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New existence and stability results for partial fractional differential inclusions with multiple delay

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Abstract. We discuss the existence of solutions and Ulam's type stability concepts for a class of partial functional fractional differential inclusions with noninstantaneous impulses and a nonconvex valued right hand side in Banach spaces. An example is provided to illustrate our results.

1. Introduction. The fractional calculus represents a powerful tool in applied mathematics to study a myriad of problems from different fields of science and engineering, with many breakthrough results found in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas et al. [ABN1, ABN2], Kilbas et al. [KST], the papers of Abbas et al. [AB1, AB2, AB3, ABC, ABG, ABH, ABV, ABZ], Darwish et al. [DHO, D, DH, DB], Diethelm [DF], Kilbas and Marzan [KM], Vityuk and Golushkov [VG], and the references therein.

The stability of functional equations was originally raised by Ulam in 1940 in a talk given at Wisconsin University (for more details see [U]). In [WZF], Wang et al. introduced some new concepts about Ulam stability of solutions of impulsive fractional differential equations. Recently, in [ABS], Abbas et al. discussed Ulam stability of solutions for a class of fractional differential inclusions with multiple delay and impulses. From the viewpoint of general theories, in [HO, POR] the authors initiated the study of some

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new classes of abstract semilinear impulsive differential equations with noninstantaneous impulses.

Differential equations with impulses were considered for the first time by Milman and Myshkis [MM] and followed by a period of active research; see the monograph by Halanay and Wexler [HW] and its references. Many phenomena and evolution processes in the fields of physics, chemical technology, population dynamics, and natural sciences may change state abruptly or be subject to short-term perturbations (see for instance [ACMAD, LBS] and the references therein). These short perturbations may be seen as impulses. Impulsive problems also arise in various applications in communications, chemical technology, mechanics (jump discontinuities in velocity), electrical engineering, medicine, and biology. These perturbations may be seen as impulses. For instance, in the periodic treatment of some diseases, impulses correspond to the administration of a drug treatment. In environmental sciences, impulses correspond to seasonal changes of the water level of artificial reservoirs. Their models are described by impulsive differential equations and inclusions. Various mathematical results (existence, asymptotic behavior, ...) have been obtained so far (see [A, BHN, GHO, LBS, PD, SP] and the references therein).

In pharmacotherapy, the above instantaneous impulses cannot describe certain dynamics of evolution processes. For example, when one considers the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption are gradual and continuous processes.

Motivated by recent works [R, POR, WFZ], we investigate the existence and Ulam–Hyers–Rassias stability of solutions of the following partial fractional differential inclusions with noninstantaneous impulses:

$$(1.1)\quad \begin{cases} {}^cD^r_{\theta_k}\Big(u(t,x)-\sum_{i=1}^n b_i(t,x)u(t-\alpha_i,x-\beta_i)\Big)\in F(t,x,u(t,x)) \\ \qquad\qquad\qquad\qquad \text{if } (t,x)\in I_k,\ k=0,\dots,m, \\ u(t,x)=g_k(t,x,u(t,x)) \quad \text{if } (t,x)\in J_k,\ k=1,\dots,m, \\ u(t,x)=\phi(t,x) \quad \text{if } (t,x)\in \tilde{J}:=[-\alpha,a]\times[-\beta,b]\setminus(0,a]\times(0,b], \end{cases}$$

where $I_0:=[0,t_1]\times[0,b]$, $I_k:=(s_k,t_{k+1}]\times[0,b]$, $J_k:=(t_k,s_k]\times[0,b]$, $a,b>0$, $\theta_k=(s_k,0)$, $k=0,\dots,m$, ${}^cD^r_{\theta_k}$ is the fractional Caputo derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1]$, $0=s_0<t_1\le s_1\le t_2<\dots<s_{m-1}\le t_m\le s_m\le t_{m+1}=a$, $F:I_k\times E\to\mathcal{P}(E)$, $k=0,\dots,m$, is a set-valued function with nonempty compact values in a (real or complex) separable Banach space E, $\mathcal{P}(E)$ is the family of all subsets of E, $g_k:J_k\times E\to E$, $k=1,\dots,m$, $b_i:I_k\to\mathbb{R}$, $k=0,\dots,m$, $i=1,\dots,n$, are given continuous functions, $\alpha_i,\beta_i\ge 0$, $i=1,\dots,n$, $\alpha=\max_{i=1,\dots,n}\{\alpha_i\}$, $\beta=\max_{i=1,\dots,n}\{\beta_i\}$,

and $\phi : \tilde{J} \to E$ is a given continuous function such that

$$(1.2) \qquad \begin{cases} \phi(t,0) = \sum_{i=1}^{n} b_i(t,0)\phi(t-\alpha_i, -\beta_i), & t \in [0,a], \\ \phi(0,x) = \sum_{i=1}^{n} b_i(0,x)\phi(-\alpha_i, x-\beta_i), & x \in [0,b]. \end{cases}$$

The present paper initiates the study of the existence of solutions and the Ulam stability for problem (1.1).

2. Preliminaries. In this section, we introduce notation, definitions, and preliminary facts which are used throughout this paper. Let $J = [0,a] \times [0,b]$, $a, b > 0$, and denote by $L^1(J)$ the space of Bochner integrable functions $u : J \to E$ with the norm

$$\|u\|_{L^1} = \int_0^a\int_0^b \|u(t,x)\|_E \, dx\, dt,$$

where $\|\cdot\|_E$ denotes the norm of E.

As usual, $\mathcal{C} := C(J)$ denotes the space of all continuous functions from J into E with the norm

$$\|u\|_\infty = \sup_{(t,x)\in J} \|u(t,x)\|_E.$$

Consider the Banach space

$$PC = \big\{u : [-\alpha,a]\times[-\beta,b] \to E : u|_{\tilde{J}} = \phi,\ u|_{J_k} = g_k,\ k = 1,\dots,m,\ u|_{I_k},\ k = 1,\dots,m, \text{ is continuous and there exist } u(s_k^-,x),\ u(s_k^+,x),\ u(t_k^-,x) \text{ and } u(t_k^+,x) \text{ with } u(s_k^+,x) = g_k(s_k,x,u(s_k)) \text{ and } u(t_k^-,x) = g_k(t_k,x,u(t_k)) \text{ for } x\in[0,b]\big\}$$

with the norm

$$\|u\|_{PC} = \sup_{(t,x)\in[-\alpha,a]\times[-\beta,b]} \|u(t,x)\|_E.$$

Let $\theta = (0,0)$, $r_1, r_2 > 0$ and $r = (r_1, r_2)$. For $u \in L^1(J)$, the expression

$$(I_\theta^r u)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t\int_0^x (t-\tau)^{r_1-1}(x-\xi)^{r_2-1} u(\tau,\xi)\, d\xi\, d\tau$$

is called the *left-sided mixed Riemann–Liouville integral of order* r, where $\Gamma(\cdot)$ is the Gamma function.

In particular,

$$(I_\theta^\theta u)(t,x) = u(t,x),$$

$$(I_\theta^\sigma u)(t,x) = \int_0^t\int_0^x u(\tau,\xi)\, d\xi\, d\tau \quad \text{for almost all } (t,x) \in J,$$

where $\sigma = (1,1)$.

For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0,\infty)$, when $u \in L^1(J)$. Note also that when $u \in C(J)$, then $I_\theta^r u \in C(J)$, and moreover

$$(I_\theta^r u)(t,0) = (I_\theta^r u)(0,x) = 0, \qquad t \in [0,a],\ x \in [0,b].$$

EXAMPLE 2.1. Let $\lambda, \omega \in (-1,0) \cup (0,\infty)$, $r = (r_1, r_2)$, $r_1, r_2 \in (0,\infty)$ and $h(t,x) = \frac{t^\lambda x^\omega}{\Gamma(1+\lambda)\Gamma(1+\omega)}$ for $(t,x) \in J$. We have $h \in L^1(J)$ and

$$(I_\theta^r h)(t,x) = \frac{t^{\lambda+r_1} x^{\omega+r_2}}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} \quad \text{for almost all } (t,x) \in J.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in [0,1) \times [0,1)$. Denote by $D_{xt}^2 := \frac{\partial^2}{\partial t \partial x}$ the mixed second order partial derivative.

DEFINITION 2.2 ([VG]). Let $r \in (0,1] \times (0,1]$ and $u \in L^1(J)$. The *Caputo fractional-order derivative* of u of order r is defined by

$$\begin{aligned} {}^cD_\theta^r u(t,x) &= (I_\theta^{1-r} D_{xt}^2 u)(t,x) \\ &= \frac{1}{\Gamma(1-r_1)\Gamma(1-r_2)} \int_0^t \int_0^x \frac{D_{\xi\tau}^2 u(\tau,\xi)}{(t-\tau)^{r_1}(x-\xi)^{r_2}}\, d\xi\, d\tau. \end{aligned}$$

The case $\sigma = (1,1)$ is included and we have

$$({}^cD_\theta^\sigma u)(t,x) = (D_{xt}^2 u)(t,x) \quad \text{for almost all } (t,x) \in J.$$

EXAMPLE 2.3. Let $\lambda, \omega \in (-1,0) \cup (0,\infty)$ and $r = (r_1, r_2) \in (0,1] \times (0,1]$. Then

$${}^cD_\theta^r \frac{t^\lambda x^\omega}{\Gamma(1+\lambda)\Gamma(1+\omega)} = \frac{t^{\lambda-r_1} x^{\omega-r_2}}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} \quad \text{for almost all } (t,x) \in J.$$

Let $a_1 \in [0,a]$, $z^+ = (a_1, 0) \in J$, $J_z = (a_1, a] \times [0,b]$, $r_1, r_2 > 0$ and $r = (r_1, r_2)$. For $u \in L^1(J_z)$, the expression

$$(I_{z^+}^r u)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{a_1^+}^t \int_0^x (t-\tau)^{r_1-1}(x-\xi)^{r_2-1} u(\tau,\xi)\, d\xi\, d\tau$$

is called the *left-sided mixed Riemann–Liouville integral* of u of order r.

DEFINITION 2.4 ([VG]). For $u \in L^1(J_z)$ where $D_{xt}^2 u$ is Bochner integrable on J_z, the *Caputo fractional order derivative* of u of order r is defined to be

$$({}^cD_{z^+}^r u)(t,x) = (I_{z^+}^{1-r} D_{xt}^2 u)(t,x).$$

Let (X,d) be a metric space. We use the following notation:

$$\begin{aligned}\mathcal{P}_{\rm cl}(X) &= \{Y\in\mathcal{P}(X): Y \text{ is closed}\},\\ \mathcal{P}_{\rm b,cl}(X) &= \{Y\in\mathcal{P}(X): Y \text{ is bounded and closed}\},\\ \mathcal{P}_{\rm cp}(X) &= \{Y\in\mathcal{P}(X): Y \text{ is compact}\}.\end{aligned}$$

A multivalued map $G: E\to\mathcal{P}(E)$ has *convex* (resp. *closed*) *values* if $G(x)$ is convex (resp. closed) for all $x\in E$. We say that G is *bounded on bounded sets* if $G(B)$ is bounded in E for each bounded set B of E, i.e.,

$$\sup_{x\in B}\big\{\sup\{\|u\|_E:\ u\in G(x)\}\big\}<\infty.$$

Finally, G has a *fixed point* if there exists $x\in E$ such that $x\in G(x)$.

For each $u\in E$ define the *set of selectors* from F by

$$S_{F,u}=\{v\in L^1(J,E): v(t,x)\in F(t,x,u) \text{ for a.e. } (t,x)\in J\}.$$

For more details on multivalued maps we refer to the books of Deimling [DE] and Górniewicz [G].

Consider $H_d:\mathcal{P}(X)\times\mathcal{P}(X)\to\mathbb{R}_+\cup\{\infty\}$ given by

$$H_d(\mathcal{A},\mathcal{B})=\max\Big\{\sup_{a\in\mathcal{A}}\ d(a,\mathcal{B}),\sup_{b\in\mathcal{B}}d(\mathcal{A},b)\Big\},$$

where $d(\mathcal{A},b)=\inf_{a\in\mathcal{A}}\ d(a,b)$, $d(a,\mathcal{B})=\inf_{b\in\mathcal{B}}\ d(a,b)$. Then $(\mathcal{P}_{\rm b,cl}(X),H_d)$ is a metric space and $(\mathcal{P}_{\rm cl}(X),H_d)$ is a generalized (complete) metric space (see [K]).

Definition 2.5. A multivalued map $G: J\to\mathcal{P}_{\rm cl}(E)$ is said to be *measurable* if for each $x\in E$, the function $Y:J\to E$ defined by

$$Y(t)=d(x,G(t))=\inf\{\|x-y\|_E: y\in G(t)\}$$

is measurable, where d is the metric induced by the norm of E.

Definition 2.6. A multivalued operator $N:X\to\mathcal{P}_{\rm cl}(X)$ is called

- γ-*Lipschitz* if there exists $\gamma>0$ such that
$$H_d(N(u),N(v))\le\gamma d(u,v)\quad \text{for each } u,v\in X,$$
- a *contraction* if it is γ-Lipschitz with $\gamma<1$.

Now, we consider the Ulam stability for problem (1.1). Let $\epsilon>0$, $\varPsi\ge0$ and let $\varPhi: J\to[0,\infty)$ be a continuous function. We consider the following conditions:

$$(2.1)\quad \begin{cases} d\Big({}^cD^r_{\theta_k}\Big(u(t,x)-\displaystyle\sum_{i=1}^n b_i(t,x)u(t-\alpha_i,x-\beta_i)\Big),F(t,x,u(t,x))\Big)\le\epsilon \\ \hfill \text{if } (t,x)\in I_k,\ k=0,\dots,m,\\ \|u(t,x)-g_k(t,x,u(t,x))\|_E\le\epsilon \quad \text{if } (t,x)\in J_k,\ k=1,\dots,m.\end{cases}$$

(2.2)

$$\begin{cases} d\Big({}^cD^r_{\theta_k}\Big(u(t,x)-\sum_{i=1}^n b_i(t,x)u(t-\alpha_i,x-\beta_i)\Big),F(t,x,u(t,x))\Big)\le \Phi(t,x) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } (t,x)\in I_k,\ k=0,\dots,m, \\ \|u(t,x)-g_k(t,x,u(t,x))\|_E\le \Psi \quad \text{if } (t,x)\in J_k,\ k=1,\dots,m. \end{cases}$$

(2.3)

$$\begin{cases} d\Big({}^cD^r_{\theta_k}\Big(u(t,x)-\sum_{i=1}^n b_i(t,x)u(t-\alpha_i,x-\beta_i)\Big),F(t,x,u(t,x))\Big)\le \epsilon\Phi(t,x) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } (t,x)\in I_k,\ k=0,\dots,m, \\ \|u(t,x)-g_k(t,x,u(t,x))\|_E\le \epsilon\Psi \quad \text{if } (t,x)\in J_k,\ k=1,\dots,m. \end{cases}$$

Definition 2.7 ([WZF]). Problem (1.1) is *Ulam–Hyers stable* if there exists a constant $c_{F,g_k}>0$ such that for each $\epsilon>0$ and for each solution $u\in PC$ of (2.1) there exists a solution $v\in PC$ of problem (1.1) with

$$\|u(t,x)-v(t,x)\|_E\le \epsilon c_{F,g_k},\qquad (t,x)\in J.$$

Definition 2.8 ([WZF]). Problem (1.1) is *generalized Ulam–Hyers stable* if there exists $c_{F,g_k}: C([0,\infty),[0,\infty))$ with $c_{F,g_k}(0)=0$ such that for each $\epsilon>0$ and for each solution $u\in PC$ of (2.1) there exists a solution $v\in PC$ of problem (1.1) with

$$\|u(t,x)-v(t,x)\|_E\le c_{F,g_k}(\epsilon),\qquad (t,x)\in J.$$

Definition 2.9 ([WZF]). Problem (1.1) is *Ulam–Hyers–Rassias stable* with respect to (Φ,Ψ) if there exists a constant $c_{F,g_k,\Phi}>0$ such that for each $\epsilon>0$ and for each solution $u\in PC$ of (2.3) there exists a solution $v\in PC$ of problem (1.1) with

$$\|u(t,x)-v(t,x)\|_E\le \epsilon c_{F,g_k,\Phi}(\Psi+\Phi(t,x)),\qquad (t,x)\in J.$$

Definition 2.10 ([WZF]). Problem (1.1) is *generalized Ulam–Hyers–Rassias stable* with respect to (Φ,Ψ) if there exists a constant $c_{F,g_k,\Phi}>0$ such that for each solution $u\in PC$ of (2.2) there exists a solution $v\in PC$ of problem (1.1) with

$$\|u(t,x)-v(t,x)\|_E\le c_{F,g_k,\Phi}(\Psi+\Phi(t,x)),\qquad (t,x)\in J.$$

Remark 2.11. It is clear that: Definition 2.7 $\Rightarrow$ Definition 2.8; Definition 2.9 $\Rightarrow$ Definition 2.10; and Definition 2.9 for $\Phi(\cdot,\cdot)=\Psi=1\Rightarrow$ Definition 2.7.

Remark 2.12. A function $u\in PC$ is a solution of (2.1) if and only if there exist a function $G\in PC$ and a sequence G_k, $k=1,\dots,m$, in E (which depend on u) such that

(i) $\|G(t,x)\|_E \le \epsilon$ and $\|G_k\|_E \le \epsilon$, $k = 1,\dots,m$,
(ii) ${}^cD^r_{\theta_k}(u(t,x) - \sum_{i=1}^n b_i(t,x)u(t-\alpha_i, x-\beta_i)) - G(t,x) \in F(t,x,u(t,x))$ if $(t,x) \in I_k$, $k = 0,\dots,m$,
(iii) $u(t,x) = g_k(t,x,u(t,x)) + G_k$ if $(t,x) \in J_k$, $k = 1,\dots,m$,

One can make similar remarks for (2.2) and (2.3). So, the Ulam stabilities for impulsive fractional differential equations are some special types of data dependence of solutions.

We need the following lemmas.

Lemma 2.13 (Covitz–Nadler [CN]). *Let (X,d) be a complete metric space. If $N : X \to \mathcal{P}_{\mathrm{cl}}(X)$ is a contraction, then N has fixed points.*

Lemma 2.14 (Gronwall lemma [P, P1]). *Let $\upsilon : J \to [0,\infty)$ and let $\omega(\cdot,\cdot)$ be a nonnegative, locally integrable function on J. If there are constants $c > 0$ and $0 < r_1, r_2 < 1$ such that*

$$\upsilon(t,x) \le \omega(t,x) + c\int_0^t\int_0^x \frac{\upsilon(\tau,\xi)}{(t-\tau)^{r_1}(x-\xi)^{r_2}}\,d\xi\,d\tau,$$

then there exists a constant $\delta = \delta(r_1,r_2)$ such that

$$\upsilon(t,x) \le \omega(t,x) + \delta c\int_0^t\int_0^x \frac{\omega(\tau,\xi)}{(t-\tau)^{r_1}(x-\xi)^{r_2}}\,d\xi\,d\tau$$

for every $(t,x) \in J$.

3. Existence and Ulam stabilities results. In this section, we present conditions for the Ulam stability of problem (1.1). As a consequence of [ABN1, Lemma 2.14], we have

Lemma 3.1. *Let $r_1, r_2 \in (0,1]$. A function $u \in PC$ is a solution of problem (1.1) if and only if there exists $f \in S_{F,u}$ such that*

$$\begin{aligned}
u(t,x) = {}& \sum_{i=1}^n b_i(t,x)u(t-\alpha_i, x-\beta_i)\\
&+ \int_0^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi)\,d\xi\,d\tau \quad \text{if } (t,x) \in [0,t_1]\times[0,b],\\
u(t,x) = {}& g_k(s_k,x,u(s_k,x)) - g_k(s_k,0,u(s_k,0))\\
&+ \sum_{i=1}^n b_i(t,x)u(t-\alpha_i, x-\beta_i) - \sum_{i=1}^n b_i(s_k,x)u(s_k-\alpha_i, x-\beta_i)\\
&+ \sum_{i=1}^n b_i(s_k,0)u(s_k-\alpha_i, -\beta_i)
\end{aligned}$$

$$+\int_{s_k}^{t}\int_{0}^{x}\frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f(\tau,\xi)\,d\xi\,d\tau \quad \text{if } (t,x)\in I_k,\ k=1,\dots,m,$$

$$u(t,x)=g_k(t,x,u(t,x)) \quad \text{if } (t,x)\in J_k,\ k=1,\dots,m,$$

$$u(t,x)=\phi(t,x) \quad \text{if } (t,x)\in \tilde{J}.$$

LEMMA 3.2. *If $u\in PC$ is a solution of (2.1), then there exists $f\in S_{F,u}$ such that*

$$\Big\|u(t,x)-\sum_{i=1}^{n}b_i(t,x)u(t-\alpha_i,x-\beta_i)-I_\theta^r f(s,t)\Big\|_E \le \frac{\epsilon a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \quad \text{if } (t,x)\in[0,t_1]\times[0,b],$$

$$\Big\|u(t,x)-g_k(s_k,x,u(s_k,x))+g_k(s_k,0,u(s_k,0)) - \sum_{i=1}^{n}b_i(t,x)u(t-\alpha_i,x-\beta_i)+\sum_{i=1}^{n}b_i(s_k,x)u(s_k-\alpha_i,x-\beta_i) - \sum_{i=1}^{n}b_i(s_k,0)u(s_k-\alpha_i,-\beta_i)-I_{\theta_k}^r f(s,t)\Big\|_E \le \frac{\epsilon a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \quad \text{if } (t,x)\in I_k,\ k=1,\dots,m,$$

$$\|u(t,x)-g_k(t,x,u(t,x))\|_E\le\epsilon \quad \text{if } (t,x)\in J_k,\ k=1,\dots,m.$$

Proof. By Remark 2.12 we have

$$\begin{cases} {}^cD_{\theta_k}^r\big(u(t,x)-\sum_{i=1}^{n}b_i(t,x)u(t-\alpha_i,x-\beta_i)\big)-G(t,x)\in F(t,x,u(t,x)) \\ \qquad\qquad \text{if } (t,x)\in I_k,\ k=0,\dots,m, \\ u(t,x)=g_k(t,x,u(t,x))+G_k \quad \text{if } (t,x)\in J_k,\ k=1,\dots,m. \end{cases}$$

Then there exists $f\in S_{F,u}$ such that

$$u(t,x)=\sum_{i=1}^{n}b_i(t,x)u(t-\alpha_i,x-\beta_i) +\int_0^t\int_0^x\frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}(f+G)(\tau,\xi)\,d\xi\,d\tau \quad \text{if } (t,x)\in[0,t_1]\times[0,b],$$

$$u(t,x)=g_k(s_k,x,u(s_k,x))-g_k(s_k,0,u(s_k,0)) +\sum_{i=1}^{n}b_i(t,x)u(t-\alpha_i,x-\beta_i)-\sum_{i=1}^{n}b_i(s_k,x)u(s_k-\alpha_i,x-\beta_i) +\sum_{i=1}^{n}b_i(s_k,0)u(s_k-\alpha_i,-\beta_i)$$

$$+\int_{s_k}^{t}\int_{0}^{x}\frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}(f+G)(\tau,\xi)\,d\xi\,d\tau \quad \text{if } (t,x)\in I_k,\ k=1,\dots,m,$$

$$u(t,x)=g_k(t,x,u(t,x)) \quad \text{if } (t,x)\in J_k,\ k=1,\dots,m.$$

Thus, it follows that

$$\Big\|u(t,x)-\sum_{i=1}^{n}b_i(t,x)u(t-\alpha_i,x-\beta_i)-I_\theta^r f(s,t)\Big\|_E$$

$$=\Big\|\int_0^t\int_0^x\frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}G(\tau,\xi)\,d\xi\,d\tau\Big\|_E \quad \text{if } (t,x)\in[0,t_1]\times[0,b],$$

$$\Big\|u(t,x)-g_k(s_k,x,u(s_k,x))+g_k(s_k,0,u(s_k,0))$$

$$-\sum_{i=1}^{n}b_i(t,x)u(t-\alpha_i,x-\beta_i)+\sum_{i=1}^{n}b_i(s_k,x)u(s_k-\alpha_i,x-\beta_i)$$

$$-\sum_{i=1}^{n}b_i(s_k,0)u(s_k-\alpha_i,-\beta_i)-I_{\theta_k}^r f(s,t)\Big\|_E$$

$$=\Big\|\int_0^t\int_0^x\frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}G(\tau,\xi)\,d\xi\,d\tau\Big\|_E \quad \text{if } (t,x)\in I_k,\ k=1,\dots,m,$$

$$\|u(t,x)-g_k(t,x,u(t,x))\|_E=\|G_k\|_E \quad \text{if } (t,x)\in J_k,\ k=1,\dots,m.$$

Hence, we obtain the conclusion.

REMARK 3.3. We have similar results for solutions of (2.2) and (2.3).

Set

$$B_k=\max_{i=1,\dots,n}\Big\{\sup_{(t,x)\in I_k}|b_i(t,x)|\Big\},\quad k=0,\dots,m,\qquad B=\max_{k=0,\dots,m}B_k.$$

THEOREM 3.4. *Assume that the following hypotheses hold:*

(H_1) *The multifunction* $F : I_k\times E\to\mathcal{P}_{\rm cp}(E)$ *has the property that* $F(\cdot,\cdot,u) : I_k\to\mathcal{P}_{\rm cp}(E)$ *is measurable for each* $u\in E$, $k=0,\dots,m$.

(H_2) *There exists a constant* $l_F>0$ *such that*

$$H_d(F(t,x,u),F(t,x,v))\le l_F\|u-v\|_E$$

for all $u,v\in E$ *and* $(t,x)\in I_k$, $k=0,\dots,m$.

(H_3) *There exist constants* $l_{g_k}>0$, $k=1,\dots,m$, *such that*

$$\|g_k(t,x,u)-g_k(t,x,\overline{u})\|_E\le l_{g_k}\|u-\overline{u}\|_E$$

for all $(t,x)\in J_k$ *and* $u,\overline{u}\in E$, $k=1,\dots,m$.

If

$$\ell := 2l_g + 3nB + \frac{l_F a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1, \tag{3.1}$$

where $l_g = \max_{k=1,\dots,m} l_{g_k}$, *then problem* (1.1) *has a solution on* J.

Assume moreover that

(H_4) *There exists* $\lambda_\Phi > 0$ *such that for all* $(t,x) \in J$ *we have*

$$I^r_{\theta_k} \Phi(t,x) \le \lambda_\Phi \Phi(t,x), \qquad k = 0,\dots,m.$$

Then problem (1.1) *is generalized Ulam–Hyers–Rassias stable.*

Proof. Consider the multivalued operator $N : PC \to \mathcal{P}(PC)$ defined by letting Nu be the set of all $h \in PC$ such that

$$h(t,x) = \begin{cases} \sum_{i=1}^{n} b_i(t,x)u(t-\alpha_i, x-\beta_i) + I^r_\theta f(t,x) & \text{if } (t,x) \in [0,t_1]\times[0,b], \\ g_k(s_k,x,u(s_k,x)) - g_k(s_k,0,u(s_k,0)) \\ \quad + \sum_{i=1}^{n} b_i(t,x)u(t-\alpha_i, x-\beta_i) \\ \quad - \sum_{i=1}^{n} b_i(s_k,x)u(s_k-\alpha_i, x-\beta_i) \\ \quad + \sum_{i=1}^{n} b_i(s_k,0)u(s_k-\alpha_i, -\beta_i) + I^r_{\theta_k} f(t,x) \\ \qquad\qquad\qquad\qquad \text{if } (t,x) \in I_k,\ k=1,\dots,m, \\ g_k(t,x,u(t,x)) & \text{if } (t,x) \in J_k, \\ \phi(t,x) & \text{if } (t,x) \in \tilde{J}, \end{cases}$$

where $f \in S_{F,u}$. Clearly, by Lemma 3.1, the fixed points of N are solutions of problem (1.1).

Remark 3.5. For each $u \in PC$, the set $S_{F,u}$ is nonempty since by (H_2), F has a measurable selection (see [CV, Theorem III.6]).

We shall show that N satisfies the assumptions of Lemma 2.13. The proof will be given in two steps.

Step 1: *$N(u) \in \mathcal{P}_{\mathrm{cl}}(PC)$ for each $u \in PC$.* Indeed, let $(u_n)_{n\ge0} \subset N(u)$ be such that $u_n \to \tilde{u}$ in PC. Then $\tilde{u} \in PC$ and there exists $f_n \in S_{F,u_n}$ such that, for each $(t,x) \in J$,

$$u_n(t,x) = \sum_{i=1}^{n} b_i(t,x)u_n(t-\alpha_i, x-\beta_i) + I^r_\theta f_n(t,x) \quad \text{if } (t,x) \in [0,t_1]\times[0,b],$$

$$u_n(t,x) = g_k(s_k, x, u_n(s_k,x)) - g_k(s_k, 0, u_n(s_k,0))$$
$$+ \sum_{i=1}^{n} b_i(t,x)u_n(t-\alpha_i, x-\beta_i) - \sum_{i=1}^{n} b_i(s_k,x)u_n(s_k-\alpha_i, x-\beta_i)$$
$$+ \sum_{i=1}^{n} b_i(s_k,0)u_n(s_k-\alpha_i, -\beta_i) + I^r_{\theta_k} f_n(t,x) \quad \text{if } (t,x) \in I_k,\ k = 1,\ldots,m,$$
$$u_n(t,x) = g_k(t,x,u_n(t,x)) \quad \text{if } (t,x) \in J_k,\ k = 1,\ldots,m.$$

Using the fact that F has compact values, and (H_2), we may pass to a subsequence if necessary to find that $f_n(\cdot,\cdot)$ converges to f in $L^1(I_k)$, $k = 0,\ldots,m$, and hence $f \in S_{F,u}$. Then, for each $(t,x) \in J$, $u_n(t,x) \to \tilde{u}(t,x)$, where

$$\tilde{u}(t,x) = \sum_{i=1}^{n} b_i(t,x)u(t-\alpha_i, x-\beta_i) + I^r_\theta f(t,x) \quad \text{if } (t,x) \in [0,t_1] \times [0,b],$$
$$\tilde{u}(t,x) = g_k(s_k, x, u(s_k,x)) - g_k(s_k, 0, u(s_k,0))$$
$$+ \sum_{i=1}^{n} b_i(t,x)u(t-\alpha_i, x-\beta_i) - \sum_{i=1}^{n} b_i(s_k,x)u(s_k-\alpha_i, x-\beta_i)$$
$$+ \sum_{i=1}^{n} b_i(s_k,0)u(s_k-\alpha_i, -\beta_i) + I^r_{\theta_k} f(t,x) \quad \text{if } (t,x) \in I_k,\ k = 1,\ldots,m,$$
$$\tilde{u}(t,x) = g_k(t,x,u(t,x)) \quad \text{if } (t,x) \in J_k,\ k = 1,\ldots,m.$$

So, $\tilde{u} \in N(u)$.

STEP 2: *N is a contraction multivalued operator.* Let $u, \overline{u} \in PC$ and $h \in N(u)$. Then there exists $f(t,x) \in F(t,x,u(t,x))$ such that

$$h(t,x) = \sum_{i=1}^{n} b_i(t,x)u(t-\alpha_i, x-\beta_i) + I^r_\theta f(t,x) \quad \text{if } (t,x) \in [0,t_1] \times [0,b],$$
$$h(t,x) = g_k(s_k, x, u(s_k,x)) - g_k(s_k, 0, u(s_k,0))$$
$$+ \sum_{i=1}^{n} b_i(t,x)u(t-\alpha_i, x-\beta_i) - \sum_{i=1}^{n} b_i(s_k,x)u(s_k-\alpha_i, x-\beta_i)$$
$$+ \sum_{i=1}^{n} b_i(s_k,0)u(s_k-\alpha_i, -\beta_i) + I^r_{\theta_k} f(t,x) \quad \text{if } (t,x) \in I_k,\ k = 1,\ldots,m,$$
$$h(t,x) = g_k(t,x,u(t,x)) \quad \text{if } (t,x) \in J_k,\ k = 1,\ldots,m.$$

From (H_2) it follows that

$$H_d\big(F(t,x,u(t,x)), F(t,x,\overline{u}(t,x))\big) \le l_F \|u(t,x) - \overline{u}(t,x)\|_E.$$

Hence, there exists $w(t,x) \in F(t,x,\overline{u}(t,x))$ such that

$$\|f(t,x) - w(t,x)\| \le l_F \|u(t,x) - \overline{u}(t,x)\|_E \quad \text{if } (t,x) \in I_k,\ k = 0,\dots,m.$$

Consider $U : I_k \to \mathcal{P}(E)$ given by

$$U(t,x) = \{w \in PC : \|f(t,x) - w(t,x)\|_E \le l_F \|u(t,x) - \overline{u}(t,x)\|_E\}.$$

Since the multivalued operator $U(t,x) \cap F(t,x,\overline{u}(t,x))$ is measurable (see [CV, Proposition III.4]), there exists a function $\overline{f}(t,x)$ which is a measurable selection for u. So, $\overline{f}(t,x) \in F(t,x,\overline{u}(t,x))$, and for each $(t,x) \in I_k$, $k = 0,\dots,m$, we have

$$\|f(t,x) - \overline{f}(t,x)\|_E \le l_F \|u(t,x) - \overline{u}(t,x)\|_E.$$

Define

$$\overline{h}(t,x) = \sum_{i=1}^{n} b_i(t,x)\overline{u}(t-\alpha_i, x-\beta_i) + I_\theta^r \overline{f}(t,x) \quad \text{if } (t,x) \in [0,t_1] \times [0,b],$$

$$\begin{aligned} \overline{h}(t,x) &= g_k(s_k,x,\overline{u}(s_k,x)) - g_k(s_k,0,\overline{u}(s_k,0)) \\ &\quad + \sum_{i=1}^{n} b_i(t,x)\overline{u}(t-\alpha_i,x-\beta_i) - \sum_{i=1}^{n} b_i(s_k,x)\overline{u}(s_k-\alpha_i,x-\beta_i) \\ &\quad + \sum_{i=1}^{n} b_i(s_k,0)\overline{u}(s_k-\alpha_i,-\beta_i) + I_{\theta_k}^r \overline{f}(t,x) \quad \text{if } (t,x) \in I_k,\ k = 1,\dots,m, \end{aligned}$$

$$\overline{h}(t,x) = g_k(t,x,\overline{u}(t,x)) \quad \text{if } (t,x) \in J_k,\ k = 1,\dots,m.$$

Then

$$\begin{aligned} \|h(t,x) - \overline{h}(t,x)\|_E \le &\sum_{i=1}^{n} |b_i(t,x)| \left\|u(t-\alpha_i,x-\beta_i) - \overline{u}(t-\alpha_i,x-\beta_i)\right\|_E \\ &+ \left\| \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} [f(\tau,\xi) - \overline{f}(\tau,\xi)]\, d\xi\, d\tau \right\|_E \\ &\qquad \text{if } (t,x) \in [0,t_1] \times [0,b], \end{aligned}$$

$$\begin{aligned} \|h(t,x) - \overline{h}(t,x)\|_E \le &\|g_k(s_k,x,u(s_k,x)) - g_k(s_k,x,\overline{u}(s_k,x))\|_E \\ &+ \|g_k(s_k,0,u(s_k,0)) - g_k(s_k,0,\overline{u}(s_k,0))\|_E \\ &+ \sum_{i=1}^{n} |b_i(t,x)| \left\|u(t-\alpha_i,x-\beta_i) - \overline{u}(t-\alpha_i,x-\beta_i)\right\|_E \\ &+ \sum_{i=1}^{n} |b_i(s_k,x)| \left\|u(s_k-\alpha_i,x-\beta_i) - \overline{u}(s_k-\alpha_i,x-\beta_i)\right\|_E \\ &+ \sum_{i=1}^{n} |b_i(s_k,0)| \left\|u(s_k-\alpha_i,-\beta_i) - \overline{u}(s_k-\alpha_i,-\beta_i)\right\|_E \end{aligned}$$

$$+\left\|\int_{s_k}^{t}\int_{0}^{x}\frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}[f(\tau,\xi)-\overline{f}(\tau,\xi)]\,d\xi\,d\tau\right\|_E$$

$$\text{if } (t,x)\in I_k,\ k=1,\dots,m,$$

$$\|h(t,x)-\overline{h}(t,x)\|_E=\|g_k(t,x,u(t,x))-g_k(t,x,\overline{u}(t,x))\|_E$$

$$\text{if } (t,x)\in J_k,\ k=1,\dots,m.$$

Thus, we get

$$\begin{aligned}\|h(t,x)-\overline{h}(t,x)\|_E &\\ &\le nB\|u-\overline{u}\|_{PC}\\ &\quad+\int_0^t\int_0^x\frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}l_F\|u-\overline{u}\|_{PC}\,d\xi\,d\tau\\ &\le\left(nB+\frac{l_Fa^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\right)\|u-v\|_{PC}\quad\text{if } (t,x)\in[0,t_1]\times[0,b],\end{aligned}$$

$$\begin{aligned}\|h(t,x)-\overline{h}(t,x)\|_E &\le 2l_g\|u-\overline{u}\|_{PC}+3nB\|u-\overline{u}\|_{PC}\\ &\quad+\int_{s_k}^{t}\int_0^x\frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}l_f\|u-\overline{u}\|_{PC}\,d\xi\,d\tau\\ &\le\left(2l_g+3nB+\frac{l_fa^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\right)\|u-v\|_{PC},\\ &\qquad\text{if } (t,x)\in I_k,\ k=1,\dots,m,\end{aligned}$$

$$\|h(t,x)-\overline{h}(t,x)\|_E\le l_g\|u-\overline{u}\|_{PC}\quad\text{if } (t,x)\in J_k,\ k=1,\dots,m.$$

Hence

$$\|h(u)-\overline{h}(v)\|_{PC}\le\ell\|u-\overline{u}\|_{PC}.$$

By an analogous relation, obtained by interchanging the roles of u and $\overline{u}$,

$$H_d(N(u),N(\overline{u}))\le\ell\|u-\overline{u}\|_{PC}.$$

From (3.1), we conclude that N is a contraction and thus, by Lemma 2.13, N has a fixed point v which is a solution to (1.1). Thus, there exists $f_v\in S_{F,v}$ such that

$$v(t,x)=\sum_{i=1}^{n}b_i(t,x)v(t-\alpha_i,x-\beta_i)+I_\theta^r f_v(t,x)\quad\text{if } (t,x)\in[0,t_1]\times[0,b],$$

$$\begin{aligned}v(t,x)&=g_k(s_k,x,v(s_k,x))-g_k(s_k,0,v(s_k,0))\\ &\quad+\sum_{i=1}^{n}b_i(t,x)v(t-\alpha_i,x-\beta_i)-\sum_{i=1}^{n}b_i(s_k,x)v(s_k-\alpha_i,x-\beta_i)\\ &\quad+\sum_{i=1}^{n}b_i(s_k,0)v(s_k-\alpha_i,-\beta_i)+I_{\theta_k}^r f_v(t,x)\quad\text{if } (t,x)\in I_k,\ k=1,\dots,m,\end{aligned}$$

$$v(t,x)=g_k(t,x,v(t,x))\quad\text{if } (t,x)\in J_k,\ k=1,\dots,m.$$

Let $u \in PC$ be a solution of (2.2). From Remark 3.3,

$$\Big\| u(t,x) - \sum_{i=1}^{n} b_i(t,x)u(t-\alpha_i, x-\beta_i) - I_\theta^r f(s,t)\Big\|_E \le \Big\| \int_0^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\Phi(\tau,\xi)\,d\xi\,d\tau\Big\|_E \quad \text{if } (t,x) \in [0,t_1]\times[0,b],$$

$$\Big\| u(t,x) - g_k(s_k,x,u(s_k,x)) + g_k(s_k,0,u(s_k,0)) - \sum_{i=1}^{n} b_i(t,x)u(t-\alpha_i,x-\beta_i) + \sum_{i=1}^{n} b_i(s_k,x)u(s_k-\alpha_i,x-\beta_i) - \sum_{i=1}^{n} b_i(s_k,0)u(s_k-\alpha_i,-\beta_i) - I_{\theta_k}^r f(s,t)\Big\|_E$$
$$\le \Big\| \int_{s_k}^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\Phi(\tau,\xi)\,d\xi\,d\tau\Big\|_E \quad \text{if } (t,x) \in I_k,\ k=1,\dots,m,$$

$$\|u(t,x) - g_k(t,x,u(t,x))\|_E \le \Psi \quad \text{if } (t,x) \in J_k,\ k=1,\dots,m,$$

where $f \in S_{F,v}$. Thus, by (H_4),

$$\Big\| u(t,x) - \sum_{i=1}^{n} b_i(t,x)u(t-\alpha_i, x-\beta_i) - I_\theta^r f(s,t)\Big\|_E \le \lambda_\Phi \Phi(t,x) \quad \text{if } (t,x) \in [0,t_1]\times[0,b],$$

$$\Big\| u(t,x) - g_k(s_k,x,u(s_k,x)) + g_k(s_k,0,u(s_k,0)) - \sum_{i=1}^{n} b_i(t,x)u(t-\alpha_i,x-\beta_i) + \sum_{i=1}^{n} b_i(s_k,x)u(s_k-\alpha_i,x-\beta_i) - \sum_{i=1}^{n} b_i(s_k,0)u(s_k-\alpha_i,-\beta_i) - I_{\theta_k}^r f(s,t)\Big\|_E \le \lambda_\Phi \Phi(t,x)$$
$$\text{if } (t,x) \in I_k,\ k=1,\dots,m,$$

$$\|u(t,x) - g_k(t,x,u(t,x))\|_E \le \Psi \quad \text{if } (t,x) \in J_k,\ k=1,\dots,m.$$

Hence

$$\|u(t,x) - v(t,x)\|_E \le \lambda_\Phi \Phi(t,x) + nB\|u(t,x)-v(t,x)\|_E + \int_0^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\|f(\tau,\xi) - f_v(\tau,\xi)\|_E\,d\xi\,d\tau$$
$$\text{if } (t,x) \in [0,t_1]\times[0,b],$$

$$\|u(t,x)-v(t,x)\|_E \le \lambda_\Phi \Phi(t,x) + (2l_g+3nB)\|u(t,x)-v(t,x)\|_E + \int_{s_k}^{t}\int_0^{x} \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\|f(\tau,\xi)-f_v(\tau,\xi)\|_E \, d\xi\, d\tau$$

if $(t,x)\in I_k$, $k=1,\dots,m$,

$$\|u(t,x)-v(t,x)\|_E \le \Psi + \|g_k(t,x,u(t,x))-g_k(t,x,v(t,x))\|_E \le \Psi + l_g\|u(t,x)-v(t,x)\|_E \quad \text{if } (t,x)\in J_k,\ k=1,\dots,m.$$

For each $(t,x)\in[0,t_1]\times[0,b]$, we have

$$\|u(t,x)-v(t,x)\|_E \le \lambda_\Phi \Phi(t,x) + nB\|u(t,x)-v(t,x)\|_E + l_F\int_0^{t}\int_0^{x} \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\|u(\tau,\xi)-v(\tau,\xi)\|_E \, d\xi\, d\tau.$$

Thus,

$$\|u(t,x)-v(t,x)\|_E \le \frac{\lambda_\Phi}{1-nB}\Phi(t,x) + \frac{l_F}{1-nB}\int_0^{t}\int_0^{x} \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\|u(\tau,\xi)-v(\tau,\xi)\|_E \, d\xi\, d\tau.$$

From Lemma 2.14, there exists a constant $\delta_1 := \delta_1(r_1,r_2)$ such that

$$\|u(t,x)-v(t,x)\|_E \le \frac{\lambda_\Phi}{1-nB}\left(\Phi(t,x) + \frac{l_F\delta_1}{1-nB} I_\theta^r \Phi(t,x)\right) \le \frac{\lambda_\Phi(1+l_F\delta_1\lambda_\Phi)}{1-nB}\Phi(t,x) =: c_{1,F,g_k,\Phi}\Phi(t,x).$$

Thus, for each $(t,x)\in[0,t_1]\times[0,b]$, we get

$$\|u(t,x)-v(t,x)\|_E \le c_{1,F,g_k,\Phi}(\Psi+\Phi(t,x)).$$

Now, for each $(t,x)\in I_k$, $k=1,\dots,m$, we have

$$\|u(t,x)-v(t,x)\|_E \le \lambda_\Phi \Phi(t,x) + (2l_g+3nB)\|u(t,x)-v(t,x)\|_E + l_F\int_{s_k}^{t}\int_0^{x} \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\|u(\tau,\xi)-v(\tau,\xi)\|_E \, d\xi\, d\tau.$$

Thus,

$$\|u(t,x)-v(t,x)\|_E \le \frac{\lambda_\Phi}{1-2l_g-3nB}\Phi(t,x) + \frac{l_F}{1-2l_g-3nB}\int_{s_k}^{t}\int_0^{x} \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\|u(\tau,\xi)-v(\tau,\xi)\|_E \, d\xi\, d\tau.$$

Again, from Lemma 2.14, there exists a constant $\delta_2 := \delta_2(r_1, r_2)$ such that

$$\begin{aligned}\|u(t,x)-v(t,x)\|_E &\le \frac{\lambda_\Phi}{1-2l_g-3nB}\left(\Phi(t,x)+\frac{l_F\delta_2}{1-2l_g-3nB}I^r_{\theta_k}\Phi(t,x)\right)\\ &\le \frac{\lambda_\Phi}{1-2l_g-3nB}\left(1+\frac{l_F\delta_2\lambda_\Phi}{1-2l_g-3nB}\right)\Phi(t,x) =: c_{2,F,g_k,\Phi}\Phi(t,x).\end{aligned}$$

Hence, for each $(t,x)\in I_k,\ k=1,\dots,m$, we get

$$\|u(t,x)-v(t,x)\|_E \le c_{2,F,g_k,\Phi}(\Psi+\Phi(t,x)).$$

Now, for each $(t,x)\in J_k,\ k=1,\dots,m$, we have

$$\|u(t,x)-v(t,x)\|_E \le \Psi + l_g\|u(t,x)-v(t,x)\|_E.$$

This gives

$$\|u(t,x)-v(t,x)\|_E \le \frac{\Psi}{1-l_g} := c_{3,F,g_k,\Phi}\Psi.$$

Thus, for each $(t,x)\in J_k,\ k=1,\dots,m$, we get

$$\|u(t,x)-v(t,x)\|_E \le c_{3,F,g_k,\Phi}(\Psi+\Phi(t,x)).$$

Set $c_{F,g_k,\Phi} := \max_{i\in\{1,2,3\}} c_{i,F,g_k,\Phi}$. Hence, for each $(t,x)\in J$, we obtain

$$\|u(t,x)-v(t,x)\|_E \le c_{F,g_k,\Phi}(\Psi+\Phi(t,x)).$$

Consequently, problem (1.1) is generalized Ulam–Hyers–Rassias stable.

4. An example. Let

$$E = l^1 = \left\{w=(w_1,w_2,\dots) : \sum_{n=1}^{\infty}|w_n|<\infty\right\}$$

be the Banach space with norm

$$\|w\|_E = \sum_{n=1}^{\infty}|w_n|.$$

Consider the following partial fractional differential inclusions with noninstantaneous impulses:

(4.1)

$$\begin{cases} {}^cD^r_{\theta_k}\Big(u(t,x)-\frac{t^2x^3}{111(1+t^2)}u(t-1,x-3)+\frac{t^4x^2}{112(1+t^4)}u(t-2,x-1/4)\\ \qquad +\frac{1}{114}u\big(t-3/2,x-2\big)\Big)\in F(t,x,u(t,x))\\ \qquad\qquad\qquad\qquad \text{if } (t,x)\in([0,1]\cup(2,3])\times[0,1],\\ u(t,x)=g(t,x,u(t,x)) \quad \text{if } (t,x)\in(1,2]\times[0,1],\\ u(t,x)=\Phi(t,x) \quad \text{if } (t,x)\in\tilde J := [-2,3]\times[-3,1]\setminus(0,3]\times(0,1], \end{cases}$$

where $k \in \{0,1\}$, $n = 3$, $r = (r_1, r_2) \in (0,1] \times (0,1]$, $\theta_0 = \theta$, $\theta_1 = (2,0)$, $0 = s_0 < t_1 = 1 < s_1 = 2 < t_2 = 3$, $u = (u_1, u_2, \ldots)$, $F = (F_1, F_2, \ldots)$, $g = (g_1, g_2, \ldots)$,

$$^cD^r_\theta u = (^cD^r_\theta u_1, ^cD^r_\theta u_2, \ldots),$$

$\Phi : \tilde{J} \to \mathbb{R}$ is a continuous function satisfying

$$\Phi(t,0) = \tfrac{1}{14}\Phi(t - 3/2, -2),\ \Phi(0,x) = \tfrac{1}{14}\Phi(-3/2, x-2),\ t \in [0,3], x \in [0,1],$$

and $F : ([0,1] \cup (2,3]) \times [0,1] \times E \to \mathcal{P}(E)$ is given by

$$F(t,x,u(t,x)) = \{v \in E : \|f_1(t,x,u(t,x))\|_E \le \|v\|_E \le \|f_2(t,x,u(t,x))\|_E\}$$

for $(t,x) \in [0,3] \times [0,1]$, where $f_1, f_2 : [0,1] \times [0,1] \times E \to E$ with

$$\begin{aligned}
&f_k = (f_{k,1}, f_{k,2}, \ldots), && k \in \{1,2\},\ n \in \mathbb{N},\\
&f_{1,n}(t,x,u_n(t,x)) = \frac{e^{t+x-4}}{111(1 + \|u_n\|_E)}, && n \in \mathbb{N},\\
&f_{2,n}(t,x,u_n(t,x)) = \frac{e^{t+x-4}}{111} u_n, && n \in \mathbb{N},\\
&g_n(t,x,u_n) = \frac{1}{(1 + 110e^{t+x})(1 + |u_n|)}, && (t,x) \in (1,2] \times [0,1],\ n \in \mathbb{N}.
\end{aligned}$$

Set

$$b_1(t,x) = \frac{t^2x^3}{111(1+t^2)}, \quad b_2(t,x) = \frac{t^4x^2}{112(1+t^4)}, \quad b_3(t,x) = \frac{1}{114};$$

then $B = 1/111$. We assume that F is closed valued. For all $n \in \mathbb{N}$, $u, \overline{u} \in E$ and $(t,x) \in ([0,1] \cup (2,3]) \times [0,1]$, we have

$$H_d(F_n(t,x,u_n) - F_n(t,x,\overline{u}_n)) \le \tfrac{1}{111}|u_n - \overline{u}_n|.$$

Thus, for all $u, \overline{u} \in E$ and $(t,x) \in [0,1] \times [0,1]$, we get

$$\begin{aligned}
&H_d\big(F(t,x,u(t,x)), F(t,x,\overline{u}(t,x))\big)\\
&\qquad = \sum_{n=1}^{\infty} H_d\big(F_n(t,x,u_n(t,x)), F_n(t,x,\overline{u}_n(t,x))\big)\\
&\qquad \le \tfrac{1}{111}\sum_{n=1}^{\infty} |u_n - \overline{u}_n| = \tfrac{1}{111}\|u - \overline{u}\|_E.
\end{aligned}$$

Also, for all $n \in \mathbb{N}$, $u, \overline{u} \in E$ and $(t,x) \in (1,2] \times [0,1]$,

$$\|g(t,x,u(t,x)) - g(t,x,\overline{u}(t,x))\|_E \le \tfrac{1}{111}\|u - \overline{u}\|_E.$$

Hence conditions (H_1)–(H_3) are satisfied with $l_F = l_g = \frac{1}{111}$. We shall show that condition (3.1) holds with $a = 3$ and $b = 1$. Indeed, for each $(r_1, r_2) \in$

$(0,1]\times(0,1]$ we get

$$\ell = 2l_g + 3nB + \frac{l_F a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}$$
$$= \frac{2}{111} + \frac{9}{111} + \frac{3^{r_1}}{111\Gamma(1+r_1)\Gamma(1+r_2)} < \frac{23}{111} < 1.$$

Finally, hypothesis (H_3) is satisfied with

$$\Phi(t,x) = tx^2, \qquad \lambda_\Phi = \frac{2\times 3^{r_1}}{\Gamma(2+r_1)\Gamma(3+r_2)}.$$

Indeed, for each $(t,x)\in[0,3]\times[0,1]$,

$$(I_\theta^r \Phi)(t,x) = \frac{\Gamma(2)\Gamma(3)t^{1+r_1}x^{2+r_2}}{\Gamma(2+r_1)\Gamma(3+r_2)} \le \frac{2\times 3^{r_1}tx^2}{\Gamma(2+r_1)\Gamma(3+r_2)} = \lambda_\Phi \Phi(t,x).$$

Consequently, Theorem 3.4 implies that problem (4.1) is generalized Ulam–Hyers–Rassias stable.

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