1}$$

where

$$\lambda = \frac{1}{2F}\mathfrak{D}_{ij}y^iy^j, \quad a_i := \frac{1}{2F^2\bar{F}^3}[2F\bar{F}^2\mathfrak{D}_{ik} - 2F\mathfrak{D}_{kl}y^ly_i - (1+\bar{F}^2)\mathfrak{D}_{kl}y^lD_j]y^k.$$

We call a Finsler metric F *generalized P-reducible* if its Landsberg curvature is given by (1), where $a_i = a_i(x,y)$ and $\lambda = \lambda(x,y)$ are scalar functions on TM. Thus every Numata-type metric is a generalized P-reducible metric. By (1), if $a_i = 0$ then F reduces to a general relatively isotropic Landsberg metric, and if $\lambda = 0$ then F is P-reducible. Thus the study of this class of Finsler spaces will enhance our understanding of the geometric meaning of P-reducible metrics.

The notion of S-curvature was originally introduced by Shen [13] for the volume comparison theorem. Finsler metrics with vanishing S-curvature are important geometric structures which deserve a deeper study [16].

An (α,β)*-metric* is a Finsler metric of the form $F := \alpha\phi(s)$, $s = \beta/\alpha$, where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. For example, $\phi = c_1\sqrt{1 + c_2s^2} + c_3s$ is called a *Randers-type metric*, where $c_1 > 0$, c_2 and c_3 are constants. In this paper, we characterize generalized P-reducible (α,β)-metrics with vanishing S-curvature and prove the following.

Theorem 1.1. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α,β)-metric on a manifold M. Suppose that F is a generalized P-reducible metric with vanishing S-curvature. Then F is a Berwald metric or a C-reducible metric.*

From Theorem 1.1, it follows that there is no concrete P-reducible (α,β)-metric with vanishing S-curvature (see Lemma 3.5).

In this paper, we use the Berwald connection. The h- and v- covariant derivatives of a Finsler tensor field are denoted by " $|$ " and " , " respectively.

2. Preliminaries. Let (M,F) be a Finsler manifold. Suppose $x \in M$ and $F_x := F|_{T_xM}$. We define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{C}_y(u,v,w) := C_{ijk}(y)u^iv^jw^k$, where

$$C_{ijk} := \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k} = \frac{1}{4}\frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k},$$

and $g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}$. The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the *Cartan torsion*. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian. For $y \in T_xM_0$, define the *mean Cartan torsion* $\mathbf{I}_y$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$.

For $y \in T_xM_0$, define the *Matsumoto torsion* $\mathbf{M}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{M}_y(u,v,w) := M_{ijk}(y)u^iv^jw^k$, where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\{I_ih_{jk} + I_jh_{ik} + I_kh_{ij}\}$$

and $h_{ij} = g_{ij} - F_{y^i}F_{y^j}$ is the angular metric. F is said to be *C-reducible* if $\mathbf{M}_y = 0$.

LEMMA 2.1 ([9]). *A Finsler metric F on a manifold M of dimension $n \geq 3$ is a Randers metric or a Kropina metric if and only if $\mathbf{M}_y = 0$ for all $y \in TM_0$.*

A Finsler metric is called *semi-C-reducible* if its Cartan tensor is given by

$$C_{ijk} = \frac{p}{n+1}\{h_{ij}I_k + h_{jk}I_i + h_{ik}I_j\} + \frac{q}{\|\mathbf{I}\|^2}I_iI_jI_k, \tag{2}$$

where $p = p(x,y)$ and $q = q(x,y)$ are scalar functions on TM satisfying $p + q = 1$ and $\|\mathbf{I}\|^2 = I^mI_m$ (see [8], [17], [18]).

LEMMA 2.2 ([8]). *Every non-Riemannian (α,β)-metric on a manifold M of dimension $n \geq 3$ is semi-C-reducible.*

The horizontal covariant derivatives of the Cartan torsion $\mathbf{C}$ and mean Cartan torsion $\mathbf{I}$ along geodesics give rise to the *Landsberg curvature* $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ and *mean Landsberg curvature* $\mathbf{J}_y : T_xM \to \mathbb{R}$, defined by $\mathbf{L}_y(u,v,w) := L_{ijk}(y)u^iv^jw^k$ and $\mathbf{J}_y(u) := J_i(y)u^i$, respectively, where

$$L_{ijk} := C_{ijk|s}y^s, \qquad J_i := I_{i|s}y^s.$$

The families $\mathbf{L} := \{\mathbf{L}_y\}_{y\in TM_0}$ and $\mathbf{J} := \{\mathbf{J}_y\}_{y\in TM_0}$ are also called the Landsberg curvature and mean Landsberg curvature, respectively. A Finsler metric is called a *Landsberg metric* or a *weakly Landsberg metric* if $\mathbf{L}=0$ or $\mathbf{J}=0$, respectively.

A Finsler metric F on an n-dimensional manifold M is called *P-reducible* if its Landsberg curvature is given by

$$L_{ijk}=\frac{1}{n+1}\{J_ih_{jk}+J_jh_{ik}+J_kh_{ij}\}.$$

It is easy to see that every C-reducible metric is P-reducible. But the converse is not true [6].

Given an n-dimensional Finsler manifold (M,F), a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i,y^i) for TM_0 is given by $\mathbf{G}=y^i\frac{\partial}{\partial x^i}-2G^i(x,y)\frac{\partial}{\partial y^i}$, where $G^i=G^i(x,y)$ are called *spray coefficients* and are given by

$$G^i:=\frac{1}{4}g^{il}\left[\frac{\partial^2F^2}{\partial x^k\partial y^l}y^k-\frac{\partial F^2}{\partial x^l}\right],\qquad y\in T_xM.$$

$\mathbf{G}$ is called the *spray* associated to F.

For $y\in T_xM_0$, define $\mathbf{B}_y:T_xM\otimes T_xM\otimes T_xM\to T_xM$ by $\mathbf{B}_y(u,v,w):={B^i}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}\big|_x$, where

$${B^i}_{jkl}:=\frac{\partial^3G^i}{\partial y^j\partial y^k\partial y^l}.$$

$\mathbf{B}$ is called the *Berwald curvature* and F is called a *Berwald metric* if $\mathbf{B}=\mathbf{0}$.

For an (α,β)-metric, let us define $b_{i|j}$ by $b_{i|j}\theta^j:=db_i-b_j\theta_i^j$, where $\theta^i:=dx^i$ and $\theta_i^j:=\Gamma^j_{ik}dx^k$ denote the Levi-Civita connection form of α. Let

$$\begin{aligned}
&r_{ij}:=\tfrac{1}{2}(b_{i|j}+b_{j|i}), && s_{ij}:=\tfrac{1}{2}(b_{i|j}-b_{j|i}),\\
&r_{i0}:=r_{ij}y^j, && r_{00}:=r_{ij}y^iy^j, && r_j:=b^ir_{ij},\\
&s_{i0}:=s_{ij}y^j, && s_j:=b^is_{ij}, && r_0:=r_jy^j, && s_0:=s_jy^j.
\end{aligned}$$

Let $G^i=G^i(x,y)$ and $G^i_\alpha=G^i_\alpha(x,y)$ denote the coefficients of F and α respectively in the same coordinate system. Then

$$G^i=G^i_\alpha+\alpha Qs^i_0+(-2Q\alpha s_0+r_{00})\left(\Theta\frac{y^i}{\alpha}+\Psi b^i\right), \tag{3}$$

where

$$\begin{aligned}
Q&:=\frac{\phi'}{\phi-s\phi'}, && \Delta:=1+sQ+(b^2-s^2)Q',\\
\Theta&:=\frac{Q-sQ'}{2\Delta}, && \Psi:=\frac{Q'}{2\Delta}.
\end{aligned}$$

The mean Landsberg curvature of an (α,β)-metric $F=\alpha\phi(s)$ is given by

$$
\begin{aligned}
(4)\qquad J_i := -\frac{1}{2\alpha^4\Delta}\bigg(&\frac{2\alpha^2}{b^2-s^2}\left[\frac{\Phi}{\Delta}+(n+1)(Q-sQ')\right](r_0+s_0)h_i \\
&+\frac{\alpha}{b^2-s^2}\left[\Psi_1+s\frac{\Phi}{\Delta}\right](r_{00}-2\alpha Q s_0)h_i+\alpha\big[-\alpha Q' s_0 h_i \\
&+\alpha Q(\alpha^2 s_i-\bar{y}_i s_0)+\alpha^2\Delta s_{i0}+\alpha^2(r_{i0}-2\alpha Q s_0) \\
&-(r_{00}-2\alpha Q s_0)\bar{y}_i\big]\frac{\Phi}{\Delta}\bigg),
\end{aligned}
$$

where

$$
\begin{aligned}
&\Psi_1 := \sqrt{b^2-s^2}\,\Delta^{1/2}\left[\frac{\sqrt{b^2-s^2}}{\Delta^{3/2}}\right]', \\
&h_i := \alpha b_i - s\bar{y}_i, \qquad \bar{y}_i := a_{ij}y^j, \\
&\Phi := -(Q-sQ')(n\Delta+1+sQ)-(b^2-s^2)(1+sQ)Q''.
\end{aligned}
$$

For more details, see [2]. We have

$$
(5)\qquad \bar{J} := b^i J_i = -\frac{1}{2\alpha^2\Delta}\{\Psi_1(r_{00}-2\alpha Q s_0)+\alpha\Psi_2(r_0+s_0)\},
$$

where

$$
\Psi_2 := 2(n+1)(Q-sQ')+3\Phi/\Delta.
$$

For a Finsler metric F on an n-dimensional manifold M, the *Busemann–Hausdorff volume form* $dV_F=\sigma_F(x)dx^1\cdots dx^n$ is defined by

$$
\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\big\{(y^i)\in\mathbb{R}^n \,\big|\, F\big(y^i\frac{\partial}{\partial x^i}\big|_x\big)<1\big\}}.
$$

Let $G^i(x,y)$ denote the geodesic coefficients of F in the same local coordinate system. The S-curvature is defined by

$$
\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x,y)-y^i\frac{\partial}{\partial x^i}[\ln\sigma_F(x)],
$$

where $\mathbf{y}=y^i\frac{\partial}{\partial x^i}\big|_x\in T_xM$. If F is a Berwald metric then $\mathbf{S}=0$.

In [4], Cheng–Shen characterized (α,β)-metrics with isotropic S-curvature.

Lemma 2.3 ([4]). *Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be a non-Riemannian (α,β)-metric on a manifold M of dimension $n\geq 3$ and $b:=\|\beta_x\|_\alpha$. Suppose that F is not a Finsler metric of Randers type. Then F is of isotropic S-curvature, $\mathbf{S}=(n+1)cF$, if and only if one of the following holds:*

(a) *β satisfies*

$$
(6)\qquad r_{ij}=\varepsilon(b^2a_{ij}-b_ib_j), \qquad s_j=0,
$$

where $\varepsilon = \varepsilon(x)$ *is a scalar function, and* $\phi = \phi(s)$ *satisfies*

$$\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2 - s^2}, \tag{7}$$

where k *is a constant. In this case,* $\mathbf{S} = (n+1)cF$ *with* $c = k\varepsilon$.

(b) β *satisfies*

$$r_{ij} = 0, \qquad s_j = 0. \tag{8}$$

In this case, $\mathbf{S} = 0$.

3. Proof of Theorem 1.1

LEMMA 3.1. *Let* $F = \alpha\phi(s)$, $s = \beta/\alpha$, *be a non-Randers type* (α,β)-*metric on a manifold* M *of dimension* $n \geq 3$. *Suppose that* F *has vanishing* S-*curvature. Then*

$$y_i s^i_0 = 0, \tag{9}$$

$$y_i s^i_{0|0} = 0, \tag{10}$$

$$y_i b^j s^i_{j|0} = \phi(\phi - s\phi')s^j_0 s_{j0}, \tag{11}$$

where $y_i := g_{ij}y^j$.

Proof. We have

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_1(b_i\alpha_j + b_j\alpha_i) + \rho_2\alpha_i\alpha_j, \tag{12}$$

where $\alpha_i := \alpha^{-1}a_{ij}y^j$ and

$$\rho := \phi(\phi - s\phi'), \tag{13}$$

$$\rho_0 := \phi\phi'' + \phi'\phi', \tag{14}$$

$$\rho_1 := -[s(\phi\phi'' + \phi'\phi') - \phi\phi'], \tag{15}$$

$$\rho_2 := s[s(\phi\phi'' + \phi'\phi') - \phi\phi']. \tag{16}$$

Then

$$y_i := \rho\bar{y}_i + \rho_0 b_i\beta + \rho_1(b_i\alpha + s\bar{y}_i) + \rho_2\bar{y}_i, \tag{17}$$

where $\bar{y}_i := a_{ij}y^j$. Since $\bar{y}_i s^i_0 = 0$, by (8) we get $b_i s^i_0 = 0$. Thus (17) implies that

$$y_i s^i_0 = 0. \tag{18}$$

Since $y_{i|0} = 0$, (18) implies that

$$y_i s^i_{0|0} = 0. \tag{19}$$

From $s_j = b^j s^i_j = 0$, we have

$$0 = (b^j s^i_j)|_0 = b^j_{|0}s^i_j + b^j s^i_{j|0} = (r^j_0 + s^j_0)s^i_j + b^j s^i_{j|0}, \tag{20}$$

or equivalently

$$b^j s^i_{j|0} = -s^j_0 s^i_j. \tag{21}$$

By (17) and (21), we get

$$y_i b^j s^i_{j|0} = -(\rho+\rho_1 s+\rho_2)s^j_0 s^0_j = (\rho+\rho_1 s+\rho_2)s^j_0 s_{j0}. \tag{22}$$

Since $\rho_1 s+\rho_2 = 0$, it follows that

$$y_i b^j s^i_{j|0} = \rho s^j_0 s_{j0} = \phi(\phi - s\phi')s^j_0 s_{j0}. \tag{23}$$

This completes the proof. ■

LEMMA 3.2. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Randers type (α,β)-metric on a manifold M of dimension $n \geq 3$. Suppose that F has vanishing S-curvature. Then*

$$b^j b^k b^l L_{jkl} = 0, \tag{24}$$

$$b^i J_i = 0. \tag{25}$$

Proof. Since F has vanishing S-curvature, (3) reduces to

$$G^i = G^i_\alpha + \alpha Q s^i_0. \tag{26}$$

Taking third order vertical derivatives of (26) with respect to y^j, y^l and y^k yields

$$\begin{aligned} B^i_{jkl} = {} & s^i_l[Q\alpha_{jk} + Q_k\alpha_j + Q_j\alpha_k + \alpha Q_{jk}] \\ & + s^i_j[Q\alpha_{lk} + Q_k\alpha_l + Q_l\alpha_k + \alpha Q_{lk}] \\ & + s^i_k[Q\alpha_{jl} + Q_j\alpha_l + Q_l\alpha_j + \alpha Q_{jl}] \\ & + s^i_0[\alpha_{jkl}Q + \alpha_{jk}Q_l + \alpha_{lk}Q_j + \alpha_{lj}Q_k \\ & + \alpha Q_{jkl} + \alpha_l Q_{jk} + \alpha_j Q_{lk} + \alpha_k Q_{jl}]. \end{aligned} \tag{27}$$

Multiplying (27) with y_i and using (9) implies that

$$\begin{aligned} -2L_{jkl} = {} & y_i {s^i}_l[Q\alpha_{jk} + Q_k\alpha_j + Q_j\alpha_k + \alpha Q_{jk}] \\ & + y_i {s^i}_j[Q\alpha_{lk} + Q_k\alpha_l + Q_l\alpha_k + \alpha Q_{lk}] \\ & + y_i {s^i}_k[Q\alpha_{jl} + Q_j\alpha_l + Q_l\alpha_j + \alpha Q_{jl}]. \end{aligned} \tag{28}$$

By (8), we have $s_j = b^j s_{ij} = 0$. Multiplying (28) with $b^j b^k b^l$ yields (24). By (5) and (8), we get (25). ■

LEMMA 3.3. *Let (M,F) be a generalized P-reducible Finsler manifold. Then the Matsumoto torsion of F satisfies*

$$M_{ijk|s}y^s = \lambda(x,y)M_{ijk}. \tag{29}$$

Proof. Let F be a generalized P-reducible metric

$$L_{ijk} = \lambda C_{ijk} + a_i h_{jk} + a_j h_{ki} + a_k h_{ij}. \tag{30}$$

Contracting (30) with $g^{ij} := (g_{ij})^{-1}$ and using the relations $g^{ij}h_{ij} = n-1$ and $g^{ij}(a_i h_{jk}) = g^{ij}(a_j h_{ik}) = a_k$ implies that

$$J_k = \lambda I_k + (n+1)a_k. \tag{31}$$

Then

$$a_i = \frac{1}{n+1}J_i - \frac{\lambda}{n+1}I_i. \tag{32}$$

Putting (32) in (30) yields

$$\begin{aligned} L_{ijk} = \lambda C_{ijk} &+ \frac{1}{n+1}\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\} \\ &- \frac{\lambda}{n+1}\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\}. \end{aligned} \tag{33}$$

By simplifying (33), we get (29). ∎

Lemma 3.4. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Randers type (α,β)-metric on a manifold M of dimension $n \geq 3$. Suppose that F is a generalized P-reducible metric with vanishing S-curvature. Then F is a P-reducible metric.*

Proof. Let F be a generalized P-reducible metric. By Lemma 3.3, we have

$$\begin{aligned} L_{ijk} &- \frac{1}{n+1}(J_i h_{jk} + J_j h_{ik} + J_k h_{ij}) \\ &= \lambda\Big[C_{ijk} - \frac{1}{n+1}(I_i h_{jk} + I_j h_{ik} + I_k h_{ij})\Big]. \end{aligned} \tag{34}$$

Contracting (34) with $b^i b^j b^k$ and using (24) and (25) implies that

$$\lambda\Big[b^i b^j b^k C_{ijk} - \frac{3}{n+1}(b^i I_i)(b^j b^k h_{jk})\Big] = 0. \tag{35}$$

By (35), we get two cases:

Case (1): $\lambda = 0$. In this case, F reduces to a P-reducible metric.

Case (2): $\lambda \neq 0$. In this case, by (35) we get

$$b^i b^j b^k C_{ijk} = \frac{3}{n+1}(b^i I_i)(b^j b^k h_{jk}). \tag{36}$$

Multiplying (2) with $b^i b^j b^k$ gives

$$b^i b^j b^k C_{ijk} = \frac{3p}{n+1}(b^i I_i)(b^j b^k h_{jk}) + \frac{q}{\|\mathbf{I}\|^2}(b^i I_i)^3. \tag{37}$$

By (36) and (37), it follows that

$$\frac{3q}{n+1}(b^i I_i)\Big[b^j b^k h_{jk} - \frac{(n+1)(b^m I_m)^2}{3\|\mathbf{I}\|^2}\Big] = 0. \tag{38}$$

By (38), we get three cases:

Case (2a): Let $b^iI_i = 0$. By a direct computation, we can obtain a formula for the mean Cartan torsion of (α,β)-metrics as follows:

$$I_i = -\frac{\Phi(\phi - s\phi')}{2\Delta\phi\alpha^2}(\alpha b_i - sy_i). \tag{39}$$

If $b^iI_i = 0$, then by contracting (39) with b^i we get

$$\frac{\Phi(\phi - s\phi')}{2\Delta\phi\alpha^3}(b^2\alpha^2 - \beta^2) = 0. \tag{40}$$

By (40), we have $\Phi = 0$ or $\phi - s\phi' = 0$, which implies that $\mathbf{I} = 0$, and thus F is a Riemannian metric. This contradicts our assumptions.

Case (2b): Suppose that

$$b^jb^kh_{jk} - \frac{n+1}{3\|\mathbf{I}\|^2}(b^iI_i)^2 = 0. \tag{41}$$

Since $h_{jk} = g_{jk} - F^{-2}g_{jm}g_{kl}y^my^l$, we have

$$b^jb^kh_{jk} = b^jb^kg_{jk} - \frac{1}{F^2}(g_{jk}b^jb^k)^2. \tag{42}$$

By (41) and (42), we obtain

$$b^jb^k\left[g_{jk} - \frac{n+1}{3\|\mathbf{I}\|^2}\,I_jI_k\right] = \left[\frac{1}{F}g_{jk}b^jb^k\right]^2. \tag{43}$$

Since $y^iI_i = 0$, by (43) we get

$$\left[\left(g_{ij} - \frac{(n+1)I_iI_j}{3\|\mathbf{I}\|^2}\right)b^i\frac{y^j}{F}\right]^2 = \left[\left(g_{ij} - \frac{(n+1)I_iI_j}{3\|\mathbf{I}\|^2}\right)b^ib^j\right]. \tag{44}$$

Set

$$G_{ij} := g_{ij} - \frac{n+1}{3\|\mathbf{I}\|^2}I_iI_j.$$

It follows from (44) that

$$\left[G_{ij}b^i\frac{y^j}{F}\right]^2 = G_{ij}b^ib^j. \tag{45}$$

Since $G_{ij}y^iy^j = F^2$, (45) implies that

$$\left[G_{ij}b^i\frac{y^j}{F}\right]^2 = [G_{ij}b^ib^j]\left[G_{ij}\frac{y^i}{F}\,\frac{y^j}{F}\right]. \tag{46}$$

By the Cauchy–Schwarz inequality and (46), we have

$$b^i = k\frac{y^i}{F}, \tag{47}$$

where k is a real constant. Multiplying (47) with b_i and $\bar{y}_i$, respectively, implies that

$$F = \frac{k\beta}{b^2} \quad \text{and} \quad F = \frac{k\alpha^2}{\beta}. \tag{48}$$

By (48), it follows that $(b^2 - s^2)\alpha^2 = 0$, which is a contradiction.

Case (2c): If $q = 0$ then $p = 1$, and from (2) it follows that F is C-reducible. In any case, F is a P-reducible Finsler metric. ∎

Now, we are going to consider P-reducible (α,β)-metrics with vanishing S-curvature.

Lemma 3.5. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Randers type (α,β)-metric on a manifold M of dimension $n \geq 3$. Suppose that F is a P-reducible metric with vanishing S-curvature. Then F reduces to a Berwald metric or a C-reducible metric.*

Proof. The Landsberg curvature of an (α,β)-metric is given by

$$L_{ijk} = \frac{-\rho}{6\alpha^5}\{h_ih_jC_k + h_jh_kC_i + h_ih_kC_j + 3E_iT_{jk} + 3E_jT_{ik} + 3E_kT_{ij}\}, \tag{49}$$

where

$$h_i := \alpha b_i - s\bar{y}_i, \tag{50}$$

$$\begin{aligned}
T_{ij} &:= \alpha^2 a_{ij} - \bar{y}_i\bar{y}_j, \qquad (51)\\
C_i &:= (X_4 r_{00} + Y_4\alpha s_0)h_i + 3\varLambda D_i,\\
E_i &:= (X_6 r_{00} + Y_6\alpha s_0)h_i + 3\mu D_i,\\
D_i &:= \alpha^2(s_{i0} + \varGamma r_{i0} + \Pi\alpha s_i) - (\varGamma r_{00} + \Pi\alpha s_0)\bar{y}_i\\
X_4 &:= \frac{1}{2\varDelta^2}\{-2\varDelta Q''' + 3(Q - sQ')Q'' + 3(b^2 - s^2)(Q'')^2\},\\
X_6 &:= \frac{1}{2\varDelta^2}\{(Q - sQ')^2 + [2(s + b^2Q) - (b^2 - s^2)(Q - sQ')]Q'\},\\
Y_4 &:= -2QX_4 + \frac{3Q'Q''}{\varDelta}, \qquad Y_6 := -2QX_6 + \frac{(Q - sQ')Q'}{\varDelta},\\
\varLambda &:= -Q'', \quad \mu := -\tfrac{1}{3}(Q - sQ'), \quad \varGamma := \frac{1}{\varDelta}, \quad \varPi := \frac{-Q}{\varDelta}.
\end{aligned}$$

For more details see [14]. Since $r_{ij} = 0$ and $s_i = 0$, (4) and (49) reduce to

$$J_i = -\frac{\varPhi}{2\alpha\varDelta}s_{i0}, \tag{52}$$

$$L_{ijk} = V_{ij}s_{k0} + V_{jk}s_{i0} + V_{ki}s_{j0}, \tag{53}$$

where

$$V_{ij} := \frac{\rho}{2\alpha^3}[Q''h_ih_j + (Q - sQ')T_{ij}].$$

We shall divide the problem into two cases: (a) $s_{i0} = 0$ and (b) $s_{i0} \neq 0$.

Case (a): Let $s_{i0}=0$. In this case, by (52) and (53), F reduces to a Landsberg metric. By Shen's Theorem of [14], F reduces to a Berwald metric.

Case (b): Let $s_{i0}\neq 0$. Then by (52) and (53), we have

$$L_{ijk}=Z_{ij}J_k+Z_{jk}J_i+Z_{ki}J_j, \tag{54}$$

where $Z_{ij}:=-(2\alpha\Delta/\Phi)V_{ij}$. Thus the Landsberg curvature of an (α,β)-metric with vanishing S-curvature satisfies (54). Set

$$A:=-\frac{\Delta\rho(Q-sQ')}{\Phi},\qquad B:=-\frac{\Delta\rho Q''}{\Phi}.$$

Then by putting (50) and (51) in the formula for Z_{ij} it follows that

$$Z_{ij}=Aa_{ij}+Bb_ib_j-sB(b_i\alpha_j+b_j\alpha_i)-(A-s^2B)\alpha_i\alpha_j. \tag{55}$$

By assumption, F is P-reducible

$$L_{ijk}=\frac{1}{n+1}(J_ih_{jk}+J_jh_{ik}+J_kh_{ij}), \tag{56}$$

where the angular metric $h_{ij}:=g_{ij}-F_{y^i}F_{y^j}$ is given by

$$h_{ij}=\phi[\phi-s\phi']a_{ij}+\phi\phi''\,b_ib_j-s\phi\phi''[b_i\alpha_j+b_j\alpha_i]-[\phi(\phi-s\phi')-s^2\phi\phi'']\,\alpha_i\alpha_j.$$

By (54) and (56), we obtain

$$\Big(Z_{ij}-\frac{1}{n+1}h_{ij}\Big)J_k+\Big(Z_{jk}-\frac{1}{n+1}h_{jk}\Big)J_i+\Big(Z_{ik}-\frac{1}{n+1}h_{ik}\Big)J_j=0. \tag{57}$$

Since $\alpha_is^i_0=0$ and $b_is^i_0=0$, we have

$$s^i_0s^j_0Z_{ij}=-\frac{\Delta\rho}{\Phi}(Q-sQ')s^m_0s_{m0},$$

$$s^i_0s^j_0h_{ij}=\phi[\phi-s\phi']s^m_0s_{m0},\qquad s^i_0J_i=-\frac{\Phi}{2\alpha\Delta}s^m_0s_{m0}.$$

Therefore, contracting (57) with $s^i_0s^j_0s^k_0$ implies that

$$\frac{1}{n+1}\phi[\phi-s\phi']=A. \tag{58}$$

By (58), it follows that

$$Z_{ij}-\frac{1}{n+1}h_{ij}=\chi[b_ib_j-s(b_i\alpha_j+b_j\alpha_i)+s^2\alpha_i\alpha_j], \tag{59}$$

where

$$\chi:=B-\frac{1}{n+1}\phi\phi''.$$

Since $J_i\neq 0$ and $b^mJ_m=0$, multiplying (57) with b^ib^j we get

$$b^ib^j\Big(Z_{ij}-\frac{1}{n+1}h_{ij}\Big)=0. \tag{60}$$

By contracting (59) with $b^i b^j$ and considering (60), it follows that

$$\chi = 0. \tag{61}$$

Then (58) and (61) imply that

$$\frac{1}{n+1}\phi[\phi - s\phi'] = -\frac{\Delta\rho}{\Phi}(Q - sQ'), \tag{62}$$

$$\frac{1}{n+1}\phi\phi'' = -\frac{\Delta\rho}{\Phi}Q''. \tag{63}$$

By (62) and (63), we obtain

$$\phi - s\phi' = c(Q - sQ'), \tag{64}$$

where c is a non-zero real constant. Solving (64) implies that

$$Q = c_1\phi + c_2 s, \tag{65}$$

where $c_1 \neq 0$ and c_2 are real constants. By (65), it follows that

$$c_2 s^2 + 2c_1 s\phi + 1 = d\phi^2, \tag{66}$$

where d is a real constant. We divide the problem into two cases: (b1) $d \neq 0$ and (b2) $d = 0$.

Subcase (b1): If $d \neq 0$, then by (66) we have

$$\phi = \frac{c_1}{d}s + \sqrt{\left[\left(\frac{c_1}{d}\right)^2 + \frac{c_2}{d}\right]s^2 + 1}, \tag{67}$$

which is a Randers-type metric. This is a contradiction.

Subcase (b2): If $d = 0$, then (66) yields

$$\phi = -\frac{1}{2c_1 s} + \frac{c_2}{2c_1}s, \tag{68}$$

which is a Randers change of a Kropina metric. It is known that Kropina metrics are C-reducible. On the other hand, every Randers change of a C-reducible metric is C-reducible [5]. Thus the Finsler metric defined by (68) is C-reducible. ∎

Proof of Theorem 1.1. Every two-dimensional Finsler surface is C-reducible. For Finsler manifolds of dimension $n \geq 3$, by Lemmas 3.4 and 3.5 the proof is complete. ∎

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