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FUNCTIONAL ANALYSIS

Equicontinuity and Convergent Sequences in the Spaces $\mathcal{O}'_C$ and $\mathcal{O}_M$

by

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Summary. Characterizations of equicontinuity and convergent sequences are given for the space $\mathcal{O}'_C(\mathbb{R}^n)$ of rapidly decreasing distributions and the space $\mathcal{O}_M(\mathbb{R}^n)$ of slowly increasing infinitely differentiable functions.

Introduction. Let $\mathcal{S}(\mathbb{R}^n)$ be the L. Schwartz space of infinitely differentiable rapidly decreasing functions on $\mathbb{R}^n$, and $\mathcal{S}'(\mathbb{R}^n)$ the space of slowly increasing distributions. Let $\mathcal{O}'_C(\mathbb{R}^n)$ be the space of rapidly decreasing distributions on $\mathbb{R}^n$, and $\mathcal{O}_M(\mathbb{R}^n)$ the space of infinitely differentiable slowly increasing functions. The space $\mathcal{O}'_C(\mathbb{R}^n)$ was introduced by L. Schwartz, and both the spaces, $\mathcal{O}_M(\mathbb{R}^n)$ and $\mathcal{O}'_C(\mathbb{R}^n)$, are discussed in [S2]. In particular,

$$\mathcal{O}'_C(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n), \quad \mathcal{O}_M(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n), \quad \text{and} \quad \mathcal{F}\mathcal{O}'_C(\mathbb{R}^n) = \mathcal{O}_M(\mathbb{R}^n)$$

where $\mathcal{F}$ is the Fourier transformation. Furthermore,

$$\mathcal{O}'_C(\mathbb{R}^n) = \{T \in \mathcal{S}'(\mathbb{R}^n) : \text{the convolution operator } T* \text{ is an endomorphism of } \mathcal{S}(\mathbb{R}^n)\},$$
$$\mathcal{O}_M(\mathbb{R}^n) = \{\phi \in C^\infty(\mathbb{R}^n) : \text{the multiplication operator } \phi\cdot \text{ is an endomorphism of } \mathcal{S}(\mathbb{R}^n)\},$$

so that $\mathcal{O}'_C(\mathbb{R}^n)$ and $\mathcal{O}_M(\mathbb{R}^n)$ may be seen as subsets of $L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$ (1). In 1938, when the theory of distributions not yet existed, I. G. Petrovskiĭ [P] noticed the significance of slowly increasing functions for the theory of

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(1) By duality $\mathcal{O}'_C(\mathbb{R}^n)$ and $\mathcal{O}_M(\mathbb{R}^n)$ are also subsets of $L(\mathcal{S}'(\mathbb{R}^n); \mathcal{S}'(\mathbb{R}^n))$.

evolutionary PDE with constant coefficients. In 1950 L. Schwartz [S1] explained how the results of Petrovskiĭ may be elucidated by placing them in the framework of $\mathcal{O}'_C$ and $\mathcal{O}_M$. In [K1–K2], for evolutionary PDO with constant coefficients, the relation is investigated between

- the properties of fundamental solution expressed in terms of $\mathcal{O}'_C(\mathbb{R}^n)$ and
- the Petrovskiĭ condition concerning zeros of the characteristic polynomial.

For the sake of further similar investigations it is convenient to collect the information about $\mathcal{O}_M(\mathbb{R}^n)$ and $\mathcal{O}'_C(\mathbb{R}^n)$ (2) in a compact form but with complete proofs, and this is the main motivation for the present paper.

1. The space $\mathcal{B}'(\mathbb{R}^n)$ of bounded distributions. Let $\mathcal{D}_{L^1}(\mathbb{R}^n)$ be the space of infinitely differentiable complex functions φ on $\mathbb{R}^n$ such that $\partial^\alpha\varphi = \partial_1^{\alpha_1}\ldots\partial_n^{\alpha_n}\varphi \in L^1(\mathbb{R}^n)$ for every multiindex $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}_0^n$. The topology in $\mathcal{D}_{L^1}(\mathbb{R}^n)$ is determined by the system of seminorms $p_\alpha(\varphi) = \int_{\mathbb{R}^n} |\partial^\alpha\varphi(x)|\,dx$, $\alpha \in \mathbb{N}_0^n$, $\varphi \in \mathcal{D}_{L^1}(\mathbb{R}^n)$. $\mathcal{D}_{L^1}(\mathbb{R}^n)$ is a Fréchet space, and $\mathcal{D}(\mathbb{R}^n)$ is densely and continuously imbedded in $\mathcal{D}_{L^1}(\mathbb{R}^n)$. A distribution T is said to be *bounded* on $\mathbb{R}^n$ if it extends to a linear functional continuous on $\mathcal{D}_{L^1}(\mathbb{R}^n)$. The space of bounded distributions on $\mathbb{R}^n$ is denoted by $\mathcal{B}'(\mathbb{R}^n)$.

Let $C_b(\mathbb{R}^n)$ be the Banach space of continuous bounded complex functions on $\mathbb{R}^n$.

THEOREM 1.1. *For any family Φ of distributions on $\mathbb{R}^n$ the following three conditions are equivalent:*

(1.1) *$\Phi \subset \mathcal{B}'(\mathbb{R}^n)$ and the distributions belonging to Φ are equicontinuous with respect to the topology of $\mathcal{D}_{L^1}(\mathbb{R}^n)$,*

(1.2) *for any fixed $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\varphi * \Phi$ is a bounded subset of $C_b(\mathbb{R}^n)$,*

(1.3) *there is a finite family $\{u_\alpha : \alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}_0^n,\ |\alpha| = \alpha_1 + \cdots + \alpha_n \le m\}$ of continuous complex functions on $\mathbb{R}^n$ with support contained in $B_1 = \{x \in \mathbb{R}^n : |x| \le 1\}$ such that*

  (i) *$T = \sum_{|\alpha|\le m} \partial^\alpha(u_\alpha * T)$ for every $T \in \mathcal{D}'(\mathbb{R}^n)$,*

  (ii) *whenever $\alpha \in \mathbb{N}_0^n$ and $|\alpha| \le m$, then $u_\alpha * \Phi$ is a bounded subset of $C_b(\mathbb{R}^n)$.*

The above theorem may be treated as a variant of Theorems XXII and XXV from Sections VI.7 and VI.8 of [S2].

(2) In [S2] the information about $\mathcal{O}_M(\mathbb{R}^n)$ and $\mathcal{O}'_C(\mathbb{R}^n)$ is in part contained in statements for which the method of proof is only indicated.

Proof. In order to prove (1.3)⇒(1.1) it is sufficient to observe that, by (1.3), for every $T \in \Phi$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$ one has

$$T(\varphi) = \sum_{|\alpha| \le m} (-1)^{|\alpha|} \langle u_\alpha * T, \partial^\alpha \varphi \rangle$$

where $u_\alpha * \Phi$, $|\alpha| \le m$, are bounded subsets of $C_b(\mathbb{R}^n)$. In order to prove (1.1)⇒(1.2) it is sufficient to note that

$$(T * \varphi)(x) = \langle T, (\varphi_x)^\vee \rangle$$

where the subscript x denotes translation by x, and the superscript $\vee$ the reflection.

The implication (1.2)⇒(1.3) is proved by a more refined argument, a part of which is based on the method of category, similarly to an argument in [S2, Sec. VI.7, p. 196]. Suppose that (1.2) holds. Since $(T * \varphi)(x) = \langle (T_x)^\vee, \varphi \rangle$, (1.2) implies that $\{(T_x)^\vee : T \in \Phi,\ x \in \mathbb{R}^n\}$ is a pointwise bounded family of continuous linear functionals on $\mathcal{D}(\mathbb{R}^n)$. Since $\mathcal{D}(\mathbb{R}^n)$ is a barrelled space, the Banach–Steinhaus theorem implies that this family is equicontinuous. Equicontinuity of $\{(T_x)^\vee : T \in \Phi,\ x \in \mathbb{R}^n\}$ implies that there are $k \in \mathbb{N}_0$ and $C \in \,]0,\infty[$ such that whenever $B_1 = \{y \in \mathbb{R}^n : |y| \le 1\}$, $\varphi \in C^\infty_{B_1}(\mathbb{R}^n)$, $T \in \Phi$ and $x \in \mathbb{R}^n$, then

$$(1.4) \qquad |(T * \varphi)(x)| = |\langle (T_x)^\vee, \varphi \rangle| \le C \|\varphi\|_{C^k_{B_1}(\mathbb{R}^n)}.$$

This estimate implies that

(1.5) whenever $\phi \in C^k_{B_1}(\mathbb{R}^n)$, then $\{\phi * T : T \in \Phi\}$ is a bounded subset of $C_b(\mathbb{R}^n)$.

Indeed, the convolution $\phi * T$ of the compactly supported distribution $\phi \in C^k_{B_1}(\mathbb{R}^n)$ with any distribution $T \in \Phi$ makes sense. If $(\phi_\nu)_{\nu=1}^\infty \subset C^\infty_{B_1}(\mathbb{R}^n)$ is a sequence such that $\lim_{\nu\to\infty} \|\phi_\nu - \phi\|_{C^k_{B_1}(\mathbb{R}^n)} = 0$, then, by (1.4), $(\phi_\nu * T)_{\nu=1}^\infty$ is a Cauchy sequence in $C_b(\mathbb{R}^n)$, and its limit coincides with the distribution $\phi * T$. Furthermore, again by (1.4), $\{\phi * T : T \in \Phi\}$ is a bounded subset of $C_b(\mathbb{R}^n)$.

Having proved (1.5), it remains to repeat the argument used by Schwartz in [S2], based on fundamental solutions for powers of the laplacian. Fix $k \in \mathbb{N}_0$ for which (1.5) holds. If $l \in \mathbb{N}$ is sufficiently large and E is the fundamental solution for Δ^l on $\mathbb{R}^n$ depending only on $|x|$, then $E \in C^k(\mathbb{R}^n)$ and $E|_{\mathbb{R}^n \setminus \{0\}} \in C^\infty(\mathbb{R}^n \setminus \{0\})$ (see [S2, Sec. VII.10, Example 2, p. 288]). Let $\gamma \in C^\infty_{B_1}(\mathbb{R}^n)$ be such that $\gamma(x) = 1$ whenever $|x| \le 1/2$. Then $\gamma E \in C^k_{B_1}(\mathbb{R}^n)$, $(1-\gamma)E \in C^\infty(\mathbb{R}^n)$, and $\Delta^l((1-\gamma)E) \in C^\infty_{B_1}(\mathbb{R}^n)$. For every $T \in \mathcal{D}'(\mathbb{R}^n)$ one has

$$T = \delta * T = (\Delta^l E) * T = [\Delta^l(\gamma E + (1-\gamma)E)] * T$$
$$= \Delta^l[(\gamma E) * T] + [\Delta^l((1-\gamma)E)] * T.$$

By (1.5), this equality, together with the fact that $\gamma E \in C^k_{B_1}(\mathbb{R}^n)$ and $\Delta^l((1-\gamma)E) \in C^\infty_{B_1}(\mathbb{R}^n)$, implies (1.3). ■

THEOREM 1.2. *For every sequence $(T_\nu)_{\nu=1}^\infty \subset \mathcal{B}'(\mathbb{R}^n)$ the following two conditions are equivalent:*

(1.6) *the sequence $(T_\nu)_{\nu=1}^\infty$ converges to zero uniformly on every bounded subset of $\mathcal{D}_{L^1}(\mathbb{R}^n)$,*
(1.7) $\lim_{\nu\to\infty} \|T_\nu * \varphi\|_{C_b(\mathbb{R}^n)} = 0$ *for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$.*

Proof of (1.6)⇒(1.7). Suppose that (1.6) holds. Fix $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Then

$$\|T_\nu * \varphi\|_{C_b(\mathbb{R}^n)} = \sup\{|\langle T_\nu * \varphi, \psi\rangle| : \psi \in L^1(\mathbb{R}^n),\ \|\psi\|_{L^1(\mathbb{R}^n)} = 1\}$$
$$= \sup\{|\langle T_\nu, \varphi^\vee * \psi\rangle| : \psi \in L^1(\mathbb{R}^n),\ \|\psi\|_{L^1(\mathbb{R}^n)} = 1\}.$$

Therefore it remains to observe that if $\varphi \in \mathcal{D}(\mathbb{R}^n)$ is fixed, then

$$\{\varphi^\vee * \psi : \psi \in L^1(\mathbb{R}^n),\ \|\psi\|_{L^1(\mathbb{R}^n)} = 1\}$$

is a bounded subset of $\mathcal{D}_{L^1}(\mathbb{R}^n)$. ■

Proof of (1.7)⇒(1.6). Suppose that (1.7) holds and put $\Phi = \{T_\nu : \nu \in \mathbb{N}\}$. Then (1.2) holds, and so, by Theorem 1.1, also the conditions (1.1) and (1.3) are satisfied. Consequently, there is $m \in \mathbb{N}_0$ and for every multiindex $\alpha \in \mathbb{N}_0^n$ of length $|\alpha| \le m$ there is a function $u_\alpha \in C_{B_1}(\mathbb{R}^n)$ such that:

$$T_\nu * u_\alpha \in C_b(\mathbb{R}^n) \quad \text{for every } \nu \in \mathbb{N}, \tag{1.8}$$

$$\sup\{\|T_\nu * u_\alpha\|_{C_b(\mathbb{R}^n)} : \nu \in \mathbb{N},\ |\alpha| \le m\} = M < \infty, \tag{1.9}$$

$$T_\nu = \sum_{|\alpha|\le m} \partial^\alpha(T_\nu * u_\alpha) \quad \text{for every } \nu \in \mathbb{N}. \tag{1.10}$$

Fix a bounded subset $\mathcal{B}$ of $\mathcal{D}_{L^1}(\mathbb{R}^n)$. Then

$$\sum_{|\alpha|\le l} \sup_{\phi\in\mathcal{B}} \|\partial^\alpha \phi\|_{L^1(\mathbb{R}^n)} = N_l < \infty \quad \text{for every } l \in \mathbb{N}_0. \tag{1.11}$$

By (1.10) for every $\nu \in \mathbb{N}$, $\phi \in \mathcal{B}$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$ one has

$$\langle T_\nu, \phi\rangle = \sum_{|\alpha|\le m} (-1)^{|\alpha|} \langle T_\nu * u_\alpha, \partial^\alpha\phi - \varphi^\vee * \partial^\alpha\phi\rangle + \sum_{|\alpha|\le m} (-1)^{|\alpha|} \langle T_\nu * (u_\alpha * \varphi), \partial^\alpha \phi\rangle,$$

whence, by (1.8), (1.9) and (1.11),

$$\sup_{\phi\in\mathcal{B}}|\langle T_\nu,\phi\rangle| \le M\sum_{|\alpha|\le m}\sup_{\phi\in\mathcal{B}}\|\partial^\alpha\phi-\varphi^\vee*\partial^\alpha\phi\|_{L^1(\mathbb{R}^n)} + N_m\sup_{|\alpha|\le m}\|T_\nu*(u_\alpha*\varphi)\|_{C_b(\mathbb{R}^n)}. \tag{1.12}$$

Take now an arbitrary $\varepsilon>0$ and choose a non-negative $\varphi\in\mathcal{D}(\mathbb{R}^n)$ such that $\operatorname{supp}\varphi\subset B_\varepsilon=\{y\in\mathbb{R}^n:|y|\le\varepsilon\}$ and $\|\varphi\|_{L^1(\mathbb{R}^n)}=1$. For every $\phi\in\mathcal{B}$ and $\alpha\in\mathbb{N}_0^n$ such that $|\alpha|\le m$ we then have

$$\begin{aligned}\|\partial^\alpha\phi-\varphi^\vee*\partial^\alpha\phi\|_{L^1(\mathbb{R}^n)} &= \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n}\Big|\int_0^1\frac{d}{dt}\partial^\alpha\phi(x+ty)\,dt\Big|\varphi(y)\,dx\,dy\\ &\le \int\limits_{\mathbb{R}^n}\Big[\int_0^1\Big[\sum_{k=1}^n|y_k|\int\limits_{\mathbb{R}^n}|\partial_k\partial^\alpha\phi(x+ty)|\,dx\Big]\,dt\Big]\varphi(y)\,dy\\ &= \int\limits_{B_\varepsilon}\Big[\sum_{k=1}^n|y_k|\,\|\partial_k\partial^\alpha\phi\|_{L^1(\mathbb{R}^n)}\Big]\varphi(y)\,dy\\ &\le n\varepsilon\sum_{k=1}^n\|\partial_k\partial^\alpha\phi\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

Consequently, by (1.12) and (1.11), for every $\nu\in\mathbb{N}$ one has

$$\sup_{\phi\in\mathcal{B}}|\langle T_\nu,\phi\rangle|\le MN_{m+1}n\varepsilon+N_m\sup_{|\alpha|\le m}\|T_\nu*(u_\alpha*\varphi)\|_{C_b(\mathbb{R}^n)}. \tag{1.13}$$

Since $u_\alpha*\varphi\in\mathcal{D}(\mathbb{R}^n)$, the last term in (1.13) tends to zero as $\nu\to\infty$, so that

$$\limsup_{\nu\to\infty}\sup_{\phi\in\mathcal{B}}|\langle T_\nu,\phi\rangle|\le MN_{m+1}n\varepsilon.$$

Hence $\lim_{\nu\to\infty}\sup_{\phi\in\mathcal{B}}|\langle T_\nu,\phi\rangle|=0$ because $\varepsilon>0$ is arbitrary. ∎

2. The space $\mathcal{O}'_C(\mathbb{R}^n)$ of rapidly decreasing distributions. Let $\mathcal{S}(\mathbb{R}^n)$ denote the space of rapidly decreasing C^∞ functions on $\mathbb{R}^n$, and $\mathcal{S}'(\mathbb{R}^n)$ the space of slowly increasing distributions on $\mathbb{R}^n$. By [S2, Sec. VII.5, Theorem IX, p. 244], for every distribution $T\in\mathcal{D}'(\mathbb{R}^n)$ the following two conditions are equivalent:

$$P\cdot T\in\mathcal{B}'(\mathbb{R}^n)\quad\text{for every polynomial } P \text{ on } \mathbb{R}^n, \tag{2.1}$$

$$T*\varphi\in\mathcal{S}(\mathbb{R}^n)\quad\text{for every } \varphi\in\mathcal{D}(\mathbb{R}^n). \tag{2.2}$$

By our Theorem 2.1 below, (2.2) is equivalent to the condition

$$T\in\mathcal{S}'(\mathbb{R}^n)\quad\text{and}\quad T*\in L(\mathcal{S}(\mathbb{R}^n);\mathcal{S}(\mathbb{R}^n)). \tag{2.3}$$

A distribution T satisfying the above equivalent conditions is called *rapidly decreasing*. The space of rapidly decreasing distributions on $\mathbb{R}^n$ is denoted by $\mathcal{O}'_C(\mathbb{R}^n)$. Notice that the implication (2.1)⇒(2.3) follows from a statement formulated without proof in [S2, Sec. VII.5, p. 248] (3).

THEOREM 2.1. *For every family Ψ of distributions on $\mathbb{R}^n$ the following three conditions are equivalent:*

(2.4) *for every polynomial P on $\mathbb{R}^n$ the family of distributions $\{P\cdot T : T \in \Psi\}$ is a subset of $\mathcal{B}'(\mathbb{R}^n)$ equicontinuous in the topology of $\mathcal{D}_{L^1}(\mathbb{R}^n)$,*

(2.5) *$\Psi \subset \mathcal{S}'(\mathbb{R}^n)$ and the set of convolution operators $\{T\, * : T \in \Psi\}$ is an equicontinuous subset of $L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$ (4),*

(2.6) *for every $k \in \mathbb{N}$ there is $m_k \in \mathbb{N}_0$ and a set of operators $\{F_{k,\beta} : \beta \in \mathbb{N}_0^n, |\beta| \le m_k\} \subset L(\mathcal{D}'(\mathbb{R}^n); \mathcal{D}'(\mathbb{R}^n))$ satisfying*

(i) $T = \sum_{|\beta|\le m_k} \partial^\beta F_{k,\beta}(T)$ *for every $T \in \mathcal{D}'(\mathbb{R}^n)$,*

(ii) *whenever $\beta \in \mathbb{N}_0^n$ and $|\beta| \le m_k$, then $F_{k,\beta}(\Psi)$ is a bounded subset of the Banach space $B_k(\mathbb{R}^n)$ of continuous complex functions f on $\mathbb{R}^n$ such that*

$$\|f\|_{B_k(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} (1+|x|^2)^k |f(x)| < \infty.$$

Proof of (2.6)⇒(2.5). Suppose that (2.6) is satisfied. Since $\partial^\alpha(T * \varphi) = T * \partial^\alpha \varphi$, (2.5) follows once it is proved that, whenever $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $k \in [n/2] + \mathbb{N}$ where $[n/2]$ is the integer part of $n/2$, then

$$\sup\{(1+\tfrac{1}{4}|x|^2)^k |(T*\varphi)(x)| : T \in \Psi,\ x \in \mathbb{R}^n\} \le 2C_k D_k p_{k,m_k}(\varphi) \tag{2.7}$$

where

$$C_k = \sup\{\|F_{k,\beta}(T)\|_{B_k(\mathbb{R}^n)} : T \in \Psi,\ |\beta| \le m_k\},$$
$$D_k = (\#\{\beta \in \mathbb{N}_0^n : |\beta| \le m_k\}) \int_{\mathbb{R}^n} (1+|y|^2)^{-k}\, dy,$$
$$p_{k,m}(\varphi) = \sup\{\|\partial^\beta \varphi\|_{B_k(\mathbb{R}^n)} : |\beta| \le m\}.$$

(3) The equivalence of (2.1)–(2.3) follows when our Theorem 2.1 is modified by extending the triple of equivalent conditions (2.4), (2.5), (2.6) to the quadruple (2.4), (2.5), (2.5)′, (2.6) where

(2.5)′ for every $r > 0$ the set of convolution operators $\{T\, * : T \in \Psi\}$ is an equicontinuous subset of $L(C^\infty_{B_r}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$, where $B_r = \{x \in \mathbb{R}^n : |x| \le r\}$.

The proof of the modified theorem follows the scheme (2.6)⇒(2.5)⇒(2.5)′⇒(2.4)⇒(2.6), where the implication (2.5)⇒(2.5)′ is trivial, and the proof of (2.5)′⇒(2.4) resembles that of (2.5)⇒(2.4).

(4) By [S, Sec. III.4, Theorems 4.1 and 4.2] the equicontinuity in (2.5) is equivalent to the boundedness of $\{T\, * : T \in \Psi\}$ in $L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$ equipped with either the topology of simple convergence or the compact-open topology.

So, take any $k \in [n/2] + \mathbb{N}$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Then, by (2.6), for every $T \in \Psi$ and $x \in \mathbb{R}^n$ one has

$$\begin{aligned} |(T * \varphi)(x)| &\le \sum_{|\beta| \le m_k} \Big(\int_{|y| \ge |x|/2} + \int_{|x-y| \ge |x|/2} \Big) |F_{k,\beta}(T)(y)| \cdot |\partial^\beta \varphi(x-y)|\, dy \\ &\le \sum_{|\beta| \le m_k} \Big(\int_{|y| \ge |x|/2} + \int_{|x-y| \ge |x|/2} \Big) C_k (1+|y|^2)^{-k} \\ &\qquad\qquad \times p_{k,m_k}(\varphi)(1+|x-y|^2)^{-k}\, dy \\ &\le (1+|x|^2/4)^{-k} 2 C_k D_k p_{k,m_k}(\varphi), \end{aligned}$$

proving (2.7). ■

Proof of (2.5)⇒(2.4). Let x_ν, $\nu = 1, \ldots, n$, denote the coordinate functions on $\mathbb{R}^n$, and for any multiindex $\alpha \in \mathbb{N}_0^n$ let $x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$. Whenever $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $T \in \mathcal{S}'(\mathbb{R}^n)$, then $x_\nu \cdot (T * \varphi) = (x_\nu \cdot T) * \varphi + T * (x_\nu \cdot \varphi)$ for $\nu = 1, \ldots, n$, which implies that

$$x^\alpha \cdot (T * \varphi) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (x^\beta \cdot T) * (x^{\alpha-\beta} \cdot \varphi)$$

for every multiindex $\alpha \in \mathbb{N}_0^n$. Suppose now that (2.5) holds. Rewriting the last formula in the form

$$(x^\alpha \cdot T) * \varphi = x^\alpha \cdot (T * \varphi) - \sum_{\substack{\beta \le \alpha \\ |\beta| < |\alpha|}} \binom{\alpha}{\beta} (x^\beta \cdot T) * (x^{\alpha-\beta} \cdot \varphi),$$

by induction with respect to $|\alpha|$ one can prove that for every $\alpha \in \mathbb{N}_0^n$ the set of operators $\{(x^\alpha \cdot T) * : T \in \Psi\}$ is an equicontinuous subset of $L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$. It follows that whenever P is a polynomial on $\mathbb{R}^n$, then $\{(P \cdot T) * : T \in \Psi\}$ is an equicontinuous subset of $L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$. Consequently, whenever P is a polynomial on $\mathbb{R}^n$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then $\{(P \cdot T) * \varphi : T \in \Psi\}$ is a bounded subset of $C_b(\mathbb{R}^n)$. By the implication (1.2)⇒(1.1) from Theorem 1.1, it follows that (2.4) is satisfied. ■

Proof of (2.4)⇒(2.6). Consider the function r on $\mathbb{R}^n$ such that $r(x) = |x|$ for every $x \in \mathbb{R}^n$. Then

$$(2.8) \qquad (1+r^2)^{a+|\alpha|/2} \partial^\alpha (1+r^2)^{-a} \in C_b(\mathbb{R}^n) \quad \text{for every } a \in]0, \infty[\text{ and } \alpha \in \mathbb{N}_0^n$$

because $\partial^\alpha (1+r^2)^{-a} = (1+r^2)^{-a-|\alpha|} P_\alpha$ where P_α is a polynomial on $\mathbb{R}^n$ of degree no greater that $|\alpha|$ [5]. Suppose that (2.4) holds. Fix $k \in \mathbb{N}$. Then

$$\Phi_k = (1+r^2)^k \Psi$$

[5] The estimation of the decay of $\partial^\alpha (1+r^2)^{-a}$ for large $|x|$ plays an important role in Sec. VII.5 of [S2].

is a subset of $\mathcal{B}'(\mathbb{R}^n)$ equicontinuous in the topology of $\mathcal{D}_{L^1}(\mathbb{R}^n)$. By the implication (1.1)⇒(1.3) from Theorem 1.1 applied to Φ_k, there is $m_k \in \mathbb{N}_0$ such that for every multiindex $\alpha \in \mathbb{N}_0^n$ of length $|\alpha| \le m_k$ there is a function $u_\alpha \in C_c(\mathbb{R}^n)$ such that

$$T = (1+r^2)^{-k} \sum_{|\alpha|\le m_k} \partial^\alpha u_\alpha * [(1+r^2)^k T] \quad \text{for every } T \in \mathcal{D}'(\mathbb{R}^n), \tag{2.9}$$

and

$$\{u_\alpha * ((1+r^2)^k T) : T \in \Psi\} = u_\alpha * \Phi_k \quad \text{is a bounded subset of } C_b(\mathbb{R}^n). \tag{2.10}$$

From (2.9) it follows that whenever $T \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then

$$\begin{aligned}\langle T, \varphi\rangle &= \sum_{|\alpha|\le m_k} \langle \partial^\alpha u_\alpha * [(1+r^2)^k T], (1+r^2)^{-k}\varphi\rangle \\ &= \sum_{|\alpha|\le m_k} \sum_{\beta\le\alpha} \left\langle u_\alpha * [(1+r^2)^k T], (-1)^{|\alpha|}\binom{\alpha}{\beta}(\partial^{\alpha-\beta}(1+r^2)^{-k})\partial^\beta\varphi\right\rangle \\ &= \sum_{|\alpha|\le m_k} \sum_{\beta\le\alpha} \left\langle (-1)^{|\alpha-\beta|}\binom{\alpha}{\beta}(\partial^{\alpha-\beta}(1+r^2)^{-k})(u_\alpha * [(1+r^2)^k T]), \right. \\ &\qquad\qquad \left. (-1)^{|\beta|}\partial^\beta\varphi\right\rangle.\end{aligned}$$

Consequently, (2.9) implies that (2.6)(i) holds for $F_{k,\beta}$ defined by the formula

$$F_{k,\beta}(T) = \sum_{\alpha\ge\beta,\, |\alpha|\le m_k} (-1)^{|\alpha-\beta|}\binom{\alpha}{\beta}(\partial^{\alpha-\beta}(1+r^2)^{-k})(u_\alpha * [(1+r^2)^k T]). \tag{2.11}$$

From (2.11), (2.10) and (2.8) it follows that whenever $\beta \in \mathbb{N}_0^n$, then $F_{k,\beta}(\Psi)$ is a bounded subset of the Banach space $B_k(\mathbb{R}^n)$. ∎

In view of (2.3) one can define in $\mathcal{O}'_C(\mathbb{R}^n)$ the topology induced by either $L_s(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$ or $L_b(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$ by means of the mapping $\mathcal{O}'_C(\mathbb{R}^n) \ni T \mapsto T* \in L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$. The subscripts s and b indicate *simple* convergence and uniform convergence on *bounded* subsets of $\mathcal{S}(\mathbb{R}^n)$. Our next theorem says that, for both these induced topologies, the class of convergent countable sequences of elements of $\mathcal{O}'_C(\mathbb{R}^n)$ is the same as for the topology in $\mathcal{O}'_C(\mathbb{R}^n)$ defined by L. Schwartz in [S2, Sec. VII.5, p. 244]. ∎

THEOREM 2.2. *For every sequence $(T_\nu)_{\nu=1}^\infty \subset \mathcal{O}'_C(\mathbb{R}^n)$ the following three conditions are equivalent:*

(2.12) *whenever P is a polynomial on $\mathbb{R}^n$, then the sequence of distributions $(P \cdot T_\nu)_{\nu=1}^\infty \subset \mathcal{B}'(\mathbb{R}^n)$ converges to zero uniformly on every bounded subset of $\mathcal{D}_{L^1}(\mathbb{R}^n)$,*

(2.13) $\lim_{\nu\to\infty}(T_\nu\, *) = 0$ *in the topology of $L_s(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$,*

(2.14) $\lim_{\nu\to\infty}(T_\nu\, *) = 0$ *in the topology of $L_b(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$.*

Proof. The equivalence (2.13)⇔(2.14) follows from the Banach–Steinhaus theorem and the fact that $\mathcal{S}(\mathbb{R}^n)$ is a Montel space. The implication (2.12)⇒(2.13) may be proved by an argument similar to that used in the case of Theorem 1.2, in the proof of (1.6)⇒(1.7).

It remains to prove that (2.13) implies (2.12). To this end, notice that the topology induced on $\mathcal{O}'_C(\mathbb{R}^n)$ by $L_s(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$ may be determined by the system of seminorms

$$p_{\alpha,\beta,\varphi}(T) = \|x^\alpha \partial^\beta (T * \varphi)\|_{C_b(\mathbb{R}^n)}, \qquad \alpha, \beta \in \mathbb{N}_0^n,\ \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Since

$$x^\alpha \partial^\beta (T * \varphi) = x^\alpha (T * \partial^\beta \varphi) = \sum_{\gamma \le \alpha} \binom{\alpha}{\gamma} (x^\gamma \cdot T) * (x^{\alpha-\gamma} \cdot \partial^\beta \varphi),$$

it follows that every seminorm of the type $p_{\alpha,\beta,\varphi}$ is no greater than a finite sum of seminorms of the type

$$p_{\alpha,\varphi}(T) = \|(x^\alpha \cdot T) * \varphi\|_{C_b(\mathbb{R}^n)}, \qquad \alpha \in \mathbb{N}_0^n,\ \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Hence, passing from the monomials x^α to arbitrary polynomials P, one finds that (2.13) holds if and only if

(2.15) $\lim_{\nu\to\infty} \|(P \cdot T_\nu) * \varphi\|_{C_b(\mathbb{R}^n)} = 0$ for every polynomial P on $\mathbb{R}^n$ and every $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

So, the implication (2.13)⇒(2.12) is a consequence of (2.15)⇒(2.12), which, in turn, follows from (1.7)⇒(1.6) of Theorem 1.2. ∎

3. The space $\mathcal{O}_M(\mathbb{R}^n)$ of infinitely differentiable slowly increasing functions. A function $\phi \in C^\infty(\mathbb{R}^n)$ is called *infinitely differentiable slowly increasing* if for every $k \in \mathbb{N}_0$ there is $m_k \in \mathbb{N}_0$ such that

$$\sup\{(1+|\xi|)^{-m_k} |\partial^\alpha \phi(\xi)| : \alpha \in \mathbb{N}_0^n,\ |\alpha| \le k,\ \xi \in \mathbb{R}^n\} < \infty.$$

The space of infinitely differentiable slowly increasing functions on $\mathbb{R}^n$ is denoted by $\mathcal{O}_M(\mathbb{R}^n)$. One has $\mathcal{O}_M(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$. A set $\Phi \subset \mathcal{O}_M(\mathbb{R}^n)$ will be called a *set of uniformly slowly increasing infinitely differentiable functions on* $\mathbb{R}^n$ if for every $k \in \mathbb{N}_0$ there is $m_k \in \mathbb{N}_0$ such that

$$\sup\{(1+|\xi|)^{-m_k} |\partial^\alpha \phi(\xi)| : \phi \in \Phi,\ \alpha \in \mathbb{N}_0^n,\ |\alpha| \le k,\ \xi \in \mathbb{R}^n\} < \infty.$$

Let $\mathcal{F}$ denote the Fourier transformation defined by the formula

$$\hat{\varphi}(\xi) = (\mathcal{F}\varphi)(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi}\varphi(x)\,dx \quad \text{for } \varphi \in \mathcal{S}(\mathbb{R}^n),\ \xi \in \mathbb{R}^n.$$

Then $\mathcal{F}$ is a continuous automorphism of $\mathcal{S}(\mathbb{R}^n)$, and $\mathcal{F}$ extends by duality, and also by continuity, to a continuous automorphism of $\mathcal{S}'_s(\mathbb{R}^n)$. (This time the subscript s indicates that $\mathcal{S}'_s(\mathbb{R}^n)$ is the *strong* dual of $\mathcal{S}(\mathbb{R}^n)$.) The extension by duality is usually discussed. The extendability by continuity follows from the form of the Fourier dual operator restricted to $\mathcal{S}(\mathbb{R}^n)$, and from the fact that $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{S}'_s(\mathbb{R}^n)$. This last denseness may be proved either by an argument based on reflexivity, similar to one in the proof [S2, Sec. III.3, Theorem XV, p. 75], or by a more elementary argument based on an analogue of [R, Proposition 4, p. 253]. In view of [S2, Sec. VII.8, Theorem XV, p. 268], one has

$$\mathcal{F}\mathcal{O}'_C(\mathbb{R}^n) = \mathcal{O}_M(\mathbb{R}^n). \tag{3.1}$$

This equality is important in what follows. The inclusion $\mathcal{F}\mathcal{O}'_C(\mathbb{R}^n) \subset \mathcal{O}_M(\mathbb{R}^n)$ follows from the fact that if $T \in \mathcal{O}'_C(\mathbb{R}^n)$, then for every multiindex $\alpha \in \mathbb{N}_0^n$ the condition (2.6) is satisfied for the singleton $\Psi = \{x^\alpha T\}$. To prove the opposite inclusion, pick $\phi \in \mathcal{O}_M(\mathbb{R}^n)$ and set $T = \mathcal{F}^{-1}\phi$, which makes sense because $\mathcal{O}_M(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$. Then $T \in \mathcal{S}'(\mathbb{R}^n)$, and so $T * \varphi \in \mathcal{S}'(\mathbb{R}^n)$ whenever $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Consequently, $\mathcal{F}(T * \varphi) = (\mathcal{F}T) \cdot \hat{\varphi} = \phi \cdot \hat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$. Since $\mathcal{F}$ is an automorphism of $\mathcal{S}(\mathbb{R}^n)$, it follows that $T * \varphi \in \mathcal{S}(\mathbb{R}^n)$ for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$. It is easy to see that $T*$ is a closed operator from $\mathcal{S}(\mathbb{R}^n)$ into itself. Consequently, by the closed graph theorem, (2.3) holds, so that $T \in \mathcal{O}'_C(\mathbb{R}^n)$ and $\phi = \mathcal{F}T \in \mathcal{F}\mathcal{O}'_C(\mathbb{R}^n)$.

There are the following characterizations of $\mathcal{O}_M(\mathbb{R}^n)$ as the space of multipliers:

(3.2) $\phi \in \mathcal{O}_M(\mathbb{R}^n)$ if and only if $\phi \in C^\infty(\mathbb{R}^n)$ and $\phi \cdot \varphi \in \mathcal{S}(\mathbb{R}^n)$ for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

(3.2)′ $\phi \in \mathcal{O}_M(\mathbb{R}^n)$ if and only if $\phi \in C^\infty(\mathbb{R}^n)$ and $\phi \cdot T \in \mathcal{S}'(\mathbb{R}^n)$ for every $T \in \mathcal{S}'(\mathbb{R}^n)$.

These characterizations of $\mathcal{O}_M(\mathbb{R}^n)$ are formulated without proof in [S2, Sec. VII.5, p. 246, remarks after Theorem X], and in [T, Chap. 25, Theorem 25.5, p. 275].

Proof of (3.2). It is easy to see that if $\phi \in \mathcal{O}_M(\mathbb{R}^n)$, then $\phi\,\cdot \in L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$. Conversely, suppose that $\phi \in C^\infty(\mathbb{R}^n)$ and $\phi \cdot \varphi \in \mathcal{S}(\mathbb{R}^n)$ for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$. It is obvious that $\phi\,\cdot$ is a closed operator of $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$, and so, by the closed graph theorem, $\phi\,\cdot \in L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$. Consequently, also $\mathcal{F}^{-1} \circ (\phi\,\cdot\,) \circ \mathcal{F} \in L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$. Moreover, since $\phi\,\cdot$ commutes with multiplication by characters of $\mathbb{R}^n$, it follows that $\mathcal{F}^{-1} \circ (\phi\,\cdot\,) \circ \mathcal{F}$ commutes with translations. Therefore, by a variant of a theorem of L. Schwartz ([S2, Sec. VI.3, Theorem X, p. 162]; [Y, Sec. VI.3, Theorem 2, p. 158]), there is a distribution $T \in \mathcal{S}'(\mathbb{R}^n)$ such that $[\mathcal{F}^{-1} \circ (\phi\,\cdot\,) \circ \mathcal{F}](\varphi) = T * \varphi$ for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$. By (2.3), this implies that $T \in \mathcal{O}'_C(\mathbb{R}^n)$. Furthermore,

$$\phi \cdot \varphi = [(\phi\,\cdot\,) \circ \mathcal{F}](\mathcal{F}^{-1}\varphi) = \mathcal{F}(T * \mathcal{F}^{-1}\varphi) = (\mathcal{F}T) \cdot (\mathcal{F}(\mathcal{F}^{-1}\varphi)) = (\mathcal{F}T) \cdot \varphi$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$, whence $\phi = \mathcal{F}T \in \mathcal{O}_M(\mathbb{R}^n)$, by (3.1). ∎

Proof of (3.2)$'$. If $\phi \in \mathcal{O}_M(\mathbb{R}^n)$, then $\phi\,\cdot \in L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$, and so, whenever $T \in \mathcal{S}'(\mathbb{R}^n)$, then $\langle \phi \cdot T, \varphi\rangle = \langle T, \phi \cdot \varphi\rangle$ for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$, where the right side of the equality extends, by continuity in the topology of $\mathcal{S}(\mathbb{R}^n)$, onto all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. This proves that also $\langle \phi \cdot T, \varphi\rangle$ extends by continuity from $\varphi \in \mathcal{D}(\mathbb{R}^n)$ onto all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and this means that $\phi \cdot T \in \mathcal{S}'(\mathbb{R}^n)$. Conversely, suppose that $\phi \cdot T \in \mathcal{S}'(\mathbb{R}^n)$ for every $T \in \mathcal{S}'(\mathbb{R}^n)$. By (3.2), in order to prove that $\phi \in \mathcal{O}_M(\mathbb{R}^n)$, it is sufficient to show that $\phi\,\cdot \in L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$. To this end, basing on reflexivity of the pair of spaces $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}'_s(\mathbb{R}^n)$, we will prove that

(3.3) the operator of multiplication $M_\phi : \mathcal{D}(\mathbb{R}^n) \ni \varphi \mapsto \phi \cdot \varphi \in \mathcal{D}(\mathbb{R}^n)$ extends by continuity to an operator belonging to $L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$.

In order to prove (3.3), it is sufficient to show that whenever

(3.4) $\varphi_\nu \in \mathcal{D}(\mathbb{R}^n)$ for every $\nu = 1, 2, \ldots$ and $\lim_{\nu\to\infty} \varphi_\nu = \varphi$ in $\mathcal{S}(\mathbb{R}^n)$,

then $\lim_{\nu\to\infty} \phi \cdot \varphi_\nu$ exists in the topology of $\mathcal{S}(\mathbb{R}^n)$. So, let $(\varphi_\nu)_{\nu=1}^\infty$ be a sequence satisfying (3.4). Then for every $T \in \mathcal{S}'(\mathbb{R}^n)$ one has $\langle T, \phi \cdot \varphi_\nu\rangle = \langle \phi\cdot T, \varphi_\nu\rangle \to \langle \phi\cdot T, \varphi\rangle$, which means that $(\phi\cdot\varphi_\nu)_{\nu=1}^\infty$ is a pointwise convergent sequence of linear functionals on $\mathcal{S}'_s(\mathbb{R}^n)$. Since $\mathcal{S}'_s(\mathbb{R}^n)$ is a barrelled space (see [Y, Appendix to Chapter V, Sec. 3, Theorem 2, p. 140]) and a Montel space ([S2, Sec. VII.4, p. 238]), from the Banach–Steinhaus theorem it follows that the sequence $(\phi \cdot \varphi_\nu)_{\nu=1}^\infty$ converges uniformly on bounded subsets of $\mathcal{S}'_s(\mathbb{R}^n)$. This means that the sequence $(\phi \cdot \varphi_\nu)_{\nu=1}^\infty$ converges in $(\mathcal{S}'_s(\mathbb{R}^n))'_s$, i.e. in $\mathcal{S}(\mathbb{R}^n)$, by reflexivity. Thus (3.3) is proved.

One can express (3.3) equivalently as

(3.5) $\phi \cdot \varphi = M_\phi \varphi$ for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$ where $M_\phi \in L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$.

From (3.5) it follows at once that $\phi \cdot \varphi = M_\phi \varphi$ not only for $\varphi \in \mathcal{D}(\mathbb{R}^n)$ but also for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Therefore $\phi\,\cdot \in L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$. ∎

Let $C_b^k(\mathbb{R}^n)$ denote the Banach space of functions continuous and bounded on $\mathbb{R}^n$ together with their partial derivatives of order no greater than k.

THEOREM 3.1. *For every family Φ of C^∞ functions on $\mathbb{R}^n$ the following three conditions are equivalent:*

(3.6) *$\Phi \subset \mathcal{O}_M(\mathbb{R}^n)$ and the functions belonging to Φ increase uniformly slowly,* (3.7) *the set of multiplication operators* $\{\phi\cdot : \phi \in \Phi\}$ *is an equicontinuous subset of* $L(\mathcal{S}(\mathbb{R}^n);\mathcal{S}(\mathbb{R}^n))$ (6),

(3.8) *for every* $k \in \mathbb{N}_0$ *there is* $m_k \in \mathbb{N}_0$ *and a set of operators* $\{G_{k,\beta} : \beta \in \mathbb{N}_0^n, |\beta| \le m_k\} \subset L(\mathcal{S}'_s(\mathbb{R}^n);\mathcal{S}'_s(\mathbb{R}^n))$ *having the two properties:*
   (i) $T = \sum_{|\beta|\le m_k} \xi^\beta G_{k,\beta}(T)$ *for every* $T \in \mathcal{S}'(\mathbb{R}^n)$ *where* $\xi_1,\ldots,\xi_n$ *denote the coordinate functions on* $\mathbb{R}^n$ *and* $\xi^\beta = \xi_1^{\beta_1}\cdot\ldots\cdot\xi_n^{\beta_n}$,
   (ii) *whenever* $\beta \in \mathbb{N}_0^n$ *and* $|\beta| \le m_k$, *then* $G_{k,\beta}(\Phi)$ *is a bounded subset of* $C_b^k(\mathbb{R}^n)$.

Proof. The implication (3.8)⇒(3.6) is obvious. To prove (3.6)⇒(3.7) it is sufficient to check that if (3.6) holds, then, for every $\alpha,\beta \in \mathbb{N}_0^n$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the set $\{x^\alpha\partial^\beta(\phi\cdot\varphi) : \phi \in \Phi\}$ is a bounded subset of $C_b(\mathbb{R}^n)$. Since

$$x^\alpha\partial^\beta(\phi\cdot\varphi) = \sum_{\gamma\le\beta}\binom{\beta}{\gamma}(\partial^\gamma\phi)\cdot(x^\alpha\partial^{\beta-\gamma}\varphi),$$

it remains to observe that if (3.6) holds, then for every $\gamma \in \mathbb{N}_0^n$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ the set $\{(\partial^\gamma\phi)\cdot\psi : \phi \in \Phi\}$ is a bounded subset of $C_b(\mathbb{R}^n)$(7).

Finally, we will prove that (3.7) implies (3.8). To this end, notice first that if (3.7) holds, then, by (3.2), $\Phi \subset \mathcal{O}_M(\mathbb{R}^n)$. Hence, by (3.1), the set of distributions $\Psi = \mathcal{F}^{-1}\Phi$ is contained in $\mathcal{O}'_C(\mathbb{R}^n)$. Since $(\mathcal{F}^{-1}\phi)*\varphi = \mathcal{F}^{-1}(\phi\cdot\mathcal{F}\varphi)$ for every $\phi \in \mathcal{O}_M(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, it follows that the set of convolution operators $(\mathcal{F}^{-1}\Phi)*$ is an equicontinuous subset of $L(\mathcal{S}(\mathbb{R}^n);\mathcal{S}(\mathbb{R}^n))$. Consequently, by Theorem 2.1, the condition (2.6) is satisfied for the subset $\Psi = \mathcal{F}^{-1}\Phi$ of $\mathcal{O}'_C(\mathbb{R}^n)$. From this one can deduce (3.8) using elementary properties of the Fourier transformation. ■

THEOREM 3.2. *For every sequence* $(\phi_\nu)_{\nu=1}^\infty \subset \mathcal{O}_M(\mathbb{R}^n)$ *the following three conditions are equivalent:*

(3.9) *the functions* ϕ_ν, $\nu = 1,2,\ldots$, *increase uniformly slowly and the sequence* $(\phi_\nu)_{\nu=1}^\infty$ *converges pointwise on* $\mathbb{R}^n$,

$(3.10)_s$ *the sequence of multiplication operators* $(\phi_\nu\,\cdot)_{\nu=1}^\infty$ *converges in the topology of* $L_s(\mathcal{S}(\mathbb{R}^n);\mathcal{S}(\mathbb{R}^n))$,

$(3.10)_b$ *the sequence of multiplication operators* $(\phi_\nu\,\cdot)_{\nu=1}^\infty$ *converges in the topology of* $L_b(\mathcal{S}(\mathbb{R}^n);\mathcal{S}(\mathbb{R}^n))$.

An argument similar to one from the proof of Theorem 2.2 shows that the topology in $\mathcal{O}_M(\mathbb{R}^n)$ defined in [S2, Sec. VII.5, pp. 243–244] is identical with the topology induced by $L_b(\mathcal{S}(\mathbb{R}^n);\mathcal{S}(\mathbb{R}^n))$. Therefore Theorem 3.2 coincides with a statement formulated without detailed proof in [S2, p. 244].

(6) Condition (3.7) is equivalent to the boundedness of $\{\phi\cdot : \phi \in \Phi\}$ in each of the l.c.v.s. $L_s(\mathcal{S}(\mathbb{R}^n);\mathcal{S}(\mathbb{R}^n))$, $L_b(\mathcal{S}(\mathbb{R}^n);\mathcal{S}(\mathbb{R}^n))$. See [S, Sec. III.4].

(7) This last can be proved by an inductive argument similar to one used in the proof of (2.5)⇒(2.4)

Proof of Theorem 3.2. The equivalence of $(3.10)_s$ and $(3.10)_b$ follows from the Banach–Steinhaus theorem and the fact that $\mathcal{S}(\mathbb{R}^n)$ is a Montel space. An elementary argument shows that (3.9) implies $(3.10)_s$. Indeed, if (3.9) holds, then for every $\alpha \in \mathbb{N}_0^n$ the sequence $(\partial^\alpha \phi_\nu)_{\nu=1}^\infty$ converges almost uniformly on $\mathbb{R}^n$, and whenever $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then $(\phi_\nu \cdot \varphi)_{\nu=1}^\infty$ is a sequence of elements of $\mathcal{S}(\mathbb{R}^n)$ convergent in the topology of $\mathcal{S}(\mathbb{R}^n)$.

It remains to prove that $(3.10)_s$ implies (3.9). So, suppose that $(3.10)_s$ holds. Then, by the Banach–Steinhaus theorem, $\{\phi_\nu \cdot : \nu = 1, 2, \ldots\}$ is an equicontinuous subset of $L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$, and hence, by the implication (3.7)⇒(3.6) from Theorem 3.1, the functions ϕ_ν, $\nu = 1, 2, \ldots$, increase uniformly slowly. Furthermore, if $(3.10)_s$ holds, then for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$ the sequence $(\phi_\nu \cdot \varphi)_{\nu=1}^\infty$ of elements of $\mathcal{S}(\mathbb{R}^n)$ converges in the topology of $\mathcal{S}(\mathbb{R}^n)$, so $(\phi_\nu)_{\nu=1}^\infty$ converges pointwise on $\mathbb{R}^n$. Hence $(3.10)_s$ implies (3.9). ∎

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