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Asymptotics for quasilinear elliptic non-positone problems

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Abstract. In the recent years, many results have been established on positive solutions for boundary value problems of the form

$$-\operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x)) = \lambda f(u(x)) \quad \text{in } \Omega,$$
$$u(x) = 0 \quad \text{on } \partial\Omega,$$

where $\lambda > 0$, Ω is a bounded smooth domain and $f(s) \geq 0$ for $s \geq 0$. In this paper, a priori estimates of positive radial solutions are presented when $N > p > 1$, Ω is an N-ball or an annulus and $f \in C^1(0,\infty) \cup C^0([0,\infty))$ with $f(0) < 0$ (non-positone).

1. Introduction. In this paper, we consider the set of positive radial solutions to the following boundary value problem for a quasilinear elliptic P.D.E.:

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda f(u) = 0 \quad \text{in } \Omega, \tag{1.1}$$
$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where Ω denotes an annulus or a ball in $\mathbb{R}^N$ $(N > p > 1)$, and $\lambda > 0$.

The problem (1.1)–(1.2) arises in the theory of quasiregular and quasiconformal mappings or in the study of non-Newtonian fluids. In the latter case, the quantity p is a characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics (see [1–2]). When $p \neq 2$, the problem becomes more complicated since certain nice properties inherent to the case $p = 2$ seem to be invalid or at least difficult to verify. The main differences between $p = 2$ and $p \neq 2$ are discussed in [6, 8]. The existence and uniqueness of positive solutions of (1.1)–(1.2) have been studied by many authors, for example, [4–10, 13–21] and the references therein.

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By a *positive solution* u of (1.1)–(1.2), we mean a function $u \in C_0^1(\Omega)$ with $u > 0$ in Ω which satisfies

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v = \lambda \int_\Omega f(u)v$$

for any $v \in C_0^\infty(\Omega)$. Thus, these solutions are considered in a weak sense. By a *small solution* u_λ of (1.1)–(1.2), we mean that $\lim_{\lambda\to 0^+} \|u_\lambda\|_\infty = 0$ (or $\lim_{\lambda\to\infty} \|u_\lambda\|_\infty = 0$). By a *large positive solution* u_λ of (1.1)–(1.2), we mean that $\lim_{\lambda\to 0^+} \|u_\lambda\|_\infty = \infty$ (or $\lim_{\lambda\to\infty} \|u_\lambda\|_\infty = \infty$).

When f is strictly increasing on $\mathbb{R}^+$, $f(0) = 0$, $\lim_{s\to 0^+} f(s)/s^{p-1} = 0$ and $f(s) \le \alpha_1 + \alpha_2 s^\mu$, where $0 < \mu < p-1$ and $\alpha_1, \alpha_2 > 0$, it has been shown in [6] that there exist at least two positive solutions for (1.1)–(1.2) when λ is sufficiently large. If $\liminf_{s\to 0^+} f(s)/s^{p-1} > 0, f(0) = 0$ and the monotonicity hypothesis $(f(s)/s^{p-1})' < 0$ holds for all $s > 0$, it has been proved in [8] that the problem (1.1)–(1.2) has a unique positive solution when λ is sufficiently large. If $f(s) > 0$ for $s \ge 0$ and $\limsup_{s\to 0^+}(f(s)/s^{p-2})' < 0$, it has been proved in [9] that the problem (1.1)–(1.2) has a unique small solution when λ is sufficiently small. It also has been proved that there exists at least one large positive radial solution of (1.1)–(1.2) for Ω being an N-ball or an annulus when λ is sufficiently small. If $f(0) < 0$, related results have been obtained in [7, 20].

A natural question is to determine how λ and $d = \max_\Omega u(\cdot,\lambda) = \|u(\cdot,\lambda)\|_\infty$ are related. When $p = 2$, $f(0) < 0$ or $f(0) = 0$ and Ω is a unit ball in $\mathbb{R}^N$, the related results have been obtained by [11, 12]. In [21], the author studied this problem for the case where Ω is a unit ball in $\mathbb{R}^N$ and $f(0) < 0$, $p > 1$. In this paper, we further study this problem for Ω being an N-ball ($N > p > 1$) or an annulus and $f(0) < 0$ (*non-positone*). This extends and complements previous results in the literature [11, 12, 21].

Consider a positive radial solution u of (1.1)–(1.2); thus $u = u(r,\lambda)$ satisfies

$$(r^{N-1}|u'|^{p-2}u')' + \lambda r^{N-1} f(u) = 0. \tag{1.3}$$

If Ω is an annulus $0 < r_1 \le r \le r_2$, we introduce the transformation of variables

$$s = r^{(p-N)/(p-1)}, \qquad u(r) = v(s). \tag{1.4}$$

Thus (1.3) becomes

$$(|v'(s)|^{p-2}v'(s))' + \lambda((p-1)/(N-p))^p s^{-p(N-1)/(N-p)} f(v(s)) = 0 \tag{1.5}$$

and the boundary conditions become

$$v(s_1) = 0, \qquad v(s_2) = 0. \tag{1.6}$$

If $\Omega = B_1(0)$, the boundary condition (1.2) becomes

$$u'(0) = 0, \quad u(1) = 0.$$

2. A priori estimates for Ω being an annulus. In this section, we consider the set of radially symmetric positive solutions to the equation

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω denotes an annulus in $\mathbb{R}^N$ $(N > p > 1)$ and $\lambda > 0$. Here $f : [0,\infty) \to \mathbb{R}$ satisfies the following assumptions:

(A) $f \in C^1(0,\infty) \cap C([0,\infty))$, $f(0) < 0$, and there exists $\alpha > 0$ such that $f(s) < 0$ for $0 < s < \alpha$, $f(\alpha) = 0$, f is increasing for $s > \alpha$ and $\lim_{s\to\infty} f(s) = \infty$.

(B) There are constants $L_0 > 0$ and $p-1 < q < ((p-1)N + p)/(N-p)$ such that $\lim_{u\to\infty} f(u)/u^q = L_0$.

THEOREM 2.1. *Suppose that conditions* (A) *and* (B) *hold. Then there exist positive constants K_1 and K_2 such that for small λ,*

$$K_1 < \lambda \|u(\cdot,\lambda)\|_\infty^{q-p+1} < K_2,$$

where $\{u(\cdot,\lambda) \mid \lambda \in (0,\lambda_0)\}$ is an arbitrary positive radially symmetric solution of (1.1)–(1.2). *Furthermore, for any sequence $\{\lambda_i\}$ with $\lim_{i\to\infty} \lambda_i = 0$, there exists a subsequence, still denoted by $\{\lambda_i\}$, a constant θ, and a positive function w such that*

(1) *w is a solution of the problem*

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \theta L_0 u^q \quad \textit{in } \Omega,$$
$$u = 0 \quad \textit{on } \partial\Omega,$$

(2) *$\{u(\cdot,\lambda_i)/\|u(\cdot,\lambda_i)\|_\infty\}$ converges to w in $C^1(\overline{\Omega})$ as $i \to \infty$.*

To obtain Theorem 2.1, the following lemma is established:

LEMMA 2.2. *Let f satisfy condition* (A) *and $u_\lambda \in C_0^1(\overline{\Omega})$ be a radially symmetric positive solution of* (1.1)–(1.2). *Then* $\lim_{\lambda\to 0^+} \|u_\lambda\|_\infty = \infty$.

Proof. On the contrary, assume that there exist sequences $\{\lambda_n\}$ and $\{u_n\} \equiv \{u_{\lambda_n}\} \in C_0^1(\overline{\Omega})$ such that $\lambda_n \to 0$ and $\|u_n\| \le M$, where $M > 0$ is independent of n. Then $\|u_n\|_\infty \not\to 0$ as $n \to \infty$. Indeed, suppose this does not hold; by the regularity of $-\operatorname{div}(|\nabla \cdot|^{p-2}\nabla \cdot)$ (see [6]), there exists $\omega \ge 0$ in Ω such that $\lambda_n^{-1/(p-1)} u_n \to \omega$ in $C^1(\Omega)$ as $n \to \infty$. Moreover, ω satisfies the problem

$$-\operatorname{div}(|\nabla\omega|^{p-2}\nabla\omega) = f(0) < 0 \quad \text{in } \Omega,$$
$$\omega = 0 \quad \text{on } \partial\Omega.$$

It follows from the maximum principle that $\omega < 0$ in Ω. This is impossible. Now, since u_n is uniformly bounded in Ω and $\lambda_n \to 0$ as $n \to \infty$, it follows from the regularity of $-\operatorname{div}(|\nabla \cdot|^{p-2}\nabla \cdot)$ again that there exists $\overline{\omega} \in C_0^1(\Omega)$ with $\overline{\omega} \geq 0$ in Ω such that $u_n \to \overline{\omega}$ in $C^1(\Omega)$ as $n \to \infty$ and $\overline{\omega}$ satisfies

$$-\operatorname{div}(|\nabla\overline{\omega}|^{p-2}\nabla\overline{\omega}) \equiv 0 \quad \text{in } \Omega,$$
$$\overline{\omega} = 0 \quad \text{on } \partial\Omega.$$

Thus, $\overline{\omega} \equiv 0$ in Ω. This also implies that $u_n \to 0$ in $C^1(\Omega)$ as $n \to \infty$. But the above argument implies that this is impossible. Hence, we conclude that $\|u_n\|_\infty \to \infty$ as $n \to \infty$.

LEMMA 2.3. *Let $a > 0$. Then, for any $\theta \leq 0$, the equation*

$$(|u'|^{p-2}u')' + au(s)^\mu = 0 \quad \textit{in } (\theta, \infty)$$

has no bounded positive solution $u \in C^1(\theta,\infty)$ with $u'(0) = 0$. Moreover, the equation

$$(|u'|^{p-2}u')' + au(s)^\mu = 0 \quad \textit{in } (-\infty, \infty)$$

has no bounded positive entire solution $u \in C^1(-\infty,\infty)$ with $u'(0) = 0$.

Proof. Suppose that such a solution $u(s)$ exists. Let $\Phi_p(y) = |y|^{p-2}y$. Then

$$\Phi_p(u'(s)) = -\int_0^s au(\xi)^\mu \, d\xi \quad \text{for } s \in (0,\infty). \tag{2.2}$$

Thus, $\Phi_p(u'(s_0)) = -k < 0$ for some $s_0 > 0$ where $k = a\int_0^{s_0} u(\xi)^\mu \, d\xi$. By (2.2), $\Phi_p(u'(s)) \leq -k$ for $s > s_0$, since $u(s) > 0$ for $s > 0$. Then

$$u'(s) \leq \Phi_p^{-1}(-k) = -k^{1/(p-1)} \quad \text{for } s > s_0. \tag{2.3}$$

Integrating (2.3) over (s_0, s), we obtain $u(s) \to -\infty$ as $s \to \infty$, contrary to the assumption that $u(s)$ is a bounded solution.

Proof of Theorem 2.1. By the standard estimates for elliptic equations and condition (B), it follows that

$$\begin{aligned}\|u(\cdot,\lambda)\|_\infty^{p-1} &\leq C(\Omega)\lambda\|f(u(\cdot,\lambda))\|_\infty \\ &= C(\Omega)\lambda\|L_0u(\cdot,\lambda)^q + \{f(u(\cdot,\lambda)) - L_0u(\cdot,\lambda)^q\}\|_\infty.\end{aligned}$$

That is,

$$\begin{aligned}1 \leq{}& C(\Omega)\lambda L_0 \frac{\|u(\cdot,\lambda)^q\|_\infty}{\|u(\cdot,\lambda)\|_\infty^{p-1}} \\ &+ C(\Omega)\lambda\left\|\frac{f(u(\cdot,\lambda)) - L_0u(\cdot,\lambda)^q}{u(\cdot,\lambda)^q + 1}\right\|_\infty \frac{\|u(\cdot,\lambda)^q + 1\|_\infty}{\|u(\cdot,\lambda)\|_\infty^{p-1}}.\end{aligned}$$

By (B), there exists a positive constant K_0 such that

$$|(f(u) - L_0 u^q)/(u^q + 1)| < K_0 \quad \text{for } u \in \mathbb{R}^+.$$

Then

$$1 \le C(\Omega)\lambda\|u(\cdot,\lambda)\|_\infty^{q-p+1} + C(\Omega)\lambda K_0\left\{\|u(\cdot,\lambda)\|_\infty^{q-p+1} + \frac{1}{\|u(\cdot,\lambda)\|_\infty^{p-1}}\right\}.$$

From $\lim_{\lambda\to 0}\|u(\cdot,\lambda)\|_\infty = \infty$, it follows that there exists a positive constant K_1 such that, for any $\lambda \in (0,\lambda_0)$, $K_1 < \lambda\|u(\cdot,\lambda)\|_\infty^{q-p+1}$.

Thus, the left-hand inequality in Theorem 2.1 is established.

To obtain the other half of Theorem 2.1, we show that $T = \lambda\|u\|_\infty^{q-p+1}$ is bounded as $\lambda \to 0$. Let u_λ be a positive radial solution of (1.1)–(1.2) satisfying $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to 0^+$. Then there exists a positive solution v_λ of (1.5)–(1.6) satisfying $\|v_\lambda\|_\infty \to \infty$ as $\lambda \to 0^+$. Let (λ_n, v_n) be a positive solution of (1.5)–(1.6) with $\lambda = \lambda_n$ satisfying $\lambda_n \to 0^+$ and $\|v_n\|_\infty \to \infty$ as $n \to \infty$. Then $w_n = v_n/\|v_n\|_\infty$ satisfies

$$-(\Phi_p(w_n'(s)))' = \lambda_n\|v_n\|^{q-p+1}\left(\frac{p-1}{N-p}\right)^p s^{-p(N-1)/(N-p)}\,\frac{f(v_n)}{\|v_n\|_\infty^q}, \tag{2.4}$$

and $w_n(s_1) = w_n(s_2) = 0$, $\|w_n\|_\infty = 1$.

Now, we show that $\{T_n\} = \{\lambda_n\|v_n\|_\infty^{q-p+1}\}$ is bounded. We prove this by a blowing up argument as in [3]. Suppose that $T_n \to \infty$ as $n \to \infty$. Let $\widehat{s}_n \in (s_1, s_2)$ be such that $w_n(\widehat{s}_n) = 1$, $y_n = T_n^{1/p}(s - \widehat{s}_n)$ and $\widehat{w}_n(y_n) = w_n(s)$. Then $\widehat{w}_n(0) = 1$, $\widehat{w}_n'(0) = 0$ and $\widehat{w}_n(y_n)$ satisfies

$$\begin{aligned} -(\Phi_p(\widehat{w}_n'))' = {} & \left(\frac{p-1}{N-p}\right)^p (y_n T_n^{-1/p} + \widehat{s}_n)^{-p(N-1)/(N-p)} \\ & \times \frac{f(\|v_n\|_\infty \widehat{w}_n(y_n))}{\|v_n\|_\infty^q}. \end{aligned} \tag{2.5}$$

Since $\widehat{s}_n \in [s_1, s_2]$ and $f(s) \le \beta_1 + \beta_2 s^q$ and $\|v_n\|_\infty \to \infty$ as $n \to \infty$, the right-hand side of (2.5) is uniformly bounded. Thus, there exist subsequences (still denoted by $\{\widehat{s}_n\}, \{\widehat{w}_n\}$ and $\{v_n\}$) such that $\widehat{w}_n \to \widehat{w}$ in $C^1_{\rm loc}(-\infty,\theta)$ (or $C^1_{\rm loc}(-\infty,\infty)$ or $C^1_{\rm loc}(\theta,\infty)$) as $n \to \infty$. Here $\theta \le 0$ is a fixed number since the limit of $\widehat{s}_n$ may be s_1 or s_2 and $T_n \to \infty$. If $\widehat{s}_n \to s_1$ as $n \to \infty$, we assume that $\lim_{n\to\infty} T_n^{1/p}(s_1 - \widehat{s}_n) = \theta \le 0$ (or $\theta = -\infty$). Otherwise, we can choose a subsequence of $\{T_n^{1/p}(s_1 - \widehat{s}_n)\}$ whose limit exists (or is $-\infty$). If the limit of $\widehat{s}_n$ is s_2, and if we set $y_n = T_n^{1/p}(\widehat{s}_n - s_2)$, it follows that $\widehat{w}_n \to \widehat{w}$ in $C^1_{\rm loc}(-\infty,\infty)$ (or $C^1_{\rm loc}(\theta,\infty)$, $\theta \le 0$) as $n \to \infty$. Therefore, we assume that $\widehat{w}_n \to \widehat{w}$ in $C^1_{\rm loc}(\theta,\infty)$ (or $C^1_{\rm loc}(-\infty,\infty)$)). Since $\widehat{w} \in C^1(\theta,\infty)$ (or $C^1(-\infty,\infty)$) satisfies $-(\Phi_p(\widehat{w}'))' \ge 0$ in (θ,∞) (or $(-\infty,\infty)$), and $\widehat{w}(0) = 1$ and $\widehat{w}'(0) = 0$, the strong maximum principle as in Lemma 2.3 of [6] implies that $\widehat{w} > 0$ in (θ,∞) (or $(-\infty,\infty)$). Thus, for any interval in (θ,∞) (or

$(-\infty,\infty)$), there exists an $\omega > 0$ such that $\widehat{w}(x) > \omega$ in this interval. This implies that

$$\frac{f(\|v_n\|_\infty \widehat{w}_n)}{\|v_n\|_\infty^q} \to L_0 \widehat{w}^q$$

in $C_{\mathrm{loc}}(\theta,\infty)$ (or $C_{\mathrm{loc}}(-\infty,\infty)$) as $n\to\infty$. Therefore, $\widehat{w}$ satisfies

$$-(\Phi_p(\widehat{w}'))' = L_0((p-1)/(N-p))^p s_*^{-p(N-1)/(N-p)} \widehat{w}^q$$

in (θ,∞) (or $(-\infty,\infty)$). Here $s_* = \lim_{n\to\infty} \widehat{s}_n$. This contradicts Lemma 2.3. Thus, $\{T_n\}$ is bounded. Therefore

$$K_1 < \lambda \|u_\lambda\|_\infty^{q-p+1} < K_2.$$

Finally, let $\{\lambda_i\}$ be a sequence with $\lim_{i\to\infty}\lambda_i = 0$ and denote the quantity $\lambda_i\|v(\cdot,\lambda_i)\|_\infty^{q-p+1}$ by θ_i. Suppose that $\theta\in[K_1,K_2]$ is any accumulation point of $\{\theta_i\}$. Thus there exists a subsequence of $\{\theta_i\}$ (still denoted by $\{\theta_i\}$ later) which converges to θ. Let $w(x,\lambda) = v(x,\lambda)/\|v(\cdot,\lambda)\|_\infty$. Then $\|w(\cdot,\lambda)\|_\infty = 1$ and

$$-\operatorname{div}(|\nabla w|^{p-2}\nabla w) = \theta_i \frac{f(v(x,\lambda_i))}{\|v(\cdot,\lambda_i)\|_\infty^q}.$$

Using the same idea as above for (2.4), we find a function $w(\cdot)$ and a subsequence of $\{w(\cdot,\lambda_i)\}$ (still denoted by $\{w(\cdot,\lambda_i)\}$) such that $\{w(\cdot,\lambda_i)\}$ converges to w in $C^1(s_1,s_2)$ as $i\to\infty$. By condition (B), it follows that

$$\lim_{i\to\infty} \frac{f(\|v(\cdot,\lambda_i)\|_\infty w(x,\lambda_i))}{\|v(\cdot,\lambda_i)\|_\infty^q} = L_0 w^q.$$

Therefore $w(\cdot)$ is a positive solution of the problem

$$-\operatorname{div}(|\nabla w|^{p-2}\nabla w) = \theta L_0 w^q \quad \text{in } \Omega,$$
$$w = 0 \quad \text{on } \partial\Omega,$$

and $\|w(\cdot)\|_\infty = 1$.

3. A priori estimates for Ω being a ball. In this section, consider the set of radially symmetric positive solutions to the equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda f(u) \quad \text{for } x\in\Omega, \tag{3.1}$$

$$u|_{\partial\Omega} = 0, \tag{3.2}$$

where Ω denotes the unit ball in $\mathbb{R}^N$ $(N>1)$, centered at the origin, and $\lambda>0$. Here $f:[0,\infty)\to\mathbb{R}$ is assumed to satisfy

(3.3) $f(0)<0$ (non-positone), $f'(u)\ge 0$, and $f(u_0)>0$ for some $u_0>0$.

Let F be defined as $F(t)=\int_0^t f(s)\,ds$, and let β and θ $(\beta<\theta)$ be the unique positive zeros of f and F, respectively.

In this section, the following theorem is proved:

THEOREM 3.1. *Let u be a radially symmetric positive solution of* (3.1)–(3.2) *with $u(0) = d$ and suppose f satisfies* (3.3). *Then for large λ,*

$$\left(\frac{p}{p-1}\right)^{p-1}(N-1) \le \frac{\lambda f(d)}{d^{p-1}} \le \frac{2Nf(d)}{d^{p-1}}\left(\frac{p}{p-1}\right)^{p-1}\left(\int_\theta^d \frac{ds}{f(s)^{1/(p-1)}}\right)^{p-1}. \tag{3.4}$$

REMARK. If $f(u) \le M$ for all u, or if $f(u) = u^\alpha - 1$ where $0 < \alpha < p-1$, then $f(d)d^{-(p-1)}(\int_\theta^d f(s)^{-1/(p-1)}\,ds)^{p-1}$ is finite.

Note that radially symmetric positive solutions of (3.1)–(3.2) are strictly decreasing in r for $r \in (0,1)$ where $r = \|x\|$. Thus, they satisfy

$$(\Phi_p(u'))' + \frac{N-1}{r}\Phi_p(u') + \lambda f(u) = 0 \quad \text{in } (0,1), \tag{3.5}$$

$$u(0) = d, \quad u'(0) = 0, \quad u(1) = 0, \quad u'(r) < 0 \quad \text{in } (0,1). \tag{3.6}$$

where $\Phi_p(s) = |s|^{p-2}s$, $p > 1$.

If u is a solution of (3.5)–(3.6), then multiplying (3.5) by r^{N-1} and integrating from 0 to t gives

$$-\int_0^t (r^{N-1}\Phi_p(u'))'\,dr = \int_0^t \lambda r^{N-1} f(u)\,dr.$$

Since u is decreasing and f is increasing, it follows that

$$-t^{N-1}\Phi_p(u') = \lambda\int_0^t r^{N-1} f(u)\,dr \ge \lambda f(u(t))\int_0^t r^{N-1}\,dr = \frac{\lambda t^{N-1} f(u)}{N}.$$

Hence

$$(-u')^{p-1} \ge \frac{\lambda t f(u)}{N}. \tag{3.7}$$

Next, multiplying (3.5) by u' and integrating over $[0,1]$ yields

$$\frac{p-1}{p}|u'(1)|^p + \int_0^1 \frac{N-1}{r}|u'|^p\,dr = \lambda F(d). \tag{3.8}$$

Note that this implies

$$d > \theta. \tag{3.9}$$

To prove Theorem 3.1, we need the following lemma:

LEMMA 3.2 (see [19]). *Let u be a radially symmetric positive solution of* (3.1)–(3.2). *Then there exists $M > 0$ such that for large λ,*

$$|u'(1)| > \lambda^{1/(p-1)}M.$$

The proof of Theorem 3.1 is based upon a modification of the method of Iaia [12].

Proof of Theorem 3.1. First, Hölder's inequality gives

$$d = u(0) - u(1) = -\int_0^1 u'(t)\,dt = \int_0^1 \frac{-u'}{t^{1/p}}\,t^{1/p}\,dt$$
$$\leq \Big(\int_0^1 \frac{|u'|^p}{t}\,dt\Big)^{1/p}\Big(\int_0^1 t^{1/(p-1)}\,dt\Big)^{(p-1)/p}.$$

Next, it follows from (3.8) that

$$d^p \leq \Big(\frac{p-1}{p}\Big)^{p-1}\int_0^1 \frac{|u'|^p}{t}\,dt \leq \Big(\frac{p-1}{p}\Big)^{p-1}\frac{\lambda F(d)}{N-1}.$$

Thus

$$\frac{\lambda F(d)}{d^p} \geq \Big(\frac{p}{p-1}\Big)^{p-1}(N-1).$$

Finally, since $f' \geq 0$,

$$F(d) = \int_0^d f(s)ds = df(d) - \int_0^d sf'(s)ds \leq df(d). \tag{3.10}$$

This proves the left-hand inequality of (3.4).

In order to establish the right-hand inequality of (3.4), from (3.7) we get

$$-u'(t) \geq \Big(\frac{\lambda t f(u)}{N}\Big)^{1/(p-1)}.$$

Let $q_\lambda \in (0,1)$ be such that $u(q_\lambda) = \theta$. Then $u(t) \geq \theta > \beta$ on $[0, q_\lambda]$. Thus $f(u(t)) \geq f(\theta) > f(\beta) = 0$ on $[0, q_\lambda]$. So, on $[0, q_\lambda]$ we have

$$\int_0^{q_\lambda} \frac{-u'}{f(u)^{1/(p-1)}}\,dt \geq \int_0^{q_\lambda}\Big(\frac{\lambda t}{N}\Big)^{1/(p-1)}dt = \Big(\frac{\lambda}{N}\Big)^{1/(p-1)}\Big(\frac{p-1}{p}\Big)q_\lambda^{p/(p-1)}.$$

Changing variables in the first integral via $s = u(t)$ gives

$$\int_\theta^d \frac{ds}{f(s)^{1/(p-1)}} \geq \Big(\frac{\lambda}{N}\Big)^{1/(p-1)}\Big(\frac{p-1}{p}\Big)q_\lambda^{p/(p-1)}.$$

Thus,

$$\frac{f(d)^{1/(p-1)}}{d}\int_\theta^d \frac{ds}{f(s)^{1/(p-1)}} \geq \frac{(\lambda f(d))^{1/(p-1)}}{N^{1/(p-1)}d}\Big(\frac{p-1}{p}\Big)q_\lambda^{p/(p-1)}. \tag{3.11}$$

Therefore, the proof of Theorem 3.1 will be completed once the following lemma is established.

LEMMA 3.3. $\lim_{\lambda\to\infty} q_\lambda = 1$.

From this lemma, for large λ, we have $q_\lambda^p \geq 1/2$. Substituting this into (3.11), one can deduce

$$\frac{\lambda f(d)}{d^{p-1}} \leq \frac{2Nf(d)}{d^{p-1}}\left(\frac{p-1}{p}\right)^{p-1}\left(\int_\theta^d \frac{ds}{f(s)^{1/(p-1)}}\right)^{p-1},$$

which completes the proof of Theorem 3.1.

Proof of Lemma 3.3. Multiplying (3.5) by u' and integrating from t to 1 gives

$$\int_t^1 \left[u'(\Phi_p(u'))' + \frac{N-1}{r}|u'|^p\right] dr = \int_t^1 (-\lambda f(u)u')\, dr.$$

Thus

$$\frac{p-1}{p}[|u'|^p(1) - |u'|^p(t)] + \int_t^1 \frac{N-1}{r}|u'|^p\, dr = -\lambda[F(u(1)) - F(u(t))].$$

Since $F(u(1)) = F(0) = 0$, it follows that

$$\frac{p-1}{p}[|u'|^p(1) - |u'|^p(t)] \leq \lambda F(u(t)).$$

Now, for $q_\lambda \leq t \leq 1$, it follows that $\theta = u(q_\lambda) \geq u(t) \geq u(1) = 0$, and then $F(u(t)) \leq 0$. Hence,

$$|u'|^p(1) \leq |u'|^p(t) \quad \text{for } t \in [q_\lambda, 1]. \tag{3.12}$$

Now Lemma 3.2 shows that there exists a $c > 0$ independent of λ such that

$$-u'(1) \geq c\lambda^{1/(p-1)} \quad \text{for large } \lambda.$$

Consequently, it follows from (3.12) that

$$(-u'(t))^p \geq (-u'(1))^p \geq c^p\lambda^{p/(p-1)} \quad \text{for } t \in [q_\lambda, 1].$$

Integrating on $[q_\lambda, 1]$ gives

$$\theta = u(q_\lambda) = -\int_{q_\lambda}^1 u'(t)\, dt \geq \int_{q_\lambda}^1 c\lambda^{1/(p-1)}\, dt = c\lambda^{1/(p-1)}(1 - q_\lambda).$$

Thus

$$0 \leq 1 - q_\lambda \leq \frac{\theta}{c\lambda^{1/(p-1)}}.$$

As $\lambda \to \infty$ the right-hand side of the above expression tends to zero; hence $\lim_{\lambda\to\infty} q_\lambda = 1$ and this completes the proof of the lemma.

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