ANNALES
POLONICI MATHEMATICI
LXXIX.1 (2002)

Solutions to some nonlinear PDE's in the form of Laplace type integrals

by Maria E. Pliś (Kraków)

Abstract. A nonlinear equation $P(D)u = \alpha u^m$ in 2 variables is considered. A formal solution as a series of Laplace integrals is constructed. It is shown that assuming some properties of $\operatorname{Char} P$, one gets the Gevrey class of such solutions. In some cases convergence "at infinity" is proved.

1. Introduction. In [P-Z] we have considered a nonlinear equation

$$P(D)u = \sum_{j=1}^{\infty} c_j u^j \tag{1}$$

with all c_j constant (complex or real), $D = (\partial/\partial x_1, \partial/\partial x_2)$, and we have found a formal solution of (1) represented at infinity as a formal sum of Laplace type integrals. In general, solutions of this type are divergent. In this work we show that in some cases these formal solutions are in some formal Gevrey class. Similar problems for some nonlinear singular partial differential equations were studied in [G-T].

This paper is a continuation of [P-Z], but we recall all definitions and notations. We restrict our attention to the case

$$P(D)u = \alpha u^m \tag{2}$$

for some fixed $m \in \mathbb{N}$, $m \geq 2$, and α a real or complex number, where P is a polynomial of two variables. In what follows, for $a = (a_1, a_2)$ and $b = (b_1, b_2)$, we write $a \leq b$ (resp. $a < b$) whenever $a_i \leq b_i$ (resp. $a_i < b_i$) for $i = 1, 2$ and ab is the scalar product. For $\eta \in \mathbb{R}$ we write ηa instead of $\eta a_1 + \eta a_2$.

The assumptions on the left-hand side of (2) are given below. Set $\operatorname{Char} P = \{z \in \mathbb{C}^2 : P(z) = 0\}$. We assume that there exists an unbounded curve $Z \subset \operatorname{Char} P \cap \mathbb{R}^2_+$ such that:

(i) $2Z \cap Z = \emptyset$;

2000 *Mathematics Subject Classification*: 35C10, 35C15, 44A10, 46F12.
Key words and phrases: Laplace distributions, convolution equations, Gevrey classes.

(ii) $kZ \subset nZ$ for $2 \le n < k$; $n, k \in \mathbb{N}$ (here $kZ = Z + \ldots + Z$ is the algebraic sum of k summands);

(iii) $|P(z)| \ge 1$ for $z \in 2Z$;

(iv) for every $\eta > 0$ there exists a constant M_η such that for every $c = (c_1, c_2)$ with $c_i \ge \eta$, $\int_Z e^{-cx}\,dx \le M_\eta$.

The integral in (iv) is a line integral over Z.

Following [S-Z], for $a \in \mathbb{R}^2$ we define

$$L_a = \{\phi \in C^\infty((\overline{\mathbb{R}}_+)^2) : \sup_{x \in (\overline{\mathbb{R}}_+)^2} |e^{-ax}(\partial/\partial x)^\nu \phi(x)| < \infty,\ \nu \in \mathbb{N}_0^2\}$$

with convergence defined by the seminorms

$$\|\phi\|_{a,\nu} = \sup_{x \in (\overline{\mathbb{R}}_+)^2} |e^{-ax}(\partial/\partial x)^\nu \phi(x)|,$$

and for $\omega \in \mathbb{R}^2$ we define

$$L_{(\omega)} = \varinjlim_{a<\omega} L_a,$$

equipped with the inductive limit topology. The dual space $L'_{(\omega)}$ is called the space of *Laplace distributions* on $(\overline{\mathbb{R}}_+)^2$.

2. Construction of solutions. The function e^{-xz} as a function of z (x fixed) belongs to the space L_a for every $a \ge -x$, so it belongs to $L_{(\omega)}$ for every $\omega > -x$. We look for a solution of (2) in the form

$$u(x) = T[e^{-xz}] \tag{3}$$

for some Laplace distribution T. Applying $P(D)$ to u of the form (3) we arrive at the convolution equation

$$P(z)T = \alpha T^{*m}. \tag{4}$$

Here T^{*m} stands for $T * \ldots * T$ with m factors.

We modify slightly this equation by multiplying the right-hand side by a factor $\varepsilon \in (0, 1]$, so we consider

$$P(z)T = \alpha\varepsilon T^{*m}, \tag{5}$$

and we look for T in the form of a formal series

$$T = \sum_{k=0}^{\infty} \varepsilon^k T_k \tag{6}$$

of Laplace distributions T_k.

Inserting (6) in (5) we obtain

$$\sum_{k=0}^{\infty} P(z)T_k\varepsilon^k = \varepsilon\alpha\sum_{k=0}^{\infty}\varepsilon^k \sum_{k_1+\ldots+k_m=k} T_{k_1} * \ldots * T_{k_m}$$
$$= \sum_{k=1}^{\infty}\varepsilon^k\alpha \sum_{k_1+\ldots+k_m=k-1} T_{k_1} * \ldots * T_{k_m},$$

therefore, we get the recurrence system

$$P(z)T_0 = 0, \tag{7}$$

$$P(z)T_k = \alpha \sum_{k_1+\ldots+k_m=k-1} T_{k_1} * \ldots * T_{k_m} \quad \text{for } k \geq 1. \tag{8}$$

Fix $\eta > 0$, $b = (b_1, b_2) \in \mathbb{R}^2$ and set $\omega = -\eta - b = (-\eta - b_1, -\eta - b_2)$. Let M_η be the constant in (iv) and Φ a function in $C^\infty(\mathbb{R}^2_+)$ satisfying

$$0 < \Phi(x) \leq Ce^{bx} \quad \text{for all } x \in \mathbb{R}^2_+, \tag{9}$$

for some constant $C < |\alpha|^{-1/(m-1)}M_\eta^{-1}$.

Exactly as in [P-Z] we define a Laplace distribution T_0 by the formula

$$T_0[\phi] = \int_Z \phi(x)\Phi(x)\,dx \tag{10}$$

for every test function $\phi \in L_{(\omega)}$.

One can see immediately that T_0 is a solution of (7), $T_0 \in L'_{(\omega)}$ and $\operatorname{supp} T_0 \subset Z$.

Definition (10) and properties of convolution imply that for $p \in \mathbb{N}$ and $\phi \in L_a$ with $a < \omega$,

$$|T_0^{*p}[\phi]| \leq (CM_\eta)^p \sup_{z\in pZ} |\phi(z)e^{-az}|. \tag{11}$$

Indeed,

$$T_0^{*p}[\phi] = \int_{Z\times\ldots\times Z} \phi(z_1 + \ldots + z_p)\Phi(z_1)\ldots\Phi(z_p)\,dz_1\ldots dz_p$$
$$= \int_{Z^p} \phi(z_1 + \ldots + z_p)e^{-a(z_1+\ldots+z_p)}\Phi(z_1)e^{az_1}\ldots\Phi(z_p)e^{az_p}\,dz_1\ldots dz_p.$$

Hence,

$$|T_0^{*p}[\phi]| \leq \sup_{z\in pZ}|\phi(z)e^{-az}|\Big(\int_Z \Phi(z)e^{az}\,dz\Big)^p,$$

which is due to the fact that $z_1 + \ldots + z_p \in pZ$ for $z_i \in Z$. This gives (11) because $\int_Z \Phi(z)e^{az}\,dz \leq C\int_Z e^{(a+b)z}\,dz \leq CM_\eta$, since $a + b < -\eta$.

Lemma 1. *Let Θ be a function in $C^\infty(\mathbb{R}^{2k}_+)$, $k \geq 2$, satisfying*

$$0 < \Theta(z) \leq He^{b(z_1+\ldots+z_k)}$$

for some constants $H > 0$, $b = (b_1, b_2) \in \mathbb{R}^2$, *and for all* $z = (z_1, \ldots, z_k)$, $z_i \in \mathbb{R}^2_+$ $(i = 1, \ldots, k)$. *Let* $T \in L'_{(\omega)}$ *be a Laplace distribution for* $\omega = -b-\eta$ *with* $\eta > 0$, *given by*

$$T[\phi] = \int\limits_{Z^k} \phi(z_1 + \ldots + z_k)\Theta(z_1, \ldots, z_k)\, dz_1 \ldots dz_k \tag{12}$$

for $\phi \in L_{(\omega)}$. *Then the Laplace distribution* S *defined by*

$$S[\phi] = \int\limits_{Z^k} \frac{\phi(z_1 + \ldots + z_k)}{P(z_1 + \ldots + z_k)}\, \Theta(z_1, \ldots, z_k)\, dz_1 \ldots dz_k, \tag{13}$$

with $|P(z)| \geq 1$ *for* $z \in 2Z$, *solves the equation* $P(z)S = T$, *and for* $\phi \in L_a$ *with* $a < \omega$,

$$|S[\phi]| \leq \sup_{u \in kZ} |\phi(u)e^{-au}| H M_\eta^k. \tag{14}$$

Proof. It follows from assumption (iii) that formula (13) makes sense. If $\phi \in L_{(\omega)}$, then also $P\phi \in L_{(\omega)}$. Therefore

$$(PS)[\phi] = S[P\phi] = \int\limits_{Z^k} \phi(z_1 + \ldots + z_k)\Theta(z_1, \ldots, z_k)\, dz_1 \ldots dz_k = T[\phi],$$

and for $\phi \in L_a$ with $a < \omega$ we obtain

$$\begin{aligned}
|S[\phi]| &\leq \int\limits_{Z^k} |\phi(z_1 + \ldots + z_k)| e^{-a(z_1+\ldots+z_k)} \\
&\qquad \times |\Theta(z_1, \ldots, z_k)| e^{a(z_1+\ldots+z_k)}\, dz_1 \ldots dz_k \\
&\leq \sup_{u \in kZ} |\phi(u)e^{-au}| H \int\limits_{Z^k} e^{(b+a)(z_1+\ldots+z_k)}\, dz_1 \ldots dz_k \\
&\leq \sup_{u \in kZ} |\phi(u)e^{-au}| H \Big(\int\limits_{Z} e^{(b+a)z}\, dz\Big)^k
\end{aligned}$$

and by (iv) we get (14), because $-(a + b) > \eta$.

If T is as in Lemma 1, then we denote by $\frac{1}{P}T$ the distribution S given by (13).

Lemma 2. *Let* $m \in \mathbb{N}$, $m \geq 2$ *and* $\{B_k\}_{k\in\mathbb{N}}$ *be a sequence of real numbers defined by the recurrence formulas*

$$\begin{aligned}
&B_1 = 1, \\
&B_k = \sum_{r=1}^{\widetilde{m}} \binom{m}{r} \sum_{\substack{m_1+\ldots+m_r=k-1 \\ m_i \geq 1}} B_{m_1} \ldots B_{m_r} \quad \text{for } k > 1,
\end{aligned} \tag{15}$$

where $\widetilde{m} = \min\{m, k - 1\}$. *Then* $B_k \leq ((m - 1)k)!$.

Proof. The proof is by induction on k. Obviously, for every $m \geq 2$, $B_1 = 1 \leq (m-1)!$. Suppose that $B_s \leq ((m-1)s)!$ for every $s < k$. From (15) it follows that

$$\begin{aligned} B_k &\leq \sum_{r=1}^{\tilde{m}} \binom{m}{r} \sum_{\substack{m_1+\ldots+m_r=k-1 \\ m_i \geq 1}} ((m-1)m_1)! \ldots ((m-1)m_r)! \\ &\leq \sum_{r=1}^{\tilde{m}} \binom{m}{r} ((m-1)(k-1))! \binom{k-2}{r-1} \\ &= ((m-1)(k-1))! \sum_{r=1}^{\tilde{m}} \binom{m}{r} \binom{k-2}{r-1}, \end{aligned}$$

due to the well known facts that the number of elements of $\{(m_1, \ldots, m_r) : m_1 + \ldots + m_r = n, m_i \geq 1, i = 1, \ldots, r\}$ is $\binom{n-1}{r-1}$, and that $p_1! \ldots p_r! \leq (p_1 + \ldots + p_r)!$. It is sufficient to show that

$$\begin{aligned} \sum_{r=1}^{\tilde{m}} \binom{m}{r} \binom{k-2}{r-1} &\leq \frac{((m-1)k)!}{((m-1)(k-1)!} \\ &= ((m-1)k - (m-2))((m-1)k - (m-3)) \ldots ((m-1)k) \\ &= \prod_{l=1}^{m-1} ((m-1)k - (l-1)). \end{aligned} \tag{16}$$

We check at once that

$$\sum_{r=1}^{\tilde{m}} \binom{m}{r} \binom{k-2}{r-1} = \binom{k+m-2}{m-1} = \prod_{l=1}^{m-1} \frac{k+l-1}{l}$$

and

$$(m-1)k - (l-1) - \frac{k+l-1}{l} = \frac{[l(m-1)-1]k - (l^2-1)}{l} \geq 0$$

for $l = 1, \ldots, m-1$ and $k \geq 1$, and (16) is proved.

Let $(k_1, \ldots, k_r) \in \mathbb{N}_0^r$ be such that $k_1 + \ldots + k_r = k-1$, $k \in \mathbb{N}$, $k_i \geq 1$ for $i = 1, \ldots, r$. Then we define $k'_j = (m-1)(k_1 + \ldots + k_j) + j$ for $j = 1, \ldots, r$, $k'_0 = 0$ and $m_k = (m-1)k + 1$. Clearly $k'_r = m_k - (m-r)$. For $z = (z_1, \ldots, z_{m_k})$ we set $u_{k_j} = (z_{k'_{j-1}+1}, \ldots, z_{k'_j})$ and $s(u_{k_j}) = z_{k'_{j-1}+1} + \ldots + z_{k'_j}$ for $j = 1, \ldots, r$. Then we define $H_k : \mathbb{R}_+^{2m_k} \to \mathbb{R}$ by the recurrence formula

$$H_1(z_1, \ldots, z_m) = \frac{\Phi(z_1) \ldots \Phi(z_m)}{P(z_1 + \ldots + z_m)}, \tag{17}$$

$$(18)\quad H_k(z_1,\ldots,z_{m_k}) = \sum_{r=1}^{\widetilde{m}} \binom{m}{r} \sum_{\substack{k_1+\ldots+k_r=k-1\\ k_i\ge 1}} \frac{H_{k_1}(u_{k_1})\ldots H_{k_r}(u_{k_r})\Phi(z_{k_r'+1})\ldots\Phi(z_{m_k})}{P(s(u_{k_1})+\ldots+s(u_{k_r})+z_{k_r'+1}+\ldots+z_{m_k})}.$$

Lemma 3. *If T_0 is the Laplace distribution given by* (10) *then the distribution T_k defined by*

$$(19)\qquad T_k = \frac{\alpha}{P}\sum_{k_1+\ldots+k_m=k-1} T_{k_1} * \ldots * T_{k_m}$$

for $k\ge 1$ satisfies the following conditions:

$$(20)\qquad T_k \in L'_{(\omega)} \quad \text{for } \omega = -b-\eta, \quad \text{and} \quad T_k \text{ solves } (8);$$

$$(21)\qquad \operatorname{supp} T_k \subset m_k Z,$$

where $m_k=(m-1)k+1$; for every $\phi\in L_a$ with $a<\omega$,

$$(22)\qquad T_k[\phi] = \alpha^k \int_{Z^{m_k}} \phi(z_1+\ldots+z_{m_k}) H_k(z_1,\ldots,z_{m_k})dz_1\ldots dz_{m_k}$$

with H_k given by (17) *and* (18);

$$(23)\qquad \int_{Z^{m_k}} |H_k(z_1,\ldots,z_{m_k})|e^{a(z_1+\ldots+z_{m_k})}\,dz_1\ldots dz_{m_k} \le B_k(CM_\eta)^{m_k},$$

and

$$(24)\qquad |T_k[\phi]| \le |\alpha|^k B_k(CM_\eta)^{m_k} \sup_{u\in m_k Z} |\phi(u)|e^{-au},$$

where the sequence B_k is given by (15).

Proof. The condition (20) follows immediately from the properties of convolution and from the assumptions on P.

In order to prove (21)–(24) we proceed by induction. For $k=1$ we have $T_1=\alpha(1/P)T_0^{*m}$, so $\operatorname{supp} T_1\subset mZ$. By (17),

$$T_1[\phi] = \alpha \int_{Z^m} \phi(z_1+\ldots+z_m)H_1(z_1,\ldots,z_m)\,dz_1\ldots dz_m,$$

and

$$|H_1(z_1,\ldots,z_m)|e^{a(z_1+\ldots+z_m)} = \frac{\Phi(z_1)e^{az_1}\ldots\Phi(z_m)e^{az_m}}{|P(z_1+\ldots+z_m)|}.$$

Hence by (iii),

$$\int_{Z^m} |H_1(z_1,\ldots,z_m)|e^{a(z_1+\ldots+z_m)}\,dz_1\ldots dz_m \le \Big(\int_Z \Phi(z)e^{az}\,dz\Big)^m.$$

Therefore

$$
\begin{aligned}
&|T_1[\phi]| \\
&= |\alpha| \Big| \int\limits_{Z^m} \phi(z_1 + \ldots + z_m) e^{-a(z_1+\ldots+z_m)} H_1(z_1, \ldots, z_m) e^{a(z_1+\ldots+z_m)}\, dz_1 \ldots dz_m \Big| \\
&\le |\alpha| \sup_{u \in mZ} |\phi(u)| e^{-au} (CM_\eta)^m.
\end{aligned}
$$

Suppose now that for T_s with $s < k$, $k > 1$, (21)–(24) hold. Then for $k_1 + \ldots + k_m = k - 1$ we have

$$
\operatorname{supp} T_{k_1} * \ldots * T_{k_m} \subset \sum_{r=1}^{m} \operatorname{supp} T_{k_r} \subset \sum_{r=1}^{m} m_{k_r} Z = m_k Z,
$$

because $m_{k_1} + \ldots + m_{k_m} = m_k$. Here $\sum$ is the algebraic sum of sets.

By the properties of convolution, assumption (iii) and by (15) we obtain

$$
\begin{aligned}
&\int\limits_{Z^{m_k}} |H_k(z_1, \ldots, z_{m_k})| e^{a(z_1+\ldots+z_{m_k})}\, dz_1 \ldots dz_{m_k} \\
&\qquad = \sum_{r=1}^{\widetilde{m}} \binom{m}{r} \sum_{\substack{k_1+\ldots+k_r=k-1\\ k_i\ge 1}} \int\limits_{Z^{k_r'}} \int\limits_{Z^{m-r}} \\
&\qquad\qquad \times \frac{\prod_{j=1}^{r} |H_{k_j}(u_{k_j})| e^{as(u_{k_j})} \varPhi(z_{k_r'+1}) e^{az_{k_r'+1}} \ldots \varPhi(z_{m_k}) e^{az_{m_k}}}{|P(\sum_{j=1}^{r} s(u_{k_j}) + z_{k_r'+1} + \ldots + z_{m_k})|}\, du\, dz \\
&\qquad \le \sum_{r=1}^{\widetilde{m}} \binom{m}{r} \sum_{\substack{k_1+\ldots+k_r=k-1\\ k_i\ge 1}} B_{k_1} (CM_\eta)^{m_{k_1}} \ldots B_{k_r} (CM_\eta)^{m_{k_r}} (CM_\eta)^{m-r} \\
&\qquad = B_k (CM_\eta)^{m_k},
\end{aligned}
$$

and

$$
\begin{aligned}
T_k[\phi] = \alpha \sum_{r=1}^{\widetilde{m}} \binom{m}{r} &\sum_{\substack{k_1+\ldots+k_r=k-1\\ k_i\ge 1}} \int\limits_{Z^{m_{k_1}}} \ldots \int\limits_{Z^{m_{k_r}}} \int\limits_{Z^{m-r}} \\
&\phi\Big(\sum_{j=1}^{r} s(u_j) + z_{k_r'+1} + \ldots + z_{m_k} \Big) \\
&\times \frac{\prod_{j=1}^{r} \alpha^{k_j} H_{k_j}(u_j) \varPhi(z_{k_r'+1}) \ldots \varPhi(z_{m_k})}{P(\sum_{j=1}^{r} s(u_j) + z_{k_r'+1} + \ldots + z_{m_k})}\, du_1 \ldots du_r\, dz \\
= \alpha^k &\int\limits_{Z^{m_k}} \phi(u) H_k(u)\, du.
\end{aligned}
$$

Moreover

$$|T_k[\phi]| \le |\alpha|^k \sup_{u\in m_k Z} |\phi(u)|e^{-au} \sum_{r=1}^{\widetilde{m}} \binom{m}{r} \sum_{\substack{k_1+\ldots+k_r=k-1\\ k_i\ge 1}} \prod_{j=1}^{r} \int_{Z^{m_{k_j}}} |H_{k_j}(u_j)|e^{as(u_j)}du_j \Big(\int_Z \Phi(z)e^{az}\,dz\Big)^{m-r}$$

$$\le |\alpha|^k \sup_{u\in m_k Z} |\phi(u)|e^{-au} \times \sum_{r=1}^{\widetilde{m}} \binom{m}{r} \sum_{\substack{k_1+\ldots+k_r=k-1\\ k_i\ge 1}} B_{k_1}\ldots B_{k_r}(CM_\eta)^{m_{k_1}+\ldots+m_{k_r}+m-r}$$

$$= |\alpha|^k \sup_{u\in m_k Z} |\phi(u)|e^{-au} B_k(CM_\eta)^{m_k}.$$

By Lemma 2 we have

Corollary 1. *The distribution T_k defined by* (19) *satisfies the estimate*

$$|T_k[\phi]| \le |\alpha|^k((m-1)k)!D^{m_k} \sup_{z\in m_k Z} |\phi(z)e^{-az}|$$

with constant $D = CM_\eta$ independent of a, for $\phi \in L_a$, $m_k = (m-1)k+1$, $-a-b > \eta$.

3. Gevrey class of solutions. Let $J = \{J_k\}_{k\in\mathbb{N}}$ be a sequence of real numbers, and $\omega \in \mathbb{R}^n$.

Definition 1. We define $G^\omega(J)$ as the set of formal series $\sum_{k=1}^{\infty} T_k[e^{-xz}]$ with Laplace distributions T_k such that the series $\sum_{k=1}^{\infty} \frac{1}{J_k} T_k[e^{-xz}]$ converges locally uniformly for $x > -\omega$.

Theorem 1. *Fix $b = (b_1, b_2) \in \mathbb{R}^2$ and $\eta > 0$, and let $\omega = -\eta - b$. Let $M_\eta = \int_Z e^{-\eta x}dx$, and Φ be a function in $C^\infty(\mathbb{R}^2_+)$ satisfying* (9) *for some constant $C < |\alpha|^{-1/(m-1)}M_\eta^{-1}$. Define Laplace distributions T_k by* (10) *and* (19) *for $k = 0, 1, \ldots,$ and set $u_k(x) = T_k[e^{-xz}]$ for $x > -\omega$. Set $J_k = ((m-1)k)!$. Then the series*

$$\sum_{k=0}^{\infty} u_k(x) = \sum_{k=0}^{\infty} T_k[e^{-xz}] \tag{25}$$

is a formal solution of (2) *and it is in the Gevrey class $G^\omega(J)$.*

Proof. The series (25) solves (2) by the construction of T_k. It remains to prove that it is in the Gevrey class. Since $e^{-xz} \in L_a$ (as a function of z) for every $a \geq -x$, from Corollary 1 it follows that if $x \geq -a > -\omega$, then

$$|u_k(x)| = |T_k[e^{-xz}]| \leq |\alpha|^k (CM_\eta)^{(m-1)k+1} J_k \sup_{z \in m_k Z} |e^{-(a+x)z}|.$$

Hence for $x \geq -a$, and in consequence for $x > -\omega$, we have

$$|u_k(x)| \leq CM_\eta J_k (|\alpha|(CM_\eta)^{m-1})^k.$$

Since $|\alpha|(CM_\eta)^{m-1} < 1$ the series $\sum_{k=1}^{\infty} u_k(x)/J_k$ is uniformly convergent on the set $\{x > -\omega\}$.

Fix $\beta > 0$, and denote by D_k the set

$$D_k = \{y \in \mathbb{R}_+^2 : yz > \beta k^2 \text{ for all } z \in kZ\}. \tag{26}$$

Theorem 2. *Assume that the sequence $\{D_k\}_k$ is decreasing, and fix $\omega \in \mathbb{R}^2$. Then, for the formal solution $\sum_{k=1}^{\infty} u_k(x)$ defined by* (25), *the sequence*

$$\beta_N = \sup_{x \in D_{m_N} - \omega} \sum_{k=1}^{N} |u_k(x)| \tag{27}$$

with $m_N = (m-1)N + 1$, for $N \in \mathbb{N}$, is bounded.

Proof. We first observe that if $\{D_k\}$ decreases then so does $\{D_k - \omega\}$. If $x \in D_{m_N} - \omega$ and $r \leq N$ then $x \in D_{m_N} - a$ for some a with $x \geq -a > -\omega$, so $x \in D_{m_r} - a$ and $(x+a)z \geq \beta m_r^2$ for $z \in m_r Z$. From the proof of Theorem 1 it follows that for such x and $k = 1, 2, \ldots$,

$$|u_k(x)| \leq CM_\eta((m-1)k)!(|\alpha|(CM_\eta)^{m-1})^k \sup_{z \in m_k Z} |e^{-(a+x)z}|.$$

Thus for all $r \leq N$,

$$|u_r(x)| \leq CM_\eta((m-1)r)!(|\alpha|^{1/(m-1)} CM_\eta)^{(m-1)r} e^{-\beta m_r^2}$$

and

$$\sum_{r=1}^{\infty} ((m-1)r)! E^{(m-1)r} e^{-\beta m_r^2} \leq \sum_{p=1}^{\infty} p! E^p e^{-\beta(p+1)^2} < \infty$$

with $E = |\alpha|^{1/(m-1)} CM_\eta$.

It is worth pointing out that the solution (25) depends strictly on the choice of the function Φ in the formula (10). But it means in fact the dependence on the choice of the constant b or $\omega = -b - \eta$, η fixed.

We can now formulate some consequence of the possibility of this choice.

Theorem 3. *Fix $\eta > 0$. For every $r, N \in \mathbb{N}$ we can choose $\omega > 0$ and a formal solution $\sum_{k=0}^{\infty} u_k(x)$ of* (2) *such that*

$$\sum_{k=0}^{N} |u_k(x)| \leq CM_\eta \sum_{k=1}^{N} ((m-1)k)! E^{(m-1)k} e^{-\beta m_k^2} \tag{28}$$

for $x \in D_r$ and $E = |\alpha|^{1/(m-1)} CM_\eta$.

Proof. If $r \geq m_N$, then obviously $D_r \subset D_{m_N} \subset D_{m_N} - \omega$ for every $\omega > 0$, and (28) follows from Theorem 2. Suppose now that $r < m_N$. Then $m_N Z \subset rZ$, therefore, for $x \in D_r$, $xz > \beta r^2$ for all $z \in m_N Z$. If we choose $\omega \in D_{m_N}$ then $(x+\omega)z = xz + \omega z > \beta r^2 + \beta m_N^2 > \beta m_N^2$ for all $z \in m_N Z$, which means that $x \in D_{m_N} - \omega$, and the solution (25) defined for such ω satisfies (28).

4. Special case. Suppose now that the set Z can be described by

$$Z = \{(t,s) \in \mathbb{R}_+^2 : s = f(t)\} = \{z(t) : t \in \mathbb{R}_+\},$$

where $z(t) = (t, f(t))$, $f : \mathbb{R}_+ \to \mathbb{R}_+$, $f \in C^\infty$, $f' < 0$, $f'' > 0$, $f^{-1} \in C^\infty$ and

$$\lim_{x\to 0} f(x) = \infty, \qquad \lim_{x\to\infty} f(x) = 0.$$

It is evident that the distribution T_0 defined by (10) with some constants $\eta > 0$, $b = (b_1, b_2) \in \mathbb{R}_+^2$ fixed and a function Φ chosen to satisfy (9), can be written as an integral over $\mathbb{R}_+$:

$$T_0[\phi] = \int_0^\infty \phi(t, f(t)) \Psi(t)\, dt \tag{29}$$

for $\phi \in L_{(\omega)}$, $\omega = -\eta - b$, with $\Psi(t) = \Phi(t, f(t))\sqrt{1 + (f'(t))^2}$. It can be seen that for some constant $C' < |\alpha|^{-1/(m-1)} M_\eta^{-1}$ we have

$$0 < \Psi(t) \leq C' e^{bz(t)} = C' e^{b_1 t + b_2 f(t)}. \tag{30}$$

Lemma 4. *For every $k \in \mathbb{N}$, $k \geq 2$,*

$$kZ = \{(x,y) \in \mathbb{R}_+^2 : y \geq k f(x/k)\} \tag{31}$$

and kZ is convex.

Proof. If $(x,y) \in kZ$, then there exist $t_1, \ldots, t_k > 0$ such that $x = t_1 + \ldots + t_k$ and $y = f(t_1) + \ldots + f(t_k)$. Hence

$$\frac{y}{k} = \frac{1}{k} f(t_1) + \ldots + \frac{1}{k} f(t_k) \geq f\left(\frac{t_1 + \ldots + t_k}{k}\right) = f\left(\frac{x}{k}\right).$$

Now fix $(x,y) \in \{y \geq k f(x/k)\}$. Writing $t_j = x/k$ for $j = 3, \ldots, k$, and $t_1 = \alpha x/k$, $t_2 = (2-\alpha)x/k$, for $\alpha \in (0,2)$ we have $x = t_1 + \ldots + t_k$. Let $F(\alpha) = f(\alpha x/k) + f((2-\alpha)x/k)$. We see that $F(1) = 2f(x/k)$, $F(\alpha) \to \infty$ as $\alpha \to 0$ or $\alpha \to 2$. Clearly $y - (k-2)f(x/k) \geq 2f(x/k)$. Then by continuity of F, there exists α such that $F(\alpha) = y - (k-2)f(x/k)$. This proves that $y = f(t_1) + \ldots + f(t_k)$, so $(x,y) \in kZ$.

Let now $(x,y), (u,v) \in kZ$, and define $(r,s) = \lambda(x,y) + (1-\lambda)(u,v)$ for $0 \leq \lambda \leq 1$. We shall show that also $(r,s) \in kZ$. We have $r = \lambda x + (1-\lambda)u$, $s = \lambda y + (1-\lambda)v$, $y \geq k f(x/k)$ and $v \geq k f(u/k)$. Therefore

$$\frac{s}{k}=\lambda\,\frac{y}{k}+(1-\lambda)\,\frac{v}{k}\geq\lambda f\left(\frac{x}{k}\right)+(1-\lambda)f\left(\frac{u}{k}\right)\geq f\left(\frac{\lambda x+(1-\lambda)u}{k}\right)=f\left(\frac{r}{k}\right).$$

This proves the convexity of kZ.

Let $\sigma\in\{1,-1\}$, $V_\sigma=\{(t_1,t_2):t_1>0,\,t_2>0,\,\sigma(t_2-t_1)>0\}$ and consider the transformation $\Psi_\sigma:V_\sigma\to 2Z$ given by

$$\Psi_\sigma(t_1,t_2)=(t_1+t_2,f(t_1)+f(t_2)) \tag{32}$$

for $t=(t_1,t_2)\in V_\sigma$. The Jacobian $J\Psi_\sigma(t)=|f'(t_2)-f'(t_1)|\neq 0$ on V_σ, hence Ψ_σ is invertible and the Jacobian $J\Psi_\sigma^{-1}(u)$ is independent of σ. We use the notation $\Psi_+^{-1}(u)=(t_1(u),t_2(u))$ and $\Psi_-^{-1}(u)=(t_2(u),t_1(u))$, and

$$K(u)=\begin{cases} J\Psi_\sigma^{-1}(u) & \text{for } u\in 2Z,\\ 0 & \text{for } u\in\mathbb{R}_+^2\setminus 2Z.\end{cases} \tag{33}$$

Define $\widehat{Z}=\{(x,y)\in\mathbb{R}_+^2: y>f(x)\}$, and for $v\in\widehat{Z}$ denote by T_v the open interval $(f^{-1}(v_2),v_1)$. It follows from the assumptions on Z that if $t\in T_v$ then $v_1-t>0$ and $v_2-f(t)>0$.

Define

$$\widehat{K}(v)=\begin{cases}\int_{T_v}K(v-z(t))\,dt & \text{for } v\in\widehat{Z},\\ 0 & \text{for } v\in\mathbb{R}_+^2\setminus\widehat{Z}.\end{cases} \tag{34}$$

LEMMA 5. *For every $k\in\mathbb{N}$, $k\geq 2$ and for $\phi\in L_{(\omega)}$ we have*

$$T_0^{*k}[\phi]=\int\limits_{\mathbb{R}_+^2}\phi(z)\Phi_k(z)\,dz$$

where

$$\begin{aligned}\Phi_2(z)&:=\begin{cases}2\Psi(t_1(z))\Psi(t_2(z))K(z) & \textit{for } z\in 2Z,\\ 0 & \textit{for } z\in\mathbb{R}_+^2\setminus 2Z,\end{cases}\\ \Phi_k(v)&:=\begin{cases}\int_{T_v}\Phi_{k-1}(v-z(t))\Psi(t)\,dt & \textit{for } v\in\widehat{Z},\\ 0 & \textit{for } v\in\mathbb{R}_+^2\setminus\widehat{Z},\end{cases}\end{aligned} \tag{35}$$

for $k\geq 3$. Moreover, $\Phi_k\in C^\infty((kZ)^\circ)$, $\operatorname{supp}\Phi_k\subset kZ$ *and*

$$\Phi_k\leq\Theta_k:=\begin{cases}\Phi_2^{*p} & \textit{for } k=2p,\\ \Phi_3*\Phi_2^{*p-1} & \textit{for } k=2p+1,\end{cases} \tag{36}$$

$$\Phi_3*\Phi_3\leq\Phi_2^{*3}. \tag{37}$$

Proof. By definition of convolution of distributions and by formula (29) we can see that

$$T_0^{*2}[\phi]=\int\limits_0^\infty\int\limits_0^\infty\phi(z(t)+z(s))\Psi(t)\Psi(s)\,ds\,dt.$$

Now we use the elementary fact that

$$\int_0^\infty\int_0^\infty = \int_0^\infty\int_0^t + \int_0^\infty\int_t^\infty$$

and we change variables using transformations (32). Thus

$$T_0^{*2}[\phi] = 2\int_{2Z}\phi(u)\Psi(t_1(u))\Psi(t_2(u))J\Psi_\sigma^{-1}(u)\,du = \int_{\mathbb{R}_+^2}\phi(u)\Phi_2(u)\,du.$$

Obviously $\operatorname{supp}\Phi_2 \subset 2Z$ and regularity of Φ_2 follows from regularity of Ψ and Ψ_σ^{-1}.

Similarly, since $T_0^{*3} = T_0^{*2} * T_0$, we get

$$T_0^{*3}[\phi] = \int_{\mathbb{R}_+^2}\int_0^\infty \phi(u+z(t))\Phi_2(u)\Psi(t)\,du\,dt.$$

The mapping

$$\chi(u_1,u_2,t) = (u_1+t, u_2+f(t), t)$$

is a homeomorphism

$$\chi : (0,\infty)^3 \to \{(v_1,v_2,\tau) : v_2 > f(v_1),\ \tau\in T_v\}$$

with Jacobian 1. Indeed, if $v_1 = u_1+t$, $v_2 = u_2+f(t)$ for $u_1>0$, $u_2>0$, $t>0$, then

$$v_2 = u_2 + f(v_1-u_1) > f(v_1-u_1) > f(v_1),$$

because f decreases. Similarly, $t = v_1-u_1 < v_1$, and since $f(t) = v_2-u_2$ we have $t = f^{-1}(v_2-u_2) > f^{-1}(v_2)$. Thus $t\in T_v$. Suppose now that $v_2 > f(v_1)$ and $t\in T_v$. Then $u_1 := v_1 - t > 0$ and $u_2 := v_2 - f(t) > 0$, so (u_1,u_2,t) belongs to the domain of χ. The injectivity and continuity of χ are obvious. Thus we can use χ to change variables in the last integral to obtain

$$T_0^{*3}[\phi] = \int_{\widehat{Z}}\phi(v)\int_{T_v}\Phi_2(v-z(t))\Psi(t)\,dt\,dv = \int_{\mathbb{R}_+^2}\phi(v)\Phi_3(v)\,dv,$$

where Φ_3 is given by (35) for $k=3$.

If $v\in(\mathbb{R}_+^2\setminus(k+1)Z)\cap\widehat{Z}$ then $v - z(t)\in\mathbb{R}_+^2\setminus kZ$ for $t\in T_v$, so $\operatorname{supp}\Phi_3\subset 3Z$ and obviously $\Phi_3\in C^\infty((3Z)^\circ)$. We now proceed by induction.

Since $T_0^{*k+1} = T_0^{*k} * T_0$, we have

$$\begin{aligned} T_0^{*k+1}[\phi] &= \int_{\mathbb{R}_+^2}\int_0^\infty \phi(u+z(t))\Phi_k(u)\Psi(t)\,dt\,du \\ &= \int_{\widehat{Z}}\phi(v)\int_{T_v}\Phi_k(v-z(t))\Psi(t)\,dt\,dv = \int_{\mathbb{R}_+^2}\phi(v)\Phi_{k+1}(v)\,dv.\end{aligned}$$

From (35) it follows that

$$\begin{aligned}\Phi_{k+1}(v) &= \int\limits_{T_v}\int\limits_{T_{v-z(t)}} \Phi_{k-1}(v-z(t)-z(s))\Psi(t)\Psi(s)\,dt\,ds\\ &= \int\limits_{\Psi_\sigma(\widehat{T}_v)} \Phi_{k-1}(v-y)\Phi_2(y)\,dy,\end{aligned}$$

where $\widehat{T}_v = \{(t,s)\in\mathbb{R}^2_+ : t\in T_v,\ s<t\}$.

The estimate (36) is obvious for $k=2$ and $k=3$. Suppose that it is true for $k=2p$ and $k=2p+1$. Then

$$\begin{aligned}\Phi_{2p+2}(v) &= \int\limits_{\Psi_\sigma(\widehat{T}_v)} \Phi_{2p}(v-y)\Phi_2(y)\,dy\\ &\le \int\limits_{\Psi_\sigma(\widehat{T}_v)} \Phi_2^{*p}(v-y)\Phi_2(y)\,dy \le \Phi_2^{*(p+1)}(v),\end{aligned}$$

and

$$\begin{aligned}\Phi_{2p+3}(v) &= \int\limits_{\Psi_\sigma(\widehat{T}_v)} \Phi_{2p+1}(v-y)\Phi_2(y)\,dy\\ &\le \int\limits_{\Psi_\sigma(\widehat{T}_v)} \Phi_3*\Phi_2^{*(p-1)}(v-y)\Phi_2(y)\,dy \le \Phi_3*\Phi_2^{*p}(v).\end{aligned}$$

By (35), using the transformations χ and Ψ_σ we get

$$\begin{aligned}\Phi_3*\Phi_3(v) &= \int\limits_0^v \Phi_3(v-u)\Phi_3(u)\,du\\ &= \int\limits_0^v\int\limits_{T_{v-u}}\int\limits_{T_u} \Phi_2(v-u-z(t))\Phi_2(u-z(s))\Psi(t)\Psi(s)\,dt\,ds\,du\\ &= \int\limits_0^v\int\limits_{T_y}\int\limits_{T_{y-z(t)}} \Phi_2(v-y)\Phi_2(y-(z(t)+z(s)))\Psi(t)\Psi(s)\,dt\,ds\,dy\\ &\le \int\limits_0^v\int\limits_{\Psi_\sigma(\widehat{T}_y)} \Phi_2(v-y)\Phi_2(y-z)\Phi_2(z)\,dz\,dy \le \Phi_2^{*3}(v).\end{aligned}$$

Corollary 2. *The function Θ_k defined by* (36) *for $k=2,3,\ldots$ satisfies, for $v\in\mathbb{R}^2_+$,*

$$\Theta_k(v) \le (\sqrt{2}\,C')^k e^{bv}\mathcal{K}_k(v)$$

where

$$\mathcal{K}_n(u) = \begin{cases} K^{*p}(u) & \text{for } n=2p,\\ \widehat{K}*K^{*p-1}(u) & \text{for } n=2p+1.\end{cases} \tag{38}$$

Proof. By the definition of Φ_2 it follows immediately that

$$\Phi_2(v) \le 2C'^2 e^{b(z(t_1(v))+z(t_2(v)))} K(v) = (\sqrt{2}\, C')^2 e^{bv} K(v),$$

since $z(t_1(v)) + z(t_2(v)) = v$. Similarly,

$$\begin{aligned}\Phi_3(v) &\le 2C'^3 \int_{T_v} e^{b(v-z(t))} K(v - z(t)) e^{bz(t)}\, dt \\ &= 2C'^3 e^{bv} \int_{T_v} K(v - z(t))\, dt < (\sqrt{2}\, C')^3 e^{bv} \widehat{K}(v).\end{aligned}$$

Hence, for $k = 2p$ we have

$$\Theta_k(v) = \Phi_2^{*p}(v) \le (\sqrt{2}\, C')^{2p} e^{bv} K^{*p}(v)$$

and for $k = 2p + 1$,

$$\Theta_k(v) = \Phi_3 * \Phi_2^{*p-1}(v) \le (\sqrt{2}\, C')^{2p+1} e^{bv} \widehat{K} * K^{*p-1}(v).$$

Exactly as in the general case, for T_0 defined by (29), we can define T_k by (19).

THEOREM 4. *The distribution T_k defined by* (19) *acts on test functions as a function $G_k \in C^\infty((m_k Z)^\circ)$ $(m_k = (m-1)k+1)$ defined on $\mathbb{R}^2_+$ by the recurrence formula*

$$G_1(u) = \alpha \frac{\Phi_m(u)}{P(u)}, \tag{39}$$

$$G_k(u) = \frac{\alpha}{P(u)} \sum_{r=1}^{\widetilde{m}} \binom{m}{r} \sum_{\substack{k_1+\ldots+k_r=k-1 \\ k_i \ge 1}} G_{k_1} * \ldots * G_{k_r} * \Phi_{m-r}(u) \tag{40}$$

for $k > 1$ and $\widetilde{m} = \min\{m, k-1\}$. Moreover, we have the estimate

$$|G_k(u)| \le |\alpha|^k ((m-1)k)! e^{bu} (\sqrt{2}\, C')^{m_k} \mathcal{K}_{m_k}(u). \tag{41}$$

Proof. The functions G_k $(k = 1, 2, \ldots)$ are well defined because Φ_m and $G_{k_1} * \ldots * G_{k_r} * \Phi_{m-r}$ vanish on some neighbourhoods of Z. By (19),

$$T_1 = \frac{\alpha}{P} T_0^{*m}$$

and

$$T_k = \frac{\alpha}{P} \sum_{r=1}^{\widetilde{m}} \binom{m}{r} \sum_{\substack{k_1+\ldots+k_r=k-1 \\ k_i \ge 1}} T_{k_1} * \ldots * T_{k_r} * T_0^{*m-r}.$$

Then for every test function ϕ we have

$$T_1[\phi] = \alpha \int_{\mathbb{R}^2_+} \phi(u) \frac{\Phi_m(u)}{P(u)}\, du = \int_{\mathbb{R}^2_+} \phi(u) G_1(u)\, du,$$

and if for $s < k$, $T_s[\phi] = \int_{\mathbb{R}^2_+} \phi(u) G_s(u)\, du$, then for $u_1, \ldots, u_r, v \in \mathbb{R}^2_+$,

$$\begin{aligned} &T_k[\phi] \\ &= \alpha \sum_{r=1}^{\widetilde{m}} \binom{m}{r} \sum_{\substack{k_1+\ldots+k_r=k-1 \\ k_i \geq 1}} \int\limits_{(\mathbb{R}^2_+)^{r+1}} \frac{\phi(\sum_{j=1}^r u_j + v)}{P(\sum_{j=1}^r u_j + v)} \prod_{j=1}^r G_{k_j}(u_j) \Phi_{m-r}(v)\, du\, dv \\ &= \alpha \sum_{r=1}^{\widetilde{m}} \binom{m}{r} \sum_{\substack{k_1+\ldots+k_r=k-1 \\ k_i \geq 1}} \int\limits_{\mathbb{R}^2_+} \phi(z) \frac{G_{k_1} * \ldots * G_{k_r} * \Phi_{m-r}(z)}{P(z)}\, dz. \end{aligned}$$

Hence we get formula (40) for G_k with $k > 1$. It remains to prove the estimate (41). By Corollary 2 it is sufficient to show that

$$|G_k(u)| \leq |\alpha|^k ((m-1)k)! \Theta_{m_k}(u), \tag{42}$$

and for this purpose we proceed by induction. Of course, for $k = 1$, (42) holds. Assume it holds for $s < k$. Then for $k_1 + \ldots + k_r = k - 1$ we have

$$\begin{aligned} &|G_{k_1} * \ldots * G_{k_r} * \Phi_{m-r}(u)| \\ &\quad \leq |\alpha|^{k_1+\ldots+k_r} ((m-1)k_1)! \ldots ((m-1)k_r)! \Theta_{m_{k_1}} * \ldots * \Theta_{m_{k_r}} * \Theta_{m-r}(u) \\ &\quad \leq |\alpha|^{k-1} ((m-1)(k-1))! \Theta_{m_k}(u). \end{aligned}$$

The rest of the reasoning runs as in the proof of Lemma 2.

Now we can state an analogue of Theorem 1.

THEOREM 5. *Fix* $b = (b_1, b_2) \in \mathbb{R}^2$ *and* $\eta > 0$ *and set* $\omega = -\eta - b$. *Assume that*, *for some constant* $A_\eta > 1$,

$$\int\limits_{\mathbb{R}^2_+} e^{-\eta z} K(z)\, dz \leq A_\eta, \qquad \int\limits_{\mathbb{R}^2_+} e^{-\eta z} \widehat{K}(z) dz \leq A_\eta,$$

where K *and* $\widehat{K}$ *are defined by* (33) *and* (34) *respectively. Let* $\Psi \in C^\infty(\mathbb{R}^2_+)$ *satisfy* (30) *for some constant* $C' < |\alpha|^{-1/(m-1)} (2A_\eta)^{-1/2}$, G_k *be given by* (39) *and* (40) *for* $k = 1, 2, \ldots$, *and let*

$$u_0(x) = \int\limits_0^\infty e^{-xz(t)} \Psi(t)\, dt, \tag{43}$$

$$u_k(x) = \int\limits_{\mathbb{R}^2_+} e^{-xz} G_k(z)\, dz \tag{44}$$

for $k \geq 1$ *and* $x > -\omega$. *Set* $J_k = ((m-1)k)!$ *and* $J = \{J_k\}$. *Then the series* $\sum_{k=0}^\infty u_k(x)$ *is a formal solution of* (2) *and is in the Gevrey class* $G^\omega(J)$.

Proof. We note that the function K depends on the curve Z only, therefore the constant A_η depends on Z and η only. By assumption, definition of

$\mathcal{K}_k$ and properties of convolution we have, for $k = 2p$,

$$\int\limits_{\mathbb{R}^2_+} e^{-\eta z}\mathcal{K}_k(z)\,dz = \int\limits_{\mathbb{R}^2_+} e^{-\eta z}K^{*p}(z)\,dz \le \Big(\int\limits_{\mathbb{R}^2_+} e^{-\eta z}K(z)\,dz\Big)^p \le A^p_\eta = A^{k/2}_\eta,$$

and for $k = 2p+1$,

$$\begin{aligned}\int\limits_{\mathbb{R}^2_+} e^{-\eta z}\mathcal{K}_k(z)\,dz &= \int\limits_{\mathbb{R}^2_+} e^{-\eta z}\widehat{K} * K^{*p-1}(z)\,dz \\ &\le \int\limits_{\mathbb{R}^2_+} e^{-\eta z}\widehat{K}(z)\,dz\Big(\int\limits_{\mathbb{R}^2_+} e^{-\eta z}K(z)\,dz\Big)^{p-1} \le A^p_\eta < A^{k/2}_\eta.\end{aligned}$$

From Theorem 4 we obtain, for $x > -\omega$,

$$\int\limits_{\mathbb{R}^2_+} e^{-xz}G_k(z)\,dz \le |\alpha|^k((m-1)k)!(\sqrt{2}\,C')^{m_k}\int\limits_{\mathbb{R}^2_+} e^{-(x-b)z}\mathcal{K}_{m_k}(z)\,dz.$$

We conclude from the definition of ω that for $x > -\omega$ we have $x - b > -\omega - b = \eta$, hence

$$\begin{aligned}\int\limits_{\mathbb{R}^2_+} e^{-xz}G_k(z)\,dz &\le |\alpha|^k((m-1)k)!(\sqrt{2A_\eta}\,C')^{(m-1)k+1} \\ &= \sqrt{2A_\eta}\,C'((m-1)k)!(|\alpha|(\sqrt{2A_\eta}\,C')^{m-1})^k.\end{aligned}$$

By the assumption on C', we have $F_\eta := |\alpha|(\sqrt{2A_\eta}\,C')^{m-1} < 1$, and

$$\frac{u_k(x)}{((m-1)k)!} \le \sqrt{2A_\eta}\,C'F^k_\eta,$$

therefore the theorem is proved.

Similarly, we can rephrase Theorem 2 as follows.

Theorem 6. *Suppose the sequence $\{D_k\}_k$ decreases and let $\omega = -\eta - b \in \mathbb{R}^2$ be as in Theorem 5. Then for the formal solution (43) and (44) the sequence (27) with $m_N = (m-1)N + 1$, for $N \in \mathbb{N}$, is bounded.*

Proof. If $x \in D_{m_k} - \omega$ then we have $(x-b)z > \eta z + \beta(m_k)^2$. Hence

$$\begin{aligned}u_k(x) = \int\limits_{\mathbb{R}^2_+} e^{-xz}G_k(z)\,dz &\le |\alpha|^k(\sqrt{2}\,C')^{m_k}((m-1)k)!\int\limits_{\mathbb{R}^2_+} e^{-(x-b)z}\mathcal{K}_{m_k}(z)\,dz \\ &\le \sqrt{2A_\eta}\,C'F^k_\eta((m-1)k)!e^{-\beta m_k^2},\end{aligned}$$

and the rest of the proof runs as in the general case.

5. Example. In this section we show that the assumptions in the present work are not vacuous, that is, there exists a polynomial P which has the asserted properties. Suppose that

$$P(z) = P(z_1, z_2) = Q(z)(z_1z_2 - 1),$$

where Q is some polynomial with $|Q| > 1$ on $\mathbb{R}^2_+$. Then

$$\operatorname{Char} P \cap \mathbb{R}^2_+ = \{z : z_2 = 1/z_1\}$$

and we can choose the curve $Z = \{(t, 1/t) : t \in \mathbb{R}_+\} \subset \operatorname{Char} P \cap \mathbb{R}^2_+$. It is easy to check that $kZ = \{x \in \mathbb{R}^2_+ : x_2 \geq k^2/x_1\}$, therefore conditions (i)–(iii) hold.

To check (iv), we observe that for $c_i \geq \eta > 0$,

$$\int_Z e^{-cz}\,dz = \int_0^\infty e^{-c_1 t - c_2 1/t}\sqrt{1+1/t^4}\,dt \leq \int_0^1 e^{-c_2 1/t}\sqrt{1+1/t^4}\,dt + \sqrt{2}\int_1^\infty e^{-c_1 t}\,dt$$
$$= \int_1^\infty e^{-c_2 s}\sqrt{1+1/s^4}\,ds + \frac{\sqrt{2}\,e^{-c_1}}{c_1} \leq \sqrt{2}\left(\frac{e^{-c_2}}{c_2} + \frac{e^{-c_1}}{c_1}\right) < 2\sqrt{2}\,\frac{e^{-\eta}}{\eta}.$$

Clearly, the function $f(t) = 1/t$ for $t \in \mathbb{R}_+$ satisfies the conditions assumed at the beginning of Section 5. We show that in this example the set D_k defined by (26) can be written as

$$D_k = \left\{ y \in \mathbb{R}^2_+ : y_2 > \beta^2 \frac{k^2}{4y_1} \right\}. \tag{45}$$

Indeed, fix $y = (y_1, y_2) \in D_k$. The function

$$F(z) = F(z_1, z_2) = yz - \beta k^2 = y_1 z_1 + y_2 z_2 - \beta k^2$$

defined for $z \in kZ$ satisfies $F(z) \geq 0$ on kZ. We have $\frac{\partial F}{\partial z_1}(z) = y_1 > 0$, $\frac{\partial F}{\partial z_2}(z) = y_2 > 0$ for all $z \in kZ$, therefore $\min\{F(z) : z \in kZ\} \geq 0$ must be attained on the boundary $\partial(kZ) = \{z_2 = k^2/z_1\}$. To calculate this minimal value we consider the function F on $\partial(kZ)$ as a function of one variable z_1, that is, as the function

$$h(z_1) = F\left(z_1, \frac{k^2}{z_1}\right) = y_1 z_1 + y_2 \frac{k^2}{z_1} - \beta k^2.$$

Since $h'(z_1) = y_1 - y_2 k^2/z_1^2 = 0$ for $z_1 = k\sqrt{y_2/y_1}$, we have

$$\begin{aligned}\min\{F(z) : z \in kZ\} &= \min\{F(z) : z \in \partial(kZ)\} = h(k\sqrt{y_2/y_1})\\ &= k(2\sqrt{y_1 y_2} - \beta k) \geq 0\end{aligned}$$

for $y_1 y_2 \geq (\beta k)^2/4$.

Conversely, if y satisfies the last inequality, then the quadratic function $g(z_1) = z_1 h(z_1) = y_1 z_1^2 - \beta k^2 z_1 + y_2 k^2$ has determinant $\Delta = \beta^2 k^4 - 4y_1 y_2 k^2 = k^2(\beta^2 k^2 - 4y_1 y_2) \leq 0$, so $h(z_1) = g(z_1)/z_1 \geq 0$ for all $z_1 > 0$. Hence, for all $z \in kZ$, $F(z) \geq h(z_1) \geq 0$ and consequently $y \in D_k$.

Thus the sequence $\{D_k\}_k$ decreases.

The transformation Ψ_σ has the form

$$t_1 + t_2 = u_1, \qquad 1/t_1 + 1/t_2 = u_2,$$

and the inverse transformation Ψ_+^{-1} is given by the formula

$$t_1(u) = \frac{1}{2}(u_1 - \sqrt{u_1(u_1 - 4/u_2)}), \qquad t_2(u) = \frac{1}{2}(u_1 + \sqrt{u_1(u_1 - 4/u_2)}).$$

Hence, we get the Jacobian

$$K(u) = \frac{u_1}{u_2^2\sqrt{u_1(u_1 - 4/u_2)}} = \frac{1}{u_2\sqrt{u_2(u_2 - 4/u_1)}}.$$

Since $\int_{2Z} = \int_0^\infty \int_{4/u_1}^\infty$ and $\int_{4/u_1}^\infty K(u_1, u_2)\, du_2 = u_1/2$, for any $\eta > 0$ we obtain

$$\int_{\mathbb{R}_+^2} e^{-\eta u} K(u)\, du = \int_{2Z} e^{-\eta u} K(u)\, du$$
$$= \int_0^\infty e^{-\eta_1 u_1} \Big[\int_{4/u_1}^\infty e^{-\eta_2 u_2} K(u_1, u_2)\, du_2 \Big] du_1 \le \frac{1}{2} \int_0^\infty e^{-\eta_1 u_1} u_1\, du_1 = \frac{1}{2\eta_1^2}.$$

Similar, but slightly more sophisticated calculations show that the function $\widehat{K}$ in this case also satisfies the assumptions of Theorem 5.

References

[G-T] R. Gerard and H. Tahara, *Singular Nonlinear Partial Differential Equations*, Aspects of Mathematics, Vieweg, Wiesbaden, 1996.

[P-Z] M. E. Pliś and B. Ziemian, *Convolution equations in the space of Laplace distributions*, Ann. Polon. Math. 69 (1998), 271–281.

[S-Z] Z. Szmydt and B. Ziemian, *Laplace distributions and hyperfunctions on* $\overline{\mathbb{R}}_+^n$, J. Math. Sci. Univ. Tokyo 5 (1998), 41–74.

Pedagogical University
Podchorążych 2
30-084 Kraków, Poland
E-mail: meplis@wsp.krakow.pl

Reçu par la Rédaction le 20.6.2001 (1276)