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On weighted composition operators acting between weighted Bergman spaces of infinite order and weighted Bloch type spaces

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Abstract. Let $\phi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to \mathbb{C}$ be analytic maps. They induce a weighted composition operator ψC_ϕ acting between weighted Bergman spaces of infinite order and weighted Bloch type spaces. Under some assumptions on the weights we give a characterization for such an operator to be bounded in terms of the weights involved as well as the functions ψ and ϕ.

1. Introduction. Let $H(\mathbb{D})$ denote the class of all analytic functions on the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$. In this note we consider an analytic self-map ϕ of $\mathbb{D}$, i.e. an analytic map on $\mathbb{D}$ such that $\phi(\mathbb{D}) \subset \mathbb{D}$. (At some points we will need that this map is also bijective, but this will then be assumed additionally.) Each such map induces through composition a linear composition operator $C_\phi : H(\mathbb{D}) \to H(\mathbb{D})$, $f \mapsto f \circ \phi$. For $\psi \in H(\mathbb{D})$ we obtain by multiplication with ψ the weighted composition operator $\psi C_\phi : H(\mathbb{D}) \to H(\mathbb{D})$, $f \mapsto \psi(f \circ \phi)$. Furthermore, let v and w be strictly positive continuous and bounded functions (*weights*) on $\mathbb{D}$. We are interested in weighted composition operators ψC_ϕ acting between weighted Bergman spaces of infinite order

$$H_v^\infty := \{f \in H(\mathbb{D});\ \|f\|_v := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\},$$

endowed with the weighted sup-norm $\|\cdot\|_v$, and the weighted Bloch type spaces

$$B_w := \{f \in H(\mathbb{D});\ \|f\|_{B_w} := \sup_{z\in\mathbb{D}} w(z)|f'(z)| < \infty\}.$$

Provided we identify functions that differ by a constant, $\|\cdot\|_{B_w}$ becomes a norm and B_w a Banach space.

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Weighted Banach spaces of holomorphic functions have important applications in functional analysis (spectral theory, functional calculus), complex analysis, partial differential equations and convolution equations, as well as distribution theory. For a deep study of these spaces we refer the reader to the articles of Bierstedt–Bonet–Galbis [1] and Bierstedt–Bonet–Taskinen [2].

The investigation of (weighted) composition operators has quite a long history. Operators of this type have been studied by many authors on various spaces of holomorphic functions such as the Bloch space, Bergman space, Hardy space and weighted Banach spaces of holomorphic functions (see e.g. [6], [15], [3], [5], [8], [12], [13], [14]). In [16] we already studied weighted composition operators acting between weighted Bergman spaces of infinite order and weighted Bloch type spaces. There we were only able to give different sufficient and necessary conditions for ψC_ϕ to be bounded resp. compact. Here, with a new approach, we give a full characterization.

2. Notation and auxiliary results. For general information on the concept of composition operators we refer the reader to the excellent monographs [6] and [15]. In the setting of weighted spaces the so-called *associated weights* play an important role. For a weight v its associated weight $\tilde{v}$ is defined as follows:

$$\tilde{v}(z) = \frac{1}{\sup\{|f(z)|;\ f \in H(\mathbb{D}),\ \|f\|_v \le 1\}} = \frac{1}{\|\delta_z\|_{H_v^{\infty\prime}}},$$

where δ_z denotes the point evaluation at z. By [2] the associated weight $\tilde{v}$ is continuous, $\tilde{v} \ge v > 0$ and for every $z \in \mathbb{D}$ we can find $f_z \in H_v^\infty$ with $\|f_z\|_v \le 1$ such that $|f_z(z)| = 1/\tilde{v}(z)$. We say that a weight v is *radial* if $v(z) = v(|z|)$ for every $z \in \mathbb{D}$. A radial, non-increasing weight is called *typical* if $\lim_{|z|\to 1} v(z) = 0$.

For a typical weight v, by [5] we know that a weighted composition operator $\psi C_\phi : H_v^\infty \to H_w^\infty$ is bounded (resp. compact) if and only if $\sup_{z\in\mathbb{D}} w(z)|\psi(z)|/\tilde{v}(\phi(z)) < \infty$ (resp. $\lim_{|z|\to 1} w(z)|\psi(z)|/\tilde{v}(\phi(z)) = 0$ and $\psi \in H_w^\infty$).

Throughout this article we consider the differentiation operator $D : H_v^\infty \to H_w^\infty$, $f \mapsto f'$. In the case that v and w are typical weights which are continuously differentiable with respect to $|z|$ such that H_w^∞ is isomorphic to ℓ^∞, in [7] Harutyunyan and Lusky showed that the condition $\lim_{r\to 1}(-w'(r)/v(r)) < \infty$ yields the boundedness of the operator $D : H_v^\infty \to H_w^\infty$, $f \mapsto f'$. Conditions ensuring that H_w^∞ is isomorphic to ℓ^∞ can be found in [11] and [7]. By [7] we know that the following weights have the desired properties:

$$w(z) = (1-|z|)^\alpha, \quad \alpha > 0, \quad \text{and} \quad w(z) = e^{-1/(1-|z|)}, \quad z \in \mathbb{D}.$$

We assume the following setting. Let ν be a holomorphic function on $\mathbb{D}$ that is non-vanishing, strictly positive and decreasing on $[0,1)$. Furthermore we suppose that $\lim_{|z|\to 1}\nu(z) = 0$ and that ν' is bounded on $\mathbb{D}$. Then we define the corresponding weight by

$$v(z) := \nu(|z|^2) \quad \text{for every } z \in \mathbb{D}.$$

Let us give some examples of weights of this type:

(i) Consider $\nu(z) = (1-z)^\alpha$, $\alpha \geq 1$, for every $z \in \mathbb{D}$. Then the corresponding weight is given by $v(z) = (1-|z|^2)^\alpha$ for every $z \in \mathbb{D}$.
(ii) For $\nu(z) = e^{-1/(1-z)^\alpha}$, $\alpha \geq 1$, for every $z \in \mathbb{D}$, we obtain $v(z) = e^{-1/(1-|z|^2)^\alpha}$ for every $z \in \mathbb{D}$.
(iii) Choose $\nu(z) = \sin(1-z)$ for every $z \in \mathbb{D}$; then $v(z) = \sin(1-|z|^2)$ for every $z \in \mathbb{D}$.

Next, we fix a point $p \in \mathbb{D}$ and introduce a function

$$v_p(z) := \nu(\overline{\phi(p)}z) \quad \text{for every } z \in \mathbb{D}.$$

Since ν is holomorphic on $\mathbb{D}$, so is the function v_p. Furthermore, $v_p(\phi(p)) = \nu(|\phi(p)|^2) = v(\phi(p))$ and $v_p'(z) = \overline{\phi(p)}\nu'(\overline{\phi(p)}z)$ for every $z \in \mathbb{D}$, i.e. $v_p'(\phi(p)) = \overline{\phi(p)}\nu'(|\phi(p)|^2)$. Moreover, we assume that there is a constant $C > 0$ with

$$\sup_{p\in\mathbb{D}}\sup_{z\in\mathbb{D}} \frac{v(z)}{|v_p(z)|} \leq C. \tag{2.1}$$

Here are some examples satisfying (2.1):

(a) Consider $v(z) = 1-|z|^2$ for every $z \in \mathbb{D}$. Then

$$\frac{v(z)}{|v_p(z)|} = \frac{1-|z|^2}{|1-\overline{\phi(p)}z|} \leq \frac{1-|z|^2}{1-|z|} \leq 1+|z| \leq 2 \quad \text{for every } z \in \mathbb{D}.$$

(b) Set $v(z) = 1/(1-\log(1-|z|^2))$ for every $z \in \mathbb{D}$. This weight has the desired property since $|1-\log(1-\overline{\phi(p)}z)| \leq 1-\log(1-|z|)$ for every $z \in \mathbb{D}$ and the function $\frac{1-\log(1-|z|)}{1-\log(1-|z|^2)}$ is continuous and tends to 1 as $|z| \to 1$.

3. Boundedness of $\psi C_\phi : H_v^\infty \to B_w$. In this section we study when an operator $\psi C_\phi : H_v^\infty \to B_w$ is bounded.

THEOREM 3.1. *Let v and w be weights. Moreover, let $\psi \in H(\mathbb{D})$ and ϕ be an analytic self-map of $\mathbb{D}$. If*

(a) *$\psi' C_\phi : H_v^\infty \to H_w^\infty$ is bounded,*
(b) *$\psi D C_\phi : H_v^\infty \to H_w^\infty$ is bounded,*

then the operator $\psi C_\phi : H_v^\infty \to B_w$ *is bounded. If we assume additionally that* v *is a weight as defined above, i.e.* $v(z) = \nu(|z|^2)$ *for every* $z \in \mathbb{D}$ *and* (2.1) *is satisfied, then the converse is also true.*

Proof. Obviously, the operator $\psi C_\phi : H_v^\infty \to B_w$ can be considered as

$$\psi' C_\phi + \psi D C_\phi : H_v^\infty \to H_w^\infty,$$

since for $f \in H_v^\infty$ we obtain

$$\begin{aligned}\|\psi C_\phi f\|_{B_w} &= \sup_{z\in\mathbb{D}} w(z)|\psi'(z) f(\phi(z)) + \psi(z)\phi'(z) f'(\phi(z))| \\ &= \sup_{z\in\mathbb{D}} w(z)|(\psi' C_\phi + \psi D C_\phi) f(z)| = \|(\psi' C_\phi + \psi D C_\phi) f\|_w.\end{aligned}$$

Hence, if (a) and (b) hold then $\psi C_\phi : H_v^\infty \to B_w$ is bounded as desired.

Conversely, to show (a), we fix a point $p \in \mathbb{D}$ and put

$$f_p(z) := \frac{1}{v_p(z)} - \frac{1}{v(\phi(p))} \qquad \text{for every } z \in \mathbb{D}.$$

Then, by hypothesis,

$$\|f_p\|_v = \sup_{z\in\mathbb{D}} \left| \frac{v(z)}{v_p(z)} - \frac{v(z)}{v(\phi(p))} \right| \le 2 \sup_{p\in\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{v(z)}{|v_p(z)|} \le 2C,$$

where C is the constant of (2.1) and thus independent of the choice of p. Hence $f_p \in H_v^\infty$. Then $f_p'(z) = -v_p'(z)/v_p(z)^2$ for every $z \in \mathbb{D}$ and hence $f_p(\phi(p)) = 0$ and $f_p'(\phi(p)) = -v_p'(\phi(p))/v(\phi(p))^2$. Thus, we obtain

$$\begin{aligned}\frac{w(p)|\psi(p)|\,|\phi'(p)|\,|v_p'(\phi(p))|}{v(\phi(p))^2} &= w(p)|\psi(p)\phi'(p) f_p'(\phi(p)) + \psi'(p) f_p(\phi(p))| \\ &= w(p)|(\psi C_\phi f_p)'(p)| \\ &\le \|\psi C_\phi f_p\|_{B_w} \le \|\psi C_\phi\|\,\|f_p\|_v \le 2C\|\psi C_\phi\|.\end{aligned}$$

Finally, $\sup_{z\in\mathbb{D}} w(z)|\psi(z)|\,|\phi'(z)|\,|v_z'(\phi(z))|/v(\phi(z))^2 < \infty$. We again fix a point $p \in \mathbb{D}$ and set

$$g_p(z) := \frac{1}{v_p(z)} \qquad \text{for every } z \in \mathbb{D}.$$

Then $\|g_p\|_v \le C$ for every $p \in \mathbb{D}$ and $g_p'(z) = -v_p'(z)/v_p(z)^2$ for every $z \in \mathbb{D}$. Hence $g_p(\phi(p)) = 1/v(\phi(p))$ and $g_p'(\phi(p)) = -v_p'(\phi(p))/v(\phi(p))^2$. We conclude that

$$\begin{aligned}&w(p)\left| \frac{\psi'(p)}{v(\phi(p))} - \frac{\psi(p)\phi'(p)v_p'(\phi(p))}{v(\phi(p))^2} \right| \\ &\quad = w(p)|\psi'(p) g_p(\phi(p)) + \psi(p)\phi'(p) g_p'(\phi(p))| = w(p)|(\psi C_\phi g_p)'(p)| \\ &\quad \le \|\psi C_\phi g_p\|_{B_w} \le \|\psi C_\phi\|\|g_p\|_v \le C\|\psi C_\phi\|.\end{aligned}$$

Since $\sup_{z\in\mathbb{D}} w(z)|\psi(z)|\,|\phi'(z)|\,|v_z'(\phi(z))|/v(\phi(z))^2 < \infty$, we get

$$\sup_{z\in\mathbb{D}} \frac{|\psi'(z)|w(z)}{v(\phi(z))} < \infty, \tag{3.1}$$

i.e. (a) is satisfied. By hypothesis, we can find a constant $M > 0$ such that

$$\|\psi C_\phi f\|_{B_w} = \sup_{z\in\mathbb{D}} w(z)|\psi(z)\phi'(z)f'(\phi(z)) + \psi'(z)f(\phi(z))| \le M\|f\|_v.$$

Since, by (a), $\psi' C_\phi : H_v^\infty \to H_w^\infty$ is bounded, there must be a constant $L > 0$ with

$$\sup_{z\in\mathbb{D}} w(z)|\psi(z)\phi'(z)f'(\phi(z))| \le L\|f\|_v.$$

Thus, (b) follows. ∎

Next, we analyze under which conditions the operator $\psi DC_\phi : H_v^\infty \to H_w^\infty$ is bounded.

THEOREM 3.2. *Let v and w be weights. Moreover, let ϕ be an analytic, bijective self-map of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$ be such that $\psi\phi' \in H^\infty$ and $|\psi(z)\phi'(z)| \ge \alpha > 0$ for every $z \in \mathbb{D}$. Then $\psi DC_\phi : H_v^\infty \to H_w^\infty$ is bounded if and only if we can find a weight u such that*

(a) *$D : H_v^\infty \to H_u^\infty$, $f \mapsto f'$, is bounded,*
(b) *$\sup_{z\in\mathbb{D}} w(z)/u(\phi(z)) < \infty$.*

Proof. First, we assume that there is a weight u such that (a) and (b) hold. Let $f \in H_v^\infty$. Then

$$\begin{aligned}\|\psi DC_\phi f\|_w &= \sup_{z\in\mathbb{D}} |\psi(z)|\,|\phi'(z)|w(z)|f'(\phi(z))| \\ &= \sup_{z\in\mathbb{D}} \frac{|\psi(z)|\,|\phi'(z)|w(z)}{u(\phi(z))} u(\phi(z))|f'(\phi(z))| \\ &\le \sup_{z\in\mathbb{D}} \frac{|\psi(z)|\,|\phi'(z)|w(z)}{u(\phi(z))} \|f'\|_u \le \sup_{z\in\mathbb{D}} \frac{|\psi(z)|\,|\phi'(z)|w(z)}{u(\phi(z))} \|D\|\,\|f\|_v\end{aligned}$$

and the claim follows.

Conversely, assume that the operator $\psi DC_\phi : H_v^\infty \to H_w^\infty$ is bounded. Define $u := w \circ \phi^{-1}$. Then

$$\begin{aligned}\sup_{z\in\mathbb{D}} \frac{|\psi(z)|\,|\phi'(z)|w(z)}{u(\phi(z))} &= \sup_{z\in\mathbb{D}} |\psi(z)|\,|\phi'(z)| < \infty, \\ \frac{|\psi(z)|\,|\phi'(z)|w(z)}{u(\phi(z))} &= |\psi(z)||\phi'(z)| \ge \alpha > 0 \quad \text{for every } z \in \mathbb{D}.\end{aligned}$$

By assumption the operator $\psi DC_\phi : H_v^\infty \to H_w^\infty$ is bounded, i.e. we can find a constant $L > 0$ such that

$$\sup_{z\in\mathbb{D}} |\psi(z)\phi'(z)|u(\phi(z))|f'(\phi(z))| = \sup_{z\in\mathbb{D}} |\psi(z)|\,|\phi'(z)|w(z)|f'(\phi(z))| \le L\|f\|_v.$$

We have

$$\begin{aligned}
&\alpha \sup_{z\in\mathbb{D}} u(\phi(z))|f'(\phi(z))| \le L\|f\|_v \\
&\quad \Leftrightarrow\ \alpha \sup_{z\in\mathbb{D}} u(\phi(\phi^{-1}(z)))|f'(\phi(\phi^{-1}(z)))| = \alpha \sup_{z\in\mathbb{D}} u(z)|f'(z)| \le L\|f\|_v \\
&\quad \Leftrightarrow\ \|Df\|_u \le \frac{L}{\alpha}\|f\|_v.
\end{aligned}$$

Thus, we conclude that the operator $D : H_v^\infty \to H_u^\infty$, $f \mapsto f'$, is bounded. ■

4. Compactness of $\psi C_\phi : H_v^\infty \to B_w$. In this section we study under which conditions an operator $\psi C_\phi : H_v^\infty \to B_w$ is compact. To do this we need the following auxiliary result taken from [6].

LEMMA 4.1 (Cowen–MacCluer, [6, Proposition 3.11]). *A bounded operator $\psi C_\phi : H_v^\infty \to B_w$ is compact if and only if for every bounded sequence $(f_n)_{n\in\mathbb{N}}$ in H_v^∞ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, we have $\psi C_\phi f_n \to 0$ in B_w.*

THEOREM 4.2. *Let v and w be weights such that the operator $\psi C_\phi : H_v^\infty \to B_w$ is bounded. Moreover, let $\psi \in H_w^\infty$ and ϕ be an analytic self-map of $\mathbb{D}$. If*

(a) *$\psi' C_\phi : H_v^\infty \to H_w^\infty$ is compact,*
(b) *$\psi D C_\phi : H_v^\infty \to H_w^\infty$ is compact,*

then the operator $\psi C_\phi : H_v^\infty \to B_w$ is compact. If we assume additionally that v is a weight as defined above, i.e. $v(z) = \nu(|z|^2)$ for every $z \in \mathbb{D}$ and (2.1) is satisfied, then the converse is also true.

Proof. Since $\psi C_\phi : H_v^\infty \to B_w$ can be considered as $\psi' C_\phi + \psi D C_\phi : H_v^\infty \to H_w^\infty$, the compactness of $\psi C_\phi : H_v^\infty \to B_w$ follows immediately from (a) and (b).

Conversely, to show (a) the idea is to use Lemma 4.1. Thus, we select a sequence $(z_n)_n \subset \mathbb{D}$ such that $|\phi(z_n)| \to 1$ as $n \to \infty$. Moreover, we fix $k \in \mathbb{N}$, $k \ge 3$, and set

$$f_{n,k}(z) := v(\phi(z_n))^{1/2k}\left(\frac{k}{k-1}\frac{1}{v_{z_n}(z)^{k-1}} - \frac{v(\phi(z_n))}{v_{z_n}(z)^k}\right)^{1/k}$$

for every $n \in \mathbb{N}$ and every $z \in \mathbb{D}$. Then

$$\begin{aligned}
\|f_{n,k}\|_v &= \sup_{z\in\mathbb{D}} v(\phi(z_n))^{1/2k}\left|\frac{k}{k-1}\frac{v(z)^k}{v_{z_n}(z)^{k-1}} - \frac{v(\phi(z_n))v(z)^k}{v_{z_n}(z)^k}\right|^{1/k} \\
&\le M^{1/2k}\left(\frac{k}{k-1}C^{k-1}M + C^k M\right)^{1/k},
\end{aligned}$$

where $M := \sup_{z\in\mathbb{D}} v(z)$ and C is the constant of (2.1). Both M and C are independent of the choice of n and k. Thus, $f_{n,k} \in H_v^\infty$ for every $n \in \mathbb{N}$. Moreover, $(f_{n,k})_n$ tends to zero uniformly on compact sets as $n \to \infty$. We obtain the following derivative:

$$f'_{n,k}(z) = v(\phi(z_n))^{1/2k}\left(\frac{k}{k-1}\,\frac{1}{v_{z_n}(z)^{k-1}} - \frac{v(\phi(z_n))}{v_{z_n}(z)^k}\right)^{-(k-1)/k} \times\left(\frac{-v'_{z_n}(z)}{v_{z_n}(z)^k} + \frac{v(\phi(z_n))v'_{z_n}(z)}{v_{z_n}(z)^{k+1}}\right)$$

for every $n \in \mathbb{N}$ and every $z \in \mathbb{D}$. Hence

$$f_{n,k}(\phi(z_n)) = \frac{1}{(k-1)^{1/k}}\,\frac{1}{v(\phi(z_n))^{1-3/2k}} \quad \text{and} \quad f'_{n,k}(\phi(z_n)) = 0$$

for every $n \in \mathbb{N}$. Thus,

$$\begin{aligned}\|(\psi C_\phi)f_{n,k}\|_{B_w} &\geq w(z_n)|\psi'(z_n)f_{n,k}(\phi(z_n)) + \psi(z_n)\phi'(z_n)f'_{n,k}(\phi(z_n))| \\ &= w(z_n)|\psi'(z_n)f_{n,k}(\phi(z_n))| \\ &= \frac{1}{(k-1)^{1/k}}\,\frac{|\psi'(z_n)|}{v(\phi(z_n))^{1-3/2k}}w(z_n).\end{aligned}$$

Since $\psi C_\phi : H_v^\infty \to B_w$ is compact, by Proposition 4.1, $(\|\psi C_\phi f_{n,k}\|_{B_w})_n$ must tend to zero. Hence

$$\limsup_{|\phi(z)|\to 1} \frac{1}{(k-1)^{1/k}}\,\frac{|\psi'(z)|}{v(\phi(z))^{1-3/2k}}w(z) = 0.$$

Next, if $k \to \infty$, we arrive at

$$\limsup_{|\phi(z)|\to 1} \frac{w(z)|\psi'(z)|}{v(\phi(z))} = 0.$$

By [5, Corollary 4.3] this implies that the operator $\psi' C_\phi : H_v^\infty \to H_w^\infty$ is compact. With the argument we used to prove the other direction we can conclude that (b) must be true. ∎

THEOREM 4.3. *Let v and w be weights, ϕ be an analytic, bijective self-map of $\mathbb{D}$, and $\psi \in H(\mathbb{D})$ be such that $\psi\phi' \in H^\infty$ and $|\psi(z)\phi'(z)| \geq \alpha > 0$ for every $z \in \mathbb{D}$. Then $\psi DC_\phi : H_v^\infty \to H_w^\infty$ is compact if and only if we can find a weight u such that*

(a) *$D : H_v^\infty \to H_u^\infty$, $f \mapsto f'$, is compact,*
(b) *$\sup_{z\in\mathbb{D}} w(z)/u(\phi(z)) < \infty$.*

Proof. First, we assume that (a) and (b) are satisfied. Let $(f_n)_n$ be a bounded sequence in H_v^∞. We have to find a subsequence $(f_{n_k})_k$ such that $(\psi DC_\phi f_{n_k})_k$ is convergent in H_w^∞. By hypothesis, we know that D :

$H_v^\infty \to H_u^\infty$, $f \mapsto f'$, is compact. Hence there is $g \in H_u^\infty$ such that for every $\varepsilon > 0$ there is $k_0 \in \mathbb{N}$ with

$$\sup_{z\in\mathbb{D}} u(z)|f'_{n_k}(z) - g(z)| < \varepsilon \quad \text{for every } k \geq k_0.$$

Next, we get

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} w(z)|\psi DC_\phi f_{n_k}(z) - \psi(z)\phi'(z)g(\phi(z))| \\
&\qquad = \sup_{z\in\mathbb{D}} w(z)|\psi(z)|\,|\phi'(z)|\,|f'_{n_k}(\phi(z)) - g(\phi(z))| \\
&\qquad = \sup_{z\in\mathbb{D}} \frac{w(z)|\psi(z)|\,|\phi'(z)|}{u(\phi(z))} u(\phi(z))|f'_{n_k}(\phi(z)) - g(\phi(z))| \leq K\varepsilon
\end{aligned}$$

for every $k \geq k_0$, where $K := \sup_{z\in\mathbb{D}} w(z)|\psi(z)|\,|\phi'(z)|/u(\phi(z))$. Since by hypothesis, the operator $\psi\phi' C_\phi : H_u^\infty \to H_w^\infty$ is bounded, we obviously have $\psi\phi' g \circ \phi \in H_w^\infty$. Hence, $(\psi DC_\phi f_{n_k})_k$ is a convergent sequence in H_w^∞ and the operator $\psi DC_\phi : H_v^\infty \to H_w^\infty$ is compact.

Conversely, assume that $\psi DC_\phi : H_v^\infty \to H_w^\infty$ is compact. Define $u := w \circ \phi^{-1}$. Then

$$\begin{aligned}
\sup_{z\in\mathbb{D}} \frac{|\psi(z)|\,|\phi'(z)|w(z)}{u(\phi(z))} &= \sup_{z\in\mathbb{D}} |\psi(z)|\,|\phi'(z)| < \infty, \\
\frac{|\psi(z)|\,|\phi'(z)|w(z)}{u(\phi(z))} &= |\psi(z)|\,|\phi'(z)| \geq \alpha > 0 \quad \text{for every } z \in \mathbb{D}.
\end{aligned}$$

Let $(f_n)_n$ be a bounded sequence in H_v^∞. By hypothesis, $\psi DC_\phi : H_v^\infty \to H_w^\infty$ is compact. This means that we can find a subsequence $(f_{n_k})_k$ of $(f_n)_n$ and a function $g \in H_w^\infty$ such that $\psi DC_\phi f_{n_k} \to g$ in H_w^∞, i.e. for every $\varepsilon > 0$ we can find $k_0 \in \mathbb{N}$ such that

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} w(z)|(\psi DC_\phi)(f_{n_k})(z) - g(z)| \\
&\qquad\qquad = \sup_{z\in\mathbb{D}} w(z)|\psi(z)\phi'(z)f'_{n_k}(\phi(z)) - g(z)| < \varepsilon
\end{aligned}$$

for every $k \geq k_0$. Next, we fix $\varepsilon > 0$ and select k_0 as above. It follows that

$$\begin{aligned}
&\alpha \sup_{z\in\mathbb{D}} u(\phi(z)) \left| f'_{n_k}(\phi(z)) - \frac{g(z)}{\psi(z)\phi'(z)} \right| \\
&\qquad \leq \sup_{z\in\mathbb{D}} \frac{|\psi(z)|w(z)|\phi'(z)|}{u(\phi(z))} u(\phi(z)) \left| f'_{n_k}(\phi(z)) - \frac{g(z)}{\psi(z)\phi'(z)} \right| \\
&\qquad = \sup_{z\in\mathbb{D}} w(z)|\psi DC_\phi f_{n_k}(z) - g(z)| < \varepsilon
\end{aligned}$$

for every $k \geq k_0$ and the claim follows. ∎

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