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FUNCTIONAL ANALYSIS

Exhaustivity in Topological Riesz Spaces with the Principal Projection Property

by

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Summary. Exhaustive and uniformly exhaustive elements are studied in the setting of locally solid topological Riesz spaces with the principal projection property. We study the structure of the order interval $[0, x]$ when x is an exhaustive element and the structure of the solid hull of a set of uniformly exhaustive elements.

1. Introduction. In functional analysis there has been a large amount of study on the embeddability of the classical Banach spaces c_0, ℓ_1 and ℓ_∞ in other Banach spaces. Because of results from vector measure theory such as the Diestel–Faires theorem [8], these studies are often done in conjunction with studies on strongly additive measures. In many of these studies the main emphasis is on normed vector spaces, or more specifically, Banach spaces. Although many of these spaces are partially ordered, less attention has been given to the properties that are inherent to the partial ordering on the space. In the 1940's mathematicians began studying these partially ordered vector spaces in more detail and many results have been obtained, especially in the study of Banach lattices. In this paper we specifically want to study the concepts of exhaustivity, (absolute) continuity, and strong additivity in the more general setting of topological Riesz spaces. Many of these results will generalize results known for Banach lattices.

As is pointed out in the introduction of [9], early interest in weak and weak* compactness was often motivated by vector measure theory. This is illustrated by the following two well-known results.

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Theorem 1.1. *A set $K \subseteq \mathrm{ca}(\Sigma)$ is weakly sequentially compact iff it is bounded and the countable additivity of μ on Σ is uniform for $\mu \in K$.*

Theorem 1.2. *A set $K \subseteq \mathrm{ca}(\Sigma)$ is weakly sequentially compact iff it is bounded and, for some positive $\lambda \in \mathrm{ca}(\Sigma)$, $\mu \ll \lambda$ uniformly for $\mu \in K$.*

As the following theorems illustrate, there is a strong connection between strong additivity and countable additivity. These theorems can be extended to the setting of topological Riesz spaces. The extensions use the lattice properties of the space of vector-valued measures and minimize set-theoretic manipulations.

Theorem 1.3 (Bell–Bilyeu–Lewis [3]). *A positive element k of a σ-Dedekind complete Banach lattice X is exhaustive if and only if the norm is countably order continuous on the order interval $[0,k]$.*

Theorem 1.4 (Brooks [5], Drewnowski [11]). *Suppose (μ_n) is a sequence of countably additive scalar functions on the σ-algebra Σ, and μ is a finitely additive (possible infinite) measure on Σ such that $\mu_n \ll \mu$ for each n. Then $\mu_n \ll \mu$ uniformly.*

A recent extension of Theorem 1.4 has been made in the study of submeasures [13]. Also, Drewnowski and Labuda [12] proved a result similar to Theorem 1.3 for a disjointly σ-Dedekind complete TRS. Their methods were quite different due to the difference in hypotheses. Also the emphasis on exhaustivity in [12] is on vector measures and not exhaustive elements in a TRS. In [12], a characterization of exhaustive vector measures is made for Lebesgue and pre-Lebesgue topologies. That connection will also be made in the setting of this paper. For more on Lebesgue and pre-Lebesgue topologies see Aliprantis and Burkinshaw [1]. Also the question was raised in [3] of whether or not $[0,k]$ must be separable whenever k is exhaustive. A counterexample will be provided in Section 3.

2. Continuity in topological Riesz spaces. If X is a Riesz space, X is said to have the *principal projection property* (PPP) provided that for each pair of x and y in $X_+ = \{z : z \geq 0\}$ the element $\bigvee_n nx \wedge y$ exists. This definition is equivalent to that found in [17]. If X is a Riesz space with the PPP, define $P_x(y) = \bigvee_n nx \wedge y$ for all $x, y \in X_+$. For arbitrary $y \in X$ define $P_x(y)$ to be $P_x(y^+) - P_x(y^-)$. These projections have proved to be useful in many different areas. For instance, if μ and ν are scalar valued measures these projections can be used to find the absolutely continuous and singular parts of ν with respect to μ only using the order properties of the reals. These projections have also been applied to abstract L-spaces [14], measure spaces [4], and more recently, submeasures [13]. Note that if $x, y \in X_+$ the following properties hold. The majority of them can be found in the results of Kakutani [14].

(a) $2P_x(y) \wedge y = P_x(y)$.
(b) $P_x(y) \wedge (y - P_x(y)) = 0$.
(c) $x \wedge (y - P_x(y)) = x \wedge y - x \wedge P_x(y) = 0$.
(d) If $x = \psi + \eta$, where $\psi \wedge \eta = 0$, then $P_\psi(y) + P_\eta(y) = P_x(y) = P_{\psi \vee \eta}(y)$.
(e) P_x is linear.
(f) $P_{P_x(y)} = P_x P_y = P_{x \wedge y}$.
(g) $|P_x(y)| = P_x(|y|)$.

Let $O = \{P_x : x \in X_+\}$. A sequence (P_i) from O is said to be *disjoint* (or pairwise disjoint) if $P_i P_j = 0$ for $i \neq j$. Assume that (X, τ) is a TRS with the PPP.

Definitions.

(1) A subset K of X is said to be (*uniformly*) *continuous with respect to an element* $m \in X$ if $P_i(u) \to 0$ (uniformly) for $u \in K$ whenever P_i is a sequence from O such that $P_i(m) \to 0$.
(2) A subset K of X is said to be (*uniformly*) *exhaustive* if $P_i(u) \to 0$ (uniformly) for $u \in K$ whenever (P_i) is a disjoint sequence from O. If $K = \{k\}$ is a singleton we say that k is *exhaustive*.

Note that if $X = \mathrm{ba}(\Sigma)$, then $\mu \in X$ is exhaustive if and only if it is strongly additive and μ is absolutely continuous with respect to $\nu \in X$ if and only if it is continuous with respect to ν using the above definition. The following lemmas will be helpful in establishing the main result of this section. The first lemma is true for any Riesz space with the PPP and can be found in [3]. For the remaining results we will assume that we have a TRS.

Lemma 2.1. *Suppose X is a Riesz space with the PPP. If x and y are in X and $0 \le y \le x$, then there is a z in X_+ so that $P_x - P_y = P_z$.*

Proof. Suppose the hypotheses are satisfied. From property (b) above we see that $P_y(x) \wedge (x - P_y(x)) = 0$ and from (d) we have $P_x = P_{P_y(x)} + P_{x - P_y(x)}$. Using property (f) and the fact that $y \le x$ we obtain $P_{P_y(x)} = P_{x \wedge y} = P_y$. Therefore $P_x = P_y + P_{x - P_y(x)}$. Since $P_y(x) \le x$ we have $P_x - P_y \in O$. ∎

Lemma 2.2. *Suppose X is a TRS with the PPP. If K is a uniformly exhaustive subset of X, then $|K| = \{|x| : x \in K\}$ is uniformly exhaustive.*

Proof. Suppose K is uniformly exhaustive. Suppose (x_n) is a disjoint sequence from X_+. Let V be a solid neighborhood of the origin. From (g) we have $|P_x(u)| = P_x(|u|)$ for all $u \in K$. Choose a natural number N such that $P_{x_n}(u) \in V$ for every $n \ge N$ and every $u \in K$. Recall that V is solid, $P_{x_n}(|u|) = |P_{x_n}(u)|$, and $P_{x_n}(u) \in V$. Therefore $P_{x_n}(|u|) \in V$ for all $n \in \mathbb{N}$ and all $u \in K$. Consequently, $|K|$ is uniformly exhaustive. ∎

THEOREM 2.3. *Suppose that X is a TRS with the PPP and K is a subset of X. If K is a uniformly exhaustive subset of X and (P_{x_i}) is a sequence from O, then for every solid neighborhood V of the origin there exists a natural number N so that $(P_{x_k} - P_{x_k \wedge \bigvee_{i=1}^{n} x_i})(u) \in V$ for all $u \in K$ whenever $k \geq n \geq N$.*

Proof. Suppose the conclusion is false. Then there is a solid neighborhood V of the origin, an increasing sequence (n_i) of positive integers, and a sequence (u_i) from K so that for all i, $(P_{x_{n_i}} - P_{x_{n_i} \wedge \bigvee_{k=1}^{n_{i-1}} x_k})(u_i)$ is not in V. A calculation using properties (c), (d), and (f) shows that the above projections are pairwise disjoint. Lemma 2.1 shows the projections are in O. This contradicts the uniform exhaustivity of K. ∎

The following theorem is the main result of this section. If X is a Banach lattice and $|x| \leq |y|$, we have by definition $\|x\| \leq \|y\|$. In many cases when working with Banach lattices, it is this property of the norm that is used and not the fact that the norm is complete. In particular, if X is a TRS and y belongs to a locally solid neighborhood V of the origin, then $x \in V$ whenever $|x| \leq |y|$. This property facilitates the proof of the following generalization of Theorem 1.4. Theorem 1.4 was established independently by L. Drewnowski [11] and James K. Brooks [5]. This argument also simplifies arguments in [6].

THEOREM 2.4. *Suppose that X is a metrizable TRS with the PPP and K is a uniformly exhaustive subset of X. If K is continuous with respect to $m \in X_+$, then K is uniformly continuous with respect to m.*

Proof. Suppose that ϱ is a metric for the TRS X. Further suppose that $m \in X_+$ and that K is a uniformly exhaustive subset of X so that K is continuous with respect to m but not uniformly continuous with respect to m. Then there exists a sequence x_i from X_+, a locally solid neighborhood V of the origin, and a sequence y_i from K so that $P_{x_i}(m) \to 0$ and

$$P_{x_i}(y_i) \notin 2V \quad \text{for all } i. \tag{I}$$

We can assume without loss of generality that

$$\sum \varrho(0, P_{x_i}(m)) < \infty. \tag{II}$$

Applying Theorem 2.3, let n_1 be a positive integer so that if $n \geq n_1$ then

$$(P_{x_n} - P_{x_n \wedge \bigvee_{k=1}^{n_1} x_k}(u)) \in \tfrac{1}{2}V \quad \text{for all } u \in K.$$

Let $z_1 = \bigvee_{k=1}^{n_1} x_k$. Then from (I) and the statement above we find that $P_{z_1 \wedge x_n}(y_n)$ is not in $\big(2 - \frac{1}{2}\big)V$ for all $n \geq n_1$. Now let $a_1 = z_1 \wedge x_{n_1}$, $a_2 = z_1 \wedge x_{n_1+1}, \ldots$. Applying Theorem 2.3 to (P_{a_i}), let n_2 $(> n_1)$ be a positive integer so that if $n \geq n_2$, then

$$(P_{a_n} - P_{a_n \wedge \bigvee_{k=1}^{n_2} a_k}(u)) \in \tfrac{1}{4}V \quad \text{for all } u \in K.$$

Let $z_2 = \bigvee_{k=1}^{n_2} a_k$. By a similar argument to that above, there is a sequence (b_n) in K such that $P_{z_2 \wedge a_n}(b_n)$ is not in $\left(2 - \frac{1}{2} - \frac{1}{4}\right)V$ for all $n \geq n_2$. Since $\left(2 - \frac{1}{2} - \frac{1}{4}\right)V$ is locally solid and $|P_{z_2}(b_n)| = P_{z_2}(|b_n|) \geq P_{a_n \wedge z_2}(b_n)$, we see that $|P_{z_2}(b_n)|$ is not in $\left(2 - \frac{1}{2} - \frac{1}{4}\right)V$ for all $n \geq n_2$.

Continue inductively to manufacture a sequence (z_k) from X_+ and a subsequence (d_k) of (y_n) such that

(III) $$z_{k+1} \leq z_k,$$

(IV) $$P_{z_k}(|d_k|) \notin V \quad \text{for each } k.$$

Next observe that if $\{q_1, \ldots, q_t\} \subseteq X_+$, $u \in X_+$, and $w = \bigvee q_i$, then $P_w(u) \leq \sum_{i=1}^{t} P_{q_i}(u)$. Therefore, using (II) and the fact that the sequence (z_k) was defined inductively in terms of (x_k) $(z_k \leq \bigvee_{i=n_{k-1}}^{n_k} x_i)$, we get

(V) $$P_{z_k}(m) \to 0.$$

Now use (IV), (V), and the fact that K (and therefore $|K|$) is continuous with respect to m to select subsequences $(P_{z_{k_i}})$ of (P_{z_k}) and (d_{k_i}) of (d_k) such that

$$(P_{z_{k_i}} - P_{z_{k_{i+1}}})(|d_{k_i}|) \notin \tfrac{1}{2}V \quad \text{for each } i.$$

But (III) implies this sequence of differences of projections is a disjoint sequence from O. This contradicts the uniform exhaustivity of $|K|$. ∎

3. Exhaustivity. The results of this section further characterize exhaustivity in a TRS. If A is a subset of X, then $\widehat{A}$ denotes the solid hull of A, i.e. $\widehat{A} = \{y \in X : |y| \leq |x| \text{ for some } x \in A\}$. The next theorem describes the structure of $\widehat{K}$ when K is a uniformly exhaustive subset of X. Recall that an *ideal* in a TRS X is a solid vector subspace of X. ∎

THEOREM 3.1. *Suppose X is a TRS with the PPP. A subset K of X is uniformly exhaustive iff each disjoint sequence in $\widehat{K}$ converges to zero. Furthermore, if I is an ideal in X and K is a subset of I so that $P_{x_i}(k) \to 0$ uniformly for $k \in K$ whenever (x_i) is a disjoint sequence from I_+, then K is uniformly exhaustive in X.*

Proof. First suppose K is uniformly exhaustive and (x_i) is a disjoint sequence from $\widehat{K}$ $(|x_i| \wedge |x_j| = 0$ if $i \neq j)$. For each i choose $y_i \in K$ so that $|x_i| \leq |y_i|$. Since $P_u(u) = u$ for all $u \in X$ and P_{x_i} is monotone on X_+, it follows that

$$0 \leq |x_i| = P_{|x_i|}(|x_i|) \leq P_{|x_i|}(|y_i|) = |P_{|x_i|}(y_i)|.$$

Since (x_i) is a disjoint sequence, property (f) ensures $(P_{|x_i|})$ is a disjoint sequence from O. Suppose V is a solid neighborhood of the origin. Choose a natural number N so that for every $i \geq N$, $P_{|x_i|}(y) \in V$ for all $y \in K$. Then clearly $P_{|x_i|}(y_i) \in V$ for all $i \geq N$. Since V is solid we infer that x_i is in V for all $i \geq N$. Consequently, $x_n \to 0$.

Conversely, suppose that each disjoint sequence in $\widehat{K}$ converges to zero. Suppose P_{x_i} is a disjoint sequence from O, and $(|u_i|)$ is a sequence from $\widehat{K}$. Then (x_i) is a disjoint sequence from X_+ and $P_{x_i}(|u_i|)$ is a disjoint sequence. Since $P_{x_i}(|u_i|) \leq |u_i|$ and $|u_i|$ is in $\widehat{K}$, we have $P_{x_i}(|u_i|) \in \widehat{K}$. Thus $P_{x_i}(|u_i|) \to 0$. Again using property (g) and the fact that X has a locally solid topology, we find that $P_{x_i}(u_i) \to 0$. Since the choice of the sequence $|u_i|$ from $\widehat{K}$ was arbitrary, it follows that K is uniformly exhaustive.

Next suppose I is an ideal in X, K is a subset of I satisfying the hypotheses of the final statement of the theorem, and (ψ_i) is a disjoint sequence in X_+. Let (u_i) be an arbitrary sequence in K. Then $\psi_i \wedge |u_i| \in I$ for all i and $(\psi_i \wedge |u_i|)$ is a disjoint sequence. By the hypothesis $P_{\psi_i \wedge |u_i|}(|u_i|) \to 0$. Also

$$P_{\psi_i}(|u_i|) = \bigvee n\psi_i \wedge |u_i| = \bigvee [n(\psi_i \wedge |u_i|) \wedge |u_i|] = P_{\psi_i \wedge |u_i|}(|u_i|).$$

Since X has a locally solid topology, $P_{\psi_i}(|u_i|) \to 0$. ∎

The next main result characterizes exhaustivity in a TRS with the PPP. The final statement of Theorem 3.3 is similar to the results in [12] for vector measures. However, the initial statement of Theorem 3.3 sheds light on the structure of the order interval $[0, k]$ when k is exhaustive. This theorem is closely akin to some of the major theorems on Banach lattices in Section 5 of Chapter 2 of [19]. If a space X is not pre-Lebesgue, the exhaustive elements in X are often of interest. Note that the set of exhaustive elements of $X = l_\infty$ is c_0. In order to prove Theorem 3.3, we use the following lemma, which can be found in [16]. It is a generalization of the Meyer-Nieberg lemma found on page 92 of [19].

LEMMA 3.2. *Let (X, τ) be a locally solid topological Riesz space. Suppose that ν_n is a sequence in X_+ with $\nu_n \not\to 0$ and $\{\sum_{i=1}^{n} \nu_i \mid n \in \mathbb{N}\}$ bounded. Suppose further that one of the following conditions is satisfied:*

(i) *(ν_n) is majorized by some $x \in X_+$.*
(ii) *X is Dedekind σ-complete with an order continuous topology on $[0, \nu_n]$ for each n.*

Then there exists a sequence $(k(n))$ of natural numbers and a disjoint sequence (x_n) in X_+ so that $x_n \not\to 0$ and $x_n \leq \nu_{k(n)}$ for every n.

THEOREM 3.3. *Suppose X is a TRS with the PPP. A positive element k is exhaustive iff (u_i) is Cauchy whenever $0 \leq u_i\uparrow \leq k$. Consequently, X is pre-Lebesgue iff every positive element of X is exhaustive.*

Proof. Suppose $0 \leq u_i\uparrow \leq k$ implies (u_n) is Cauchy, and suppose k is not exhaustive. Then there is a disjoint sequence (x_n) from X_+ and a solid neighborhood V of the origin so that $P_{x_i}(k) \notin V$ for all $i \in \mathbb{N}$. Let

$$u_n = \sum_{i=1}^{n} P_{x_i}(k) = P_{\sum_{i=1}^{n} x_i}(k) \leq k.$$

Then $0 \leq u_i\uparrow \leq k$. By our hypothesis, (u_n) is a Cauchy sequence. Choose an N so that if $m, n \geq N$, then $u_n - u_m \in V$. If $n \geq m$, then $u_n - u_m = \sum_{i=m+1}^{n} P_{x_i}(k) \in V$. Since V is solid, $P_{x_n}(k) \in V$, which contradicts the assumption.

Conversely, suppose k is exhaustive and suppose there is a sequence (u_n) so that $0 \leq u_i\uparrow \leq k$ and (u_n) is not Cauchy. Choose a solid neighborhood V of the origin and intertwining sequences (n_i) and (m_i) so that $y_i = u_{n_i} - u_{m_i} \notin V$ for each i. Then for all n, $\sum_{i=1}^{n} y_i \leq k$. Now apply Lemma 3.2 to find a disjoint sequence (x_n) in X_+ and $(k(n))$ so that $x_n \notin V$ for all n and $x_n \leq y_{k(n)} \leq k$. Consequently, $P_{x_n}(k) \geq P_{x_n}(x_n) = x_n \notin V$ and k is not exhaustive. ■

We can now use the above theorem to establish a generalization of Theorem 1.3. It is known that X is Lebesgue iff each element of X is exhaustive whenever X is a sequentially complete and Dedekind σ-complete TRS. The following result investigates the structure of the order interval $[0, k]$ if k is exhaustive.

COROLLARY 3.4. *Suppose X is a sequentially complete and Dedekind σ-complete TRS. A positive element k is exhaustive iff $y_i \to 0$ whenever $y_i \downarrow 0$ in the order interval $[0, k]$.*

Proof. Suppose k is exhaustive and (y_n) is a sequence from $[0, k]$ so that $y_n \downarrow 0$. Then $k - y_n$ is an increasing sequence in $[0, k]$. By Theorem 3.3, $k - y_n$ is Cauchy. Since X is sequentially complete, $k - y_n$ converges. Therefore (y_n) converges. Since $y_n \downarrow 0$, we have $y_n \to 0$.

Now suppose for every sequence (y_n) in $[0, k]$ with $y_n \downarrow 0$ we have $y_n \to 0$. Suppose that k is not exhaustive. Then there is a disjoint sequence (P_{x_i}) from O and a solid neighborhood V of the origin so that $P_{x_i}(k) \notin V$ for each i. Let $l_n = P_{\bigvee_{i=1}^{n} x_i}(k)$ and $l = \bigvee_n l_n$. Then $l - l_n \downarrow 0$. By our assumption $l - l_n \to 0$. Then l_n is a Cauchy sequence. However, $l_{n+1} - l_n = P_{x_{n+1}}(k) \notin V$. Therefore we have a contradiction and k is exhaustive. ■

When studying uniform absolute continuity and uniform exhaustivity, it is natural to consider the following two classical results from measure theory.

Theorem 3.5 (Vitali–Hahn–Saks). *Let (S, Σ, μ) be a measure space and (λ_n) a sequence of μ-continuous vector or scalar valued additive set functions on Σ. If the limit $\lim \lambda_n(E)$ exists for each E in Σ then λ_n is uniformly absolutely continuous with respect to μ.*

Theorem 3.6 (Brooks–Jewett, [6]). *Let (λ_n) be a sequence of strongly additive vector or scalar valued set functions defined on Σ. If the limit $\lim \lambda_n(E)$ exists for each E in Σ, then the sequence (λ_n) is uniformly exhaustive.*

Recall that in $\mathrm{ca}(\Sigma)$ weak convergence and setwise convergence are equivalent. Consider the unit vector basis (e_i) in c_0. Each e_i is exhaustive and (e_i) is weakly convergent, but $P_{e_i}(e_i) = e_i$, which implies (e_i) is not uniformly exhaustive. Therefore, the Brooks–Jewett theorem does not hold in arbitrary Banach lattices if we replace setwise convergence with weak convergence. If X is a Schur space then weak and norm convergence will coincide. Therefore in a Schur space, the weak convergence of the sequence (x_i) is sufficient to guarantee that the sequence is uniformly exhaustive. Again consider (e_i) in c_0 and let $x = (1/n)_n$ be in c_0. Each e_i is continuous with respect to x. However, $P_{e_i}(x) \to 0$ and $P_{e_i}(e_i) = e_i \not\to 0$. Therefore the continuity is not uniform. Thus the Vitali–Hahn–Saks theorem does not hold in arbitrary Banach lattices if we replace setwise convergence with weak convergence.

Now note that by combining Lemma II.5.4 of [19], and Theorem 3.1 and Corollary 3.4 above, we find that a positive element x of the Dedekind σ-complete Banach lattice X is exhaustive whenever $[0, x]$ is separable. This is due to the fact that separability of the order interval $[0, x]$ yields the separability of the principal ideal I_x generated by x. Therefore I_x is Lebesgue and each element of I_x is exhaustive. The question was raised in [3] of whether or not it is also true that $[0, x]$ is separable whenever x is exhaustive. The following counterexample resolves this question.

Let $\mathbb{R}$ represent the real numbers. Also let λ represent Lebesgue measure on the interval $[0, 1]$. Using the results of Kakutani in [15], we can define a countably additive measure m on the product space $([0, 1], \lambda)^{\mathbb{R}}$. Let Σ denote the measurable sets in $([0, 1], \lambda)^{\mathbb{R}}$. Then m is an exhaustive element of $\mathrm{ca}(\Sigma)$. However, $[0, m]$ is not separable in $\mathrm{ca}(\Sigma)$. To see this let $m_\alpha = m|_{E_\alpha}$ where E_α is the element of Σ whose αth projection is $[0, 1/2]$ and whose βth projection is $[0, 1]$ if $\alpha \neq \beta$. Therefore we have an uncountable number of elements of $[0, m]$ with $\|m_\gamma - m_\beta\| \geq 1/4$ whenever $\gamma \neq \beta$. Consequently, $[0, m]$ is not separable.

Finally, Theorem 2.4(vi) in [3] is false as stated. It states that if X is a Dedekind σ-complete Banach lattice and K is a subset of the exhaustive elements of X, then K is continuous with respect to some exhaustive element in X iff each pairwise disjoint subset of K is countable. For a counterexample again let $\mathbb{R}$ be the set of real numbers and let $X = l_\infty(\mathbb{R})$. Let $K = \{e_0 + e_\alpha : \alpha \in \mathbb{R}\}$. Then each element of K is exhaustive. Vacuously, each disjoint subset of K is countable. However, K is not continuous with respect to any exhaustive element of $l_\infty(\mathbb{R})$. The theorem should have read as follows: If X is a Dedekind σ-complete Banach lattice and K is a subset of the exhaustive elements of X, then K is continuous with respect to some exhaustive element in X iff each pairwise disjoint subset of $\widehat{K}$ is countable. The proof runs as before.

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