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PROBABILITY THEORY AND STOCHASTIC PROCESSES

A Note on the Men'shov–Rademacher Inequality

by

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Summary. We improve the constants in the Men'shov–Rademacher inequality by showing that for $n \geq 64$,

$$\mathbf{E}\Big(\sup_{1\leq k\leq n}\Big|\sum_{i=1}^{k} X_i\Big|^2\Big) \leq 0.11(6.20+\log_2 n)^2$$

for all orthogonal random variables $X_1,\dots,X_n$ such that $\sum_{k=1}^{n}\mathbf{E}|X_k|^2 = 1$.

1. Introduction. We consider real or complex orthogonal random variables $X_1,\dots,X_n$, i.e.

$$\mathbf{E}|X_i|^2 < \infty, \quad 1\leq i\leq n, \quad \text{and} \quad \mathbf{E}(X_i X_j) = 0, \quad 1\leq i,j\leq n.$$

Set $S_j := X_1+\dots+X_j$ for $1\leq j\leq n$, and $S_0 = 0$. Clearly

$$\mathbf{E}|S_j - S_i|^2 = \sum_{k=i+1}^{j}\mathbf{E}|X_k|^2 \quad \text{for } i\leq j.$$

The best constant in the Men'shov–Rademacher inequality is defined by

$$D_n := \sup \mathbf{E}\sup_{1\leq i\leq n}|S_i|^2,$$

where the supremum is taken over all orthogonal systems $X_1,\dots,X_n$ which satisfy $\sum_{k=1}^{n}\mathbf{E}|X_k|^2 = 1$. We also define

$$C := \limsup_{n\to\infty}\frac{D_n}{\log_2^2 n}.$$

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Rademacher [6] in 1922 and independently Men'shov [5] in 1923 proved that there exists $K > 0$ such that for $n \geq 2$,

$$D_n \leq K \log_2^2 n, \quad \text{hence} \quad C \leq K.$$

By now there are several different proofs of the above inequality. The traditional proof of the Men'shov–Rademacher inequality uses the bisection method (see Doob [1] and Loève [4]), which leads to

$$D_n \leq (2 + \log_2 n)^2, \quad n \geq 2, \quad \text{hence} \quad C \leq 1.$$

In 1970 Kounias [3] used a trisection method to get a finer inequality

$$D_n \leq \left(\frac{\log_2 n}{\log_2 3} + 2\right)^2, \quad n \geq 2, \quad \text{hence} \quad C \leq \left(\frac{\log_2 2}{\log_2 3}\right)^2.$$

S. Chobanyan, S. Levental and H. Salehi [2] proved the following result:

$$D_{2n} \leq \frac{4}{3} D_n \quad \text{if } D_n \leq 3; \qquad D_{2n} \leq \left(\left(D_n - \frac{3}{4}\right)^{1/2} + \frac{1}{2}\right)^2, \tag{1}$$

and as a consequence they got the estimate $D_n \leq \frac{1}{4}(3 + \log_2^2 n)$, $C \leq \frac{1}{4}$. An example given in [2] shows that $D \geq \frac{\log_2^2 n}{\pi^2 \log_2^2 e}$ and thus $C \geq 0.0487$. The aim of this paper is to improve the bisection method and together with (1) show that $C \leq 0.1107 < \frac{1}{9}$.

2. Results

THEOREM 1. *For each $n, m \in \mathbb{N}$ and $l > 2$,*

$$\sqrt{D_{n(2m+l)}} \leq \sqrt{D_n} + \sqrt{\max\{D_m, 2D_{l-1}\}}.$$

If $l = 2$ then an even stronger inequality holds:

$$\sqrt{D_{n(2m+l)}} \leq \sqrt{D_n} + \sqrt{D_m}.$$

Proof. Set $p := 2m + l$. We can assume that $\mathbf{E}|S_{pn}|^2 = 1$. The triangle inequality yields

$$|S_i| \leq |S_i - S_{pj}| + |S_{pj}|.$$

Consequently,

$$\max_{1 \leq i \leq pn} |S_i| \leq \max_{1 \leq i \leq pn} \min_{0 \leq j \leq n} |S_i - S_{pj}| + \max_{0 \leq j \leq n} |S_{pj}|.$$

Thus

$$\mathbf{E} \max_{1 \leq i \leq pn} |S_i|^2 \leq \mathbf{E}\Big(\max_{1 \leq i \leq pn} \min_{0 \leq j \leq n} |S_i - S_{pj}| + \max_{0 \leq j \leq n} |S_{pj}|\Big)^2.$$

The definition of D_n together with the classical norm inequality implies

$$\sqrt{D_{pn}} \leq \sqrt{D_n} + \sqrt{\mathbf{E} \max_{1 \leq i \leq pn} \min_{0 \leq j \leq n} |S_i - S_{pj}|^2}.$$

It remains to show that

$$\mathbf{E} \max_{1\le i\le pn} \min_{0\le j\le n} |S_i - S_{pj}|^2 \le \begin{cases} \max\{D_m, 2D_{l-1}\} & \text{if } l > 2, \\ D_m & \text{if } l = 2. \end{cases}$$

Define

$$\begin{aligned} A_j &:= \max\{|S_i - S_{pj}| : pj \le i \le pj + m\}, \\ B_j &:= \max\{|S_{p(j+1)} - S_i| : pj + m + l \le i \le p(j+1)\}, \\ C_j &:= \max\{|S_i - S_{pj+m}| : pj + m < i < pj + m + l\}, \\ D_j &:= \max\{|S_{pj+m+l} - S_i| : pj + m < i < pj + m + l\}, \end{aligned}$$

for each $j \in \{0, 1, \ldots, n-1\}$. Each $0 \le i \le pn$ can be written in the form $i = pj + r$, where $j \in \{0, \ldots, n-1\}$, $r \in \{1, \ldots, p\}$. If $r \le m$, then

$$|S_i - S_{pj}|^2 \le A_j^2.$$

If $r \ge m + l$, then

$$|S_{p(j+1)} - S_i|^2 \le B_j^2.$$

The last case is when $i = pj + m + r$, $r \in \{1, \ldots, l-1\}$. Set

$$\begin{aligned} P_j &:= S_{pj+m} - S_{pj}, & V_j &:= S_{pj+m+r} - S_{pj+m}, \\ Q_j &:= S_{p(j+1)} - S_{pj+m+l}, & W_j &:= S_{pj+m+l} - S_{pj+m+r}. \end{aligned}$$

Clearly ($i = pj + m + r$, $r \in \{1, \ldots, l-1\}$)

$$\min\{|S_i - S_{pj}|^2, |S_{p(j+1)} - S_i|^2\} = \min\{|P_j + V_j|^2, |Q_j + W_j|^2\}.$$

For all complex numbers a, b, c, d we have

$$\frac{1}{2}\,|a+b|^2 \le |a|^2 + |b|^2, \qquad \frac{1}{2}\,|c+d|^2 \le |c|^2 + |d|^2.$$

Since

$$\min\{|a+b|^2, |c+d|^2\} \le \frac{1}{2}\,|a+b|^2 + \frac{1}{2}\,|c+d|^2$$

we obtain

$$\min\{|a+b|^2, |c+d|^2\} \le |a|^2 + |b|^2 + |c|^2 + |d|^2.$$

Hence

$$\min\{|S_i - S_{pj}|^2, |S_{p(j+1)} - S_i|^2\} \le |P_j|^2 + |Q_j|^2 + |V_j|^2 + |W_j|^2,$$

and consequently for each $pj < i \le p(j+1)$, $j \in \{0, 1, \ldots, n-1\}$,

$$\min\{|S_i - S_{pj}|^2, |S_{p(j+1)} - S_i|^2\} \le A_j^2 + B_j^2 + C_j^2 + D_j^2.$$

In fact we have proved that

$$\mathbf{E} \max_{1\le i\le pn} \min_{0\le j\le n} |S_i - S_{pj}|^2 \le \mathbf{E} \sum_{j=0}^{n-1} (A_j^2 + B_j^2 + C_j^2 + D_j^2).$$

Observe that

$$\mathbf{E}A_j^2 \le D_m \sum_{k=1}^{m} \mathbf{E}|X_{pj+k}|^2, \qquad \mathbf{E}B_j^2 \le D_m \sum_{k=1}^{m} \mathbf{E}|X_{pj+m+l+k}|^2,$$

$$\mathbf{E}(C_j^2 + D_j^2) \le D_{l-1}\Big(\mathbf{E}|X_{pj+m+1}|^2 + \mathbf{E}|X_{pj+m+l}|^2 + 2\sum_{k=2}^{l-1} \mathbf{E}|X_{pj+m+k}|^2\Big),$$

Notice that if $l = 2$ then

$$\mathbf{E}(C_j^2 + D_j^2) \le D_1(\mathbf{E}|X_{pj+m+1}|^2 + \mathbf{E}|X_{pj+m+1}|^2).$$

Hence, if $l > 2$ then

$$\mathbf{E} \max_{1 \le i \le pn} \min_{0 \le j \le n} |S_i - S_{pj}|^2 \le \max\{D_m, 2D_{l-1}\}$$

and if $l = 2$ then

$$\mathbf{E} \max_{1 \le i \le pn} \min_{0 \le j \le n} |S_i - S_{pj}|^2 \le D_m.$$

This ends the proof. ∎

Corollary 1. *For each $n \ge m$,*

$$D_n \le D_m \bigg(2 + \frac{\log_2 n - \log_2 m}{\log_2(2m+2)}\bigg)^2.$$

Proof. Taking $l = 2$ in Theorem 1 we obtain

$$D_{m(2m+l)^k} \le (k+1)^2 D_m.$$

For each $n \ge m$ there exists $k \ge 0$ such that $m(2m+l)^k \le n < m(2m+l)^{k+1}$. Hence

$$k \le 1 + \frac{\log_2 n - \log_2 m}{\log_2(2m+2)}.$$

Consequently,

$$D_n \le D_m \bigg(2 + \frac{\log_2 n - \log_2 m}{\log_2(2m+2)}\bigg)^2. \ ∎$$

This result implies

$$C = \limsup_{n \to \infty} \frac{D_n}{\log_2^2 n} \le \frac{D_m}{\log_2^2(2m+2)}.$$

Putting $l > 2$ in Theorem 1 and proceeding as in the proof of Corollary 1 we get the following result.

Corollary 2. *For each $l > 2$ and $n \ge m$,*

$$D_n \le \max\{D_m, 2D_{l-1}\} \bigg(2 + \frac{\log_2 n - \log_2 m}{\log_2(2m+l)}\bigg)^2.$$

Consequently,

$$C \le \frac{\max\{D_m, 2D_{l-1}\}}{\log_2^2(2m+l)}.$$

As mentioned in the introduction, $D_2 = 4/3$ (by the result of Chobanyan, Levental, and Salehi [2]). Hence applying Corollary 1 with $m = 2$ we get

$$D_n \leq \frac{4}{3\log_2^2 6}\,(2\log_2 6 - \log_2 n)^2, \quad \text{and} \quad C \leq \frac{4}{3\log_2^2 6} < \frac{1}{5}.$$

It follows by (1) that

$$D_2 = \frac{4}{3}, \quad D_4 \leq \left(\frac{4}{3}\right)^2, \quad D_8 \leq \left(\frac{4}{3}\right)^3, \quad D_{16} \leq \left(\frac{4}{3}\right)^4$$

and

$$D_{32} \leq \left(\left(D_{16} - \frac{3}{4}\right)^{1/2} + \frac{1}{2}\right)^2, \quad D_{64} \leq \left(\left(D_{32} - \frac{3}{4}\right)^{1/2} + \frac{1}{2}\right)^2.$$

Hence

$$D_8 \leq 2.3704, \quad D_{64} \leq 5.5741.$$

Taking $m = 64$, $l = 9$ we get

$$\frac{\max\{D_m, 2D_{l-1}\}}{\log_2^2(2m+l)} \leq 0.1107 < 1/9.$$

Thus applying Corollary 2 (with $m = 64$, $l = 9$) we find that for each $n \geq 64$,

$$D_n \leq 0.1107(2\log_2(137) - 8 + \log_2 n)^2 \leq 0.1107(6.1960 + \log_2 n)^2.$$

This gives the estimate $C \leq 0.1107 < \frac{1}{9}$.

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