BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 59, No. 3, 2011

FUNCTIONAL ANALYSIS

Real Interpolation between Row and Column Spaces

by

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Summary. We give an equivalent expression for the K-functional associated to the pair of operator spaces (R, C) formed by the rows and columns respectively. This yields a description of the real interpolation spaces for the pair $(M_n(R), M_n(C))$ (uniformly over n). More generally, the same result is valid when M_n (or $B(\ell_2)$) is replaced by any semi-finite von Neumann algebra. We prove a version of the non-commutative Khintchine inequalities (originally due to Lust-Piquard) that is valid for the Lorentz spaces $L_{p,q}(\tau)$ associated to a non-commutative measure τ, simultaneously for the whole range $1 \leq p, q < \infty$, regardless of whether $p < 2$ or $p > 2$. Actually, the main novelty is the case $p = 2$, $q \neq 2$. We also prove a certain simultaneous decomposition property for the operator norm and the Hilbert–Schmidt norm.

1. Introduction. Let $B(\ell_2)$ denote the space of all bounded operators on ℓ_2. Let $R \subset B(\ell_2)$ (resp. $C \subset B(\ell_2)$) be the row (resp. column) operator spaces, defined by $R = \overline{\mathrm{span}}[e_{1j} \mid j \geq 1]$ (resp. $C = \overline{\mathrm{span}}[e_{i1} \mid i \geq 1]$). The couple (R, C) plays an important rôle in operator space theory. In particular, it is known that the complex interpolation space $(R, C)_{1/2}$ coincides with the (self-dual) operator space OH. See [24] for details. We refer to [30] for the real interpolation method in the operator space framework. In particular, Xu proved in [30] that $(R, C)_{1/2,2}$ is completely isomorphic to OH.

This paper studies three problems concerning real interpolation for several pairs of Banach spaces associated to (R, C).

In §3, we consider the pair $(M(R), M(C))$ when $M = B(\ell_2)$. The space $M(R)$ consists of those $x = (x_n)$ with $x_n \in B(\ell_2)$ such that $\sum x_n x_n^*$ converges in the weak operator topology (w.o.t. for short) and $\|x\|_{M(R)} := \|(\sum x_n x_n^*)^{1/2}\|$. Then $M(C)$ consists of those $x = (x_n)$ such that $(x_n^*) \in M(R)$ with norm $\|x\|_{M(C)} := \|(\sum x_n^* x_n)^{1/2}\|$.

2010 *Mathematics Subject Classification*: 46L07, 47L25, 46B70.
Key words and phrases: real interpolation method, row and column operator space, K-functional, non-commutative Khintchine inequality, non-commutative Lorentz space.

DOI: 10.4064/ba59-3-6

The main result of §3 is an equivalent expression for the K-functional for this pair $(M(R), M(C))$. Our result extends to more general (semi-finite) von Neumann algebras. As an application we can describe the interpolation space $X(\theta) = (M(R), M(C))_{\theta,\infty}$ for $0 < \theta < 1$. We find that if x is in the latter space, then $\|x\|^2_{X(\theta)}$ is equivalent to the norm of the associated completely positive map $T_x : T \mapsto \sum x_n T x_n^*$ as an operator of "very weak type (p, p)" on the L_p-space associated to the trace of M with $p = 1/\theta$. The analogous result for the complex interpolation method was obtained in our previous works (see [20, 21]). Our result can be interpreted as a description of the operator space structure of $(R, C)_{\theta,\infty}$ in the sense of [30]. Our approach is based on a non-commutative version of a lemma originally due to Varopoulos, which we extended with a different proof in a separate paper [25].

In §4, we present a version of the non-commutative Khintchine inequalities (originally due to Lust-Piquard [14]) that is valid for the Lorentz spaces $L_{p,q}(\tau)$ associated to (M, τ). This provides an equivalent for the average over all signs of the norm in $L_{p,q}(\tau)$ of a series of the form $\sum \pm x_n$ ($x_n \in L_{p,q}(\tau)$). The main interest of our result is the case of $L_{2,q}(\tau)$ which seemed out of reach of previous works (see [9]). Here again our study concentrates on a pair of Banach spaces, but this time it is the pair (A_0, A_1) where $A_0 = M(R) \cap M(C)$ and where A_1 is the natural predual of $M(R) \cap M(C)$, which we describe as the sum of the preduals of $M(R)$ and $M(C)$ and denote by $A_1 = M_*(R) + M_*(C)$.

In §5, we study another pair, namely (A_0, A_2) where $A_2 = (A_0, A_1)_{1/2,2}$. When $M = B(\ell_2)$, the space A_2 is nothing but $\ell_2(S_2)$ where S_2 is the Hilbert–Schmidt class. We formulate our result using the notions of "K-closed" and "J-closed" introduced in [18], which are isomorphic versions of Peetre's notion of "subcouple". To give a more concrete statement, the following can be viewed as the main point of §5:

There is a constant c such that for any x in $(M(R) + M(C)) \cap \ell_2(S_2)$ there is a decomposition $x = x_1 + x_2$ such that we have simultaneously

$$\|x_1\|_{M(R)} + \|x_2\|_{M(C)} \le c\|x\|_{M(R)+M(C)}, \tag{1.1}$$

$$\|x_1\|_{\ell_2(S_2)} + \|x_2\|_{\ell_2(S_2)} \le c\|x\|_{\ell_2(S_2)}. \tag{1.2}$$

In our more abstract terminology, this says that the pair (A_0, A_2) is J-closed when viewed as sitting inside (via the diagonal embedding) the pair $(M(R)\oplus M(C), \ell_2(S_2)\oplus\ell_2(S_2))$. This is analogous to what was proved in [11] (resp. [18]) for the pair (H^∞, H^2) inside (L^∞, L^2) (resp. $(H^\infty(B(\ell_2)), H^2(S_2))$ inside $(L^\infty(B(\ell_2)), L^2(S_2))$).

In §6, we include a brief comparative discussion of what the non-commutative Khintchine inequalities become in free probability and how it relates to our interpolation problems.

2. Notation and background. We will use the real interpolation method. We refer to [1] for all undefined terms. We just recall that if (A_0, A_1) is a compatible couple of Banach spaces, then for any $x \in A_0 + A_1$ the K-functional is defined for all $t > 0$ by

$$K_t(x; A_0, A_1) = \inf\{\|x_0\|_{A_0} + t\|x_1\|_{A_1} \mid x = x_0 + x_1,\ x_0 \in A_0,\ x_1 \in A_1\}.$$

Recall that the ("real" or "Lions–Peetre" interpolation) space $(A_0, A_1)_{\theta,q}$ is defined, for $0 < \theta < 1$ and $1 \le q \le \infty$, as the space of all x in $A_0 + A_1$ such that $\|x\|_{\theta,q} < \infty$ where

$$\|x\|_{\theta,q} = \Big(\int (t^{-\theta} K_t(x, A_0, A_1))^q \, dt/t\Big)^{1/q},$$

with the usual convention when $q = \infty$.

Let M be a von Neumann algebra equipped with a semi-finite faithful normal trace τ. The basic example is $M = B(\ell_2)$, equipped with the usual trace, and, to improve readability, we will present most of our results in this special case with mere indications for the extension to the general case.

Let M_* be the predual of M. As is well known (see e.g. [26]), M_* can be identified with the non-commutative L_1-space associated to τ, usually denoted by $L_1(\tau)$. When $M = B(\ell_2)$, M_* is the classical "trace class" S_1. More generally, for any $1 \le p < \infty$ we denote by $L_p(\tau)$ the associated non-commutative L_p-space. By convention we set $L_\infty(\tau) = M$. When $M = B(\ell_2)$, M_* (resp. $L_p(\tau)$) is the classical "trace class" S_1 (resp. the Schatten class S_p). See e.g. [4] or [26] for more information on non-commutative L_p-spaces. We will first mainly use the case $p = 2$ and we denote its norm simply by $\|\cdot\|_2$.

We always denote by p' the conjugate of $1 \le p \le \infty$ defined by $p^{-1} + p'^{-1} = 1$.

We denote by $\mathcal{P}(M)$ or simply $\mathcal{P}$ the set of all (self-adjoint) projections in M.

We denote by $M(R)$ (resp. $M(C)$) the space of sequences $x = (x_n)$ with $x_n \in M$ such that $(\sum_{n=1}^\infty x_n x_n^*)^{1/2} \in M$ (resp. $(\sum_{n=1}^\infty x_n^* x_n)^{1/2} \in M$). Here we implicitly assume that the series $\sum_{n=1}^\infty x_n x_n^*$ (resp. $\sum_{n=1}^\infty x_n^* x_n$) converge in (say) the w.o.t. We equip $M(R)$ (resp. $M(C)$) with the natural norm

$$\|x\|_{M(R)} = \Big\|\Big(\sum x_n x_n^*\Big)^{1/2}\Big\|_M \quad \Big(\text{resp. } \|x\|_{M(C)} = \Big\|\Big(\sum x_n^* x_n\Big)^{1/2}\Big\|_M\Big).$$

Note that $M(R)$ (resp. $M(C)$) can be identified with $M \mathbin{\overline{\otimes}} R$ (resp. $M \mathbin{\overline{\otimes}} C$), i.e. the weak-$*$ closure of $M \otimes R$ (resp. $M \otimes C$) in the (von Neumann sense) tensor product $M \mathbin{\overline{\otimes}} B(\ell_2)$.

We will consider $(M(R), M(C))$ as an interpolation couple in the obvious way using the inclusions $M(R) \subset M^{\mathbb{N}}$, $M(C) \subset M^{\mathbb{N}}$.

Similarly, we denote by $M_*(R)$ (resp. $M_*(C)$) the space of sequences $x = (x_n)$ with $x_n \in M_*$ such that $(\sum_{n=1}^{\infty} x_n x_n^*)^{1/2} \in M_*$ (resp. $(\sum_{n=1}^{\infty} x_n^* x_n)^{1/2} \in M_*$). Here we assume that the sequence $(\sum_{n=1}^{N} x_n x_n^*)^{1/2}$ (resp. $(\sum_{n=1}^{N} x_n^* x_n)^{1/2}$) norm-converges in M_* as $N \to \infty$. We equip $M_*(R)$ (resp. $M_*(C)$) with the natural norm

(2.1)

$$\|x\|_{M_*(R)} = \Big\|\Big(\sum x_n x_n^*\Big)^{1/2}\Big\|_{M_*} \quad \Big(\text{resp. } \|x\|_{M_*(C)} = \Big\|\Big(\sum x_n^* x_n\Big)^{1/2}\Big\|_{M_*}\Big).$$

Note that $M(R) = M_*(C)^*$ and $M(C) = M_*(R)^*$ isometrically with respect to the duality defined by $\langle x, y\rangle = \sum \tau(x_n y_n)$. (Equivalently, $\overline{M(R)} = M_*(R)^*$ and $\overline{M(C)} = M_*(C)^*$ with respect to the duality defined by $\langle x, \overline{y}\rangle = \sum \tau(x_n y_n^*)$, with the bar denoting complex conjugation.)

3. K-functional between R and C. Our main result is:

THEOREM 3.1. *For any $a = (a_n) \in M(R) + M(C)$ we have*

$$\forall t > 0 \qquad k_t(a) \le K_t(a; M(R), M(C)) \le 2k_t(a)$$

where

$$k_t(a) = \sup\Big\{\Big(\sum \|Pa_nQ\|_2^2\Big)^{1/2} \max\{\tau(P)^{1/2}, t^{-1}\tau(Q)^{1/2}\}^{-1} \;\Big|\; P, Q \in \mathcal{P}\Big\}.$$

REMARK. The following was pointed out by Q. Xu (by a modification of the proof of Lemma 3.2 below). Let $\widehat{k}_t(a)$ be the same as $k_t(a)$ except that when $t > 1$ (resp. $t < 1$) we restrict to pairs of projections P, Q such that $P \le Q$ (resp. $Q \le P$), and when $t = 1$ we restrict to pairs such that $P = Q$. Then

$$\widehat{k}_t(a) \le k_t(a) \le 2^{1/2}\widehat{k}_t(a).$$

We merely indicate a quick argument for $t > 1$. Let $Q' = P \vee Q$. Then $P \le Q'$ and $\tau(Q') \le \tau(P) + \tau(Q)$. With the above notation we have $(\sum \|Pa_nQ\|_2^2)^{1/2} \le (\sum \|Pa_nQ'\|_2^2)^{1/2}$ and also $\tau(P)^{1/2} \vee t^{-1}\tau(Q')^{1/2} \le \tau(P)^{1/2} \vee t^{-1}(\tau(P) + \tau(Q))^{1/2} \le (t^{-2}+1)^{1/2}(\tau(P)^{1/2} \vee t^{-1}\tau(Q)^{1/2}$ and since $(t^{-2}+1)^{1/2} \le 2^{1/2}$ we obtain $k_t(a) \le 2^{1/2}\widehat{k}_t(a)$. We leave the other cases to the reader.

First part of the proof of Theorem 3.1. Consider $a^0 \in M(R)$ and $P, Q \in \mathcal{P}$. We have

$$\sum \|Pa_n^0Q\|_2^2 = \sum \tau(Pa_n^0Qa_n^{0*}P) \le \sum \tau(Pa_n^0a_n^{0*}P) \le \Big\|\sum a_n^0a_n^{0*}\Big\|\tau(P)$$

and hence

$$\Big(\sum \|Pa_n^0Q\|_2^2\Big)^{1/2} \le \|a^0\|_{M(R)}\tau(P)^{1/2}.$$

Similarly for any $a^1 \in M(C)$ we have

$$\Big(\sum \|Pa^1_n Q\|_2^2\Big)^{1/2} \le \|a^1\|_{M(C)}\tau(Q)^{1/2}.$$

Therefore if $a = a^0 + a^1$ we find by the triangle inequality

$$\begin{aligned}\Big(\sum \|Pa_n Q\|_2^2\Big)^{1/2} &\le \|a^0\|_{M(R)}\tau(P)^{1/2} + \|a^1\|_{M(C)}\tau(Q)^{1/2} \\ &\le (\|a^0\|_{M(R)} + t\|a^1\|_{M(C)}) \max\{\tau(P)^{1/2}, t^{-1}\tau(Q)^{1/2}\}.\end{aligned}$$

So we obtain

$$k_t(a) \le K_t(a; M(R), M(C)). \ \blacksquare$$

To prove the converse we will use duality, via the following lemma.

Lemma 3.2. *Let $x \in M_*(R) \cap M_*(C)$ be such that*

$$J_t(x) := \max\bigg\{\Big\|\Big(\sum x_n^* x_n\Big)^{1/2}\Big\|_1, \frac{1}{t}\Big\|\Big(\sum x_n x_n^*\Big)^{1/2}\Big\|_1\bigg\} \le 1.$$

Let $\mathcal{C}_t$ be the subset of $M_*(R) \cap M_*(C)$ consisting of all sequences $\chi = (\chi_n)$ of the form

$$\chi_n = Q y_n P \cdot (\tau(P) + t^{-2}\tau(Q))^{-1/2}$$

with $\sum \|y_n\|_2^2 \le 1$ and where P, Q are commuting projections. Then $x \in 2\,\overline{\mathrm{conv}}(\mathcal{C}_t)$ where the closure is in $M_*(R) \cap M_*(C)$.

We will need the following simple

Lemma 3.3. *Let $\varphi : [1, \ldots, N]^2 \to \mathbb{R}_+$ be defined by $\varphi(i,j) = g(i) \wedge f(j)$ where $g \ge 0$ and $f \ge 0$ satisfy $\sum f(j) \le 1$ and $\sum g(i) \le t^2$. Then $\varphi \in 2\,\mathrm{conv}(\Phi)$ where Φ is the set of functions on $[1, \ldots, N]^2$ of the form*

$$\frac{1_{E\times F}}{t^{-2}|E| + |F|} \tag{3.1}$$

where $E, F \subset [1, \ldots, N]$ are arbitrary subsets.

Proof. We may write

$$\varphi = \int_0^\infty 1_{\{\varphi>c\}}\, dc = \int_0^\infty 1_{\{g>c\}\times\{f>c\}}\, dc = \int_0^\infty m(c)\frac{1_{\{g>c\}\times\{f>c\}}}{m(c)}\, dc$$

where $m(c) = t^{-2}|\{g > c\}| + |\{f > c\}|$. But since $\int_0^\infty m(c)\, dc = t^{-2}\sum g(i) + \sum f(j) \le 2$, Lemma 3.3 follows. $\blacksquare$

Remark. A simple verification shows that if a function $\varphi \in \Phi$ is of the form (3.1), we have $\sup_j \varphi(i,j) = 1_E(i)(t^{-2}|E| + |F|)^{-1}$ and $\sup_i \varphi(i,j) = 1_F(j)(t^{-2}|E| + |F|)^{-1}$. Therefore we find

$$t^{-2}\sum_i \sup_j \varphi(i,j) + \sum_j \sup_i \varphi(i,j) \le 1. \tag{3.2}$$

REMARK. Let (Ω,μ) and (Ω',μ') be measure spaces. Let $f:\Omega'\to\mathbb{R}_+$ and $g:\Omega\to\mathbb{R}_+$ be step functions. Assume that $\int f\,d\mu'\le 1$ and $\int g\,d\mu\le t^2$ $(t>0)$. Let $\varphi(\omega,\omega')=g(\omega)\wedge f(\omega')$ on $\Omega\times\Omega'$. Then $\varphi\in 2\operatorname{conv}(\Phi)$ where Φ is the set of functions of the form

$$\varphi=\frac{1_{E\times F}}{t^{-2}\mu(E)+\mu'(F)},$$

where $E\subset\Omega$ and $F\subset\Omega'$ are arbitrary measurable subsets.

Proof of Lemma 3.2. As is well known, if we truncate the sequence (x_n) and replace it by $x(N)$ defined by $x_n(N)=x_n1_{\{n\le N\}}$, then we have $\|x(N)-x\|_{M_*(R)}\to 0$ and similarly for $M_*(C)$. Indeed, since for all $N\le m$ the norms in $M_*(R)$ and $M_*(R)$ both satisfy

$$\|x(N)-x(m)\|^2+\|x(N)\|^2\le\|x(m)\|^2,$$

and since $\|x(m)\|\to\|x\|$, this fact follows easily.

Thus it suffices to prove the lemma for finite sequences $x=(x_1,\dots,x_N)$. Let us first assume that $M=B(\ell_2)$, $M_*=S_1$ (trace class) and τ is the ordinary trace on S_1. Assume $J_t(x)<1$. Let $f=(\sum x_n^*x_n)^{1/2}$ and $g=t(\sum x_nx_n^*)^{1/2}$. We have $\operatorname{tr}f<1$, $\operatorname{tr}g<t^2$ and moreover if $a_n=g^{-1}x_n$ and $b_n=x_nf^{-1}$ then $\sum a_na_n^*=g^{-1}t^{-2}g^2g^{-1}=t^{-2}$, and similarly $\sum b_n^*b_n=1$, so we have

$$\Big\|\Big(\sum a_na_n^*\Big)^{1/2}\Big\|\le t^{-1},\quad \Big\|\sum b_n^*b_n\Big\|^{1/2}\le 1, \tag{3.3}$$

with $x_n=ga_n=b_nf$.

Note that by a simple perturbation argument we may assume $f>0$ and $g>0$ so that f and g are invertible.

Let us now consider the matrix representation of x_n with respect to the bases that diagonalize respectively f (for the column index) and g (for the row index). We then have

$$x_n(i,j)=g_ia_n(i,j)=b_n(i,j)f_j.$$

Moreover we know from (3.3) that for all $\xi\in\ell_2$,

$$\sum\|a_n^*\xi\|^2\le t^{-2}\|\xi\|^2\quad\text{and}\quad\sum\|b_n\xi\|^2\le\|\xi\|^2,$$

and hence taking for ξ either the ith or the jth basis vector we find

$$\sup_i\sum_n\sum_j|a_n(i,j)|^2\le t^{-2}\quad\text{and}\quad\sup_j\sum_n\sum_i|b_n(i,j)|^2\le 1. \tag{3.4}$$

Let $\gamma_n(i,j)=\frac{1}{g_i\wedge f_j}x_n(i,j)$. Then $|\gamma_n(i,j)|\le\max\{|a_n(i,j)|,|b_n(i,j)|\}\le |a_n(i,j)|+|b_n(i,j)|$ and $x_n(i,j)=(g_i\wedge f_j)\gamma_n(i,j)$. By Lemma 3.3, since $x_n(i,j)=g_i\wedge f_j\,\gamma_n(i,j)$, we know that $x/2$ is in the convex hull of

$$(\varphi(i,j)\gamma_n(i,j))=(\varphi(i,j)^{1/2}\varphi(i,j)^{1/2}\gamma_n(i,j))$$

where $\varphi \in \Phi$. We then note that if

$$y_n(i,j) = \varphi(i,j)^{1/2}\gamma_n(i,j)$$

then using (3.4) and (3.2) we have, since $|\gamma_n(i,j)| \le |a_n(i,j)| \vee |b_n(i,j)|$,

$$\begin{aligned}\sum_n \|y_n\|_2^2 = \sum_{n,i,j} |\xi_n(i,j)|^2 &\le \sum_n \sum_{i,j} \varphi(i,j)(|a_n(i,j)|^2 + |b_n(i,j)|^2) \\ &\le \sum_i \sup_j \varphi(i,j) t^{-2} + \sum_j \sup_i \varphi(i,j) \le 1.\end{aligned}$$

Thus we find that $x/2$ can be written as a convex combination of elements of the form

$$(\varphi(i,j)^{1/2} y_n)$$

with $\varphi \in \Phi$ and $\sum \|y_n\|_2^2 \le 1$. Let Q, P be the projections associated to 1_E and 1_F. Then

$$[\varphi(i,j)^{1/2} y_n] = (t^{-2} \operatorname{tr} Q + \operatorname{tr} P)^{-1/2} Q y_n P.$$

This completes the proof in the case $M = B(H)$. The case of a general semi-finite von Neumann algebra $M \subset B(H)$ can easily be reduced (by density) to the case when M is finite. In that case, the densities f and g in the preceding argument can be replaced by $f_\varepsilon = f + \varepsilon 1$ and $g_\varepsilon = g + \varepsilon 1$ in order to obtain f, g invertible. We are thus left with a finite sequence $x_1, \ldots, x_N$ in M and $f, g \ge 0$ in M_* invertible such that $\tau(f) < 1$ and $\tau(g) < t^2$ and moreover

$$x_n = g a_n = b_n f \quad (n \le N) \tag{3.5}$$

with $a_n, b_n \in M$ such that (3.3) holds. Just like we do for functions and step functions we may approximate f, g by elements of the form $\sum_{i=1}^k f_i P_i$ and $\sum_{j=1}^k g_j Q_j$ where P_i (resp. Q_j) are orthogonal projections in M with $\sum P_i = \sum Q_j = 1$. This modification leads by (3.5) to a perturbation of x_n so it suffices to complete the proof for this special case. If we then denote $x_n(i,j) = P_i x_n Q_j$, and similarly for a_n and b_n, we can essentially repeat the preceding argument using the remark following the above lemma with the measures $\mu = \sum g_j \tau(Q_j)\delta_j$ and $\mu' = \sum f_i \tau(P_i)\delta_i$. ∎

End of the proof of Theorem 3.1. Assume $k_t(a) \le 1$, so that for all $P, Q \in \mathcal{P}$,

$$\sum \|P a_n Q\|_2^2 \le \tau(P) \vee t^{-2}\tau(Q).$$

This implies by Cauchy–Schwarz

$$\begin{aligned}\Big|\sum \tau(P a_n Q y_n)\Big| &\le (\tau(P) \vee t^{-2}\tau(Q))^{1/2} \Big(\sum \|y_n\|_2^2\Big)^{1/2} \\ &\le (t^{-2}\tau(Q) + \tau(P))^{1/2} \Big(\sum \|y_n\|_2^2\Big)^{1/2}\end{aligned}$$

and hence by Lemma 3.2,

$$\Big|\sum \tau(a_n x_n)\Big| \le 2J_t(x).$$

Then by duality we conclude

$$K_t(a; M(R), M(C)) \le 2. \ \blacksquare$$

REMARK. The preceding proof reveals the following slightly surprising fact: Consider a sequence $x = (x_n)$ of operators $x_n \in B(H)$, with say $H = \ell_2$. Assume that for any pair of orthonormal bases (e_i), (f_j) in H the matrix $a_{ij} = \langle e_i, x_n f_j\rangle$ belongs to $\ell_\infty(i; \ell_2(n, j)) + \ell_\infty(j; \ell_2(n, i))$. Then $x = (x_n) \in M(R) + M(C)$ with $M = B(\ell_2)$. Indeed, if E (resp. F) is a finite subset of $\mathbb{N}$, and if P (resp. Q) is the orthogonal projection into span(E) (resp. span(F)), then

$$\Big(\sum \|Pa_nQ\|^2\Big)^{1/2} \max(\tau(P)^{1/2}, \tau(Q)^{1/2})^{-1} = \Big(\sum_{E\times F}\sum |a_n(i,j)|^2 \max\{|E|, |F|\}^{-1}\Big)^{1/2}.$$

So Theorem 3.1 (compare with the Varopoulos Lemma in [28]) implies the preceding fact. Moreover the converse implication also holds since $x \in M(R)$ (resp. $M(C)$) implies $a \in \ell_\infty(i; \ell_2(n, j))$ (resp. $\ell_\infty(j; \ell_2(n, i))$.

REMARK. Let $N \subset M$ be a von Neumann subalgebra such that $\tau_{|N}$ is still semi-finite, so that the conditional expectation $\mathbb{E} : M \to N$ is well defined. It is easy to check that the main result remains valid if we replace the norms of $M(R)$ (resp. $M(C)$) by their conditional versions:

$$\Big\|\sum \mathbb{E}(x_n x_n^*)\Big\|^{1/2} \quad \Big(\text{resp. } \Big\|\sum \mathbb{E}(x_n^* x_n)\Big\|^{1/2}\Big).$$

The formula for k_t has now to be modified by restricting p, q to lie in N.

To put the next corollary in a proper perspective, recall that, according to [20], the elements $a = (a_n)$ in the complex interpolation space $(M(R), M(C))^\theta$ are precisely those such that the operator

$$T_a : x \mapsto \sum a_n x a_n^*$$

is bounded on $L_p(\tau)$, where $p = 1/\theta$.

In the commutative case, a bounded operator $T : L_p(\Omega_1, \mu_1) \to L_p(\Omega_2, \mu_2)$ is called *of strong type* p, and the classical Riesz interpolation theorem says that, in the complex case, if $1 \le p_0 < p_1 \le \infty$ and T is of strong type p_j for $j = 0, 1$ then T is of strong type p for any intermediate p such that $p_0 < p < p_1$. The latter theorem is the founding result for the "complex interpolation method" (see [1]), while the classical Marcinkiewicz theorem is the basis for the "real method". In that context, an operator that is bounded from $L_{p,1}(\mu_1)$ to $L_{p,\infty}(\mu_2)$ is called *of very weak type p*. The generalized version of the Marcinkiewicz theorem then says that, if T is of very weak type p_j for $j = 0, 1$, then T is of strong type p for any $p_0 < p < p_1$.

Let (Ω, μ) be a measure space. Recall that the "weak L_p" space $L_{p,\infty}(\mu)$ consists of all measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\|f\|_{p,\infty} = \sup_{c>0} (c^p \mu\{|f| > c\})^{1/p} < \infty.$$

When $p > 1$, the quasi-norm $\|\cdot\|_{p,\infty}$ is equivalent to the norm

$$\|f\|_{[p,\infty]} = \sup\Big\{ \int_E |f| \frac{d\mu}{\mu(E)^{1/p'}} \;\Big|\; E \subset \Omega \Big\}. \tag{3.6}$$

When $p > 1$, $L_{p',\infty}(\mu)$ is the dual of the "Lorentz space" $L_{p,1}(\mu)$ that can be defined (see e.g. [1]) as consisting of those f such that

$$[f]_{p,1} = \int_0^\infty \mu\{|f| > c\}^{1/p}\, dc < \infty. \tag{3.7}$$

Using the generalized s-numbers from [4], it is easy to define the spaces $L_{p,\infty}(\tau)$ and $L_{p,1}(\tau)$ (or more generally $L_{p,q}(\tau)$ for $1 \le q \le \infty$) associated to (M, τ). The simplest way to describe those is as follows. Given a τ-measurable operator x (in the sense of [4]), let $M_{|x|} \subset M$ denote the von Neumann subalgebra generated by the spectral projections of $|x|$. Then $M_{|x|} \simeq L_\infty(\Omega_{|x|}, \mu_{|x|})$ for some measure space $(\Omega_{|x|}, \mu_{|x|})$ in such a way that the restriction of τ to $M_{|x|}$ coincides with $\mu_{|x|}$ in this identification. Moreover the space $L_0(\Omega_{|x|}, \mu_{|x|})$ of scalar valued measurable functions (that are bounded outside a set of finite measure) can be identified with that of τ-measurable operators affiliated with $M_{|x|}$. The space $L_{p,\infty}(\tau)$ (resp. $L_{p,q}(\tau)$ for $1 \le q \le \infty$) then consists of those x such that, in the latter identification, $|x| \in L_{p,\infty}(\mu_{|x|})$ (resp. $L_{p,q}(\mu_{|x|})$. The duality extends to this setting: we have $L_{p',\infty}(\tau) = L_{p,1}(\tau)^*$ for any $1 \le p < \infty$ (see [4, 26]).

The following two statements appear as analogues for real interpolation of the complex case already treated in [22, 20].

Corollary 3.4. *Let $0 < \theta < 1$. An element $a = (a_n)$ $(x_n \in M)$ belongs to the space $(M(R), M(C))_{\theta,\infty}$ iff the mapping*

$$T_\alpha : x \mapsto \sum a_n x a_n^*$$

is bounded from $L_{p,1}(\tau)$ to $L_{p,\infty}(\tau)$ where $\frac{1}{p} = \frac{1-\theta}{\infty} + \frac{\theta}{1}$, i.e. $p = 1/\theta$. Moreover, the norm in $(M(R), M(C))_{\theta,\infty}$ is equivalent to $a \mapsto \|T_a : L_{p,1}(\tau) \to L_{p,\infty}(\tau)\|^{1/2}$.

Proof. Recall

$$\forall a_0, a_1 > 0 \qquad a_0^{1-\theta} a_1^\theta = \inf_{t>0}\{(1-\theta)a_0 t^\theta + \theta a_1 t^{\theta-1}\}. \tag{3.8}$$

By definition

$$\|a\|_{\theta,\infty} = \sup_{t>0} t^{-\theta} K_t(a; M(R), M(C)).$$

By Theorem 3.1 this is equivalent to $\sup_{t>0} t^{-\theta} k_t(a)$. Note that $(1-\theta)\xi+\theta\eta \le \max(\xi,\eta) \le \max\{\theta^{-1},(1-\theta)^{-1}\}((1-\theta)\xi+\theta\eta)$ for all $\xi,\eta>0$. Therefore

$$\sup_{t>0} t^{-\theta}\big(\max(\tau(P)^{1/2}, t^{-1}\tau(Q)^{1/2})\big)^{-1}$$

is equivalent to

$$\Big(\inf_{t>0}(1-\theta)t^{2\theta}\tau(P)+\theta t^{2\theta-2}\tau(Q)\Big)^{-1/2}$$

or equivalently (by (3.8)) to $\inf_{s>0}\big((1-\theta)s^{\theta}\tau(P)+\theta s^{\theta-1}\tau(Q)\big)^{-1/2} = (\tau(P)^{1-\theta}\tau(Q)^{\theta})^{-1/2}$. Thus we find that $\|a\|^2_{\theta,\infty}$ is equivalent to

$$\begin{aligned}\sup\Big\{\sum \|Pa_nQ\|_2^2(\tau(P)^{1-\theta}\tau(Q)^{\theta})^{-1} \;\Big|\; P,Q\in\mathcal{P}\Big\}\\ =\sup\Big\{\sum \tau(Pa_nQa_n^*)(\tau(P)^{1-\theta}\tau(Q)^{\theta})^{-1} \;\Big|\; P,Q\in\mathcal{P}\Big\}\\ =\sup\{\langle T_a(Q\tau(Q)^{-\theta}), P\tau(P)^{-(1-\theta)}\rangle \mid P,Q\in\mathcal{P}\}.\end{aligned}$$

This last expression is equivalent to

$$\sup\{\langle T_a(x),y\rangle \mid x\in B_{L_{p,1}(\tau)},\ y\in B_{L_{p',1}(\tau)}\}.$$

Indeed, using convex combinations of elements of the form $Q\tau(Q)^{-\theta}$ yields the case of $x\ge 0$ in $B_{L_{p,1}(\tau)}$; then the decomposition $x=x_1-x_2+i(x_3-x_4)$ yields the general case up to a factor of 4. The same reasoning applies to y. So we conclude that $\|a\|^2_{\theta,\infty}$ is equivalent to $\|T_a : L_{p,1}(\tau)\to L_{p,\infty}(\tau)\|$. ∎

Remark. In particular, when $M=B(\ell_2)$ or M_n with n arbitrary, the preceding corollary yields a description of the operator space structure of $(R,C)_{\theta,\infty}$ according to Xu's definition in [30].

Let $1<p\le\infty$. When $p=\infty$, by convention we identify $L_{p,\infty}(\tau)$ with M. Let $M(R;p,\infty)$ denote the space of sequences $a=(a_n)$ with $a_n\in L_{p,\infty}(\tau)$ such that $(\sum a_na_n^*)^{1/2}\in L_{p,\infty}(\tau)$, equipped with the "norm" $\|a\| = \|(\sum a_na_n^*)^{1/2}\|_{p,\infty}$. Similarly, we define $M(C;p,\infty)=\{a=(a_n)\mid (a_n^*)\in M(R;p,\infty)\}$ with $\|a\|_{M(C;p,\infty)} = \|(\sum a_n^*a_n)^{1/2}\|_{p,\infty}$.

Using a simple non-commutative adaptation of the results in [25] along the lines of the proof of Theorem 3.1, we find

Theorem 3.5. *Let $2<p_0,p_1\le\infty$. Let $a=(a_n)$ be a sequence with $a_n\in L_{p_0,\infty}(\tau)+L_{p_1,\infty}(\tau)$ for all n. To abbreviate, set*

$$K_t(a) = K_t(a; M(R;p_0,\infty), M(C;p_1,\infty)),$$

$$k_t(a) = \sup\Big\{\Big(\sum\|Pa_nQ\|_2^2\Big)^{1/2}\max\{\tau(P)^{\alpha_0}, t^{-1}\tau(Q)^{\alpha_1}\}^{-1} \;\Big|\; P,Q\in\mathcal{P}\Big\}$$

where $\alpha_0 = 1/2 - 1/p_0$ *and* $\alpha_1 = 1/2 - 1/p_1$. *Then there are positive constants* c *and* C *(depending only on* p_0, p_1*) such that*

$$\forall t > 0 \quad ck_t(a) \le K_t(a) \le Ck_t(a). \tag{3.9}$$

Proof. Let (Ω, μ) be any measure space. For any measurable $f : \Omega \to \mathbb{R}_+$ we have obviously $\|f\|_{p,\infty}^p = \|f^2\|_{p/2,\infty}^{p/2}$. Therefore, using (3.6), $\|f\|_{p,\infty}$ is equivalent to

$$\sup_{E \in \Omega} \Big(\int_E |f|^2 \, d\mu \Big)^{1/2} \mu(E)^{1/p - 1/2}.$$

Thus we may use on $M(R; p_0, \infty)$ the following equivalent norm:

$$\sup \Big\{ \Big(\tau \Big(\sum x_n x_n^* P \Big) \Big)^{1/2} \tau(P)^{-\alpha_0} \;\Big|\; P \in \mathcal{P} \Big\}.$$

Similarly, we may equip $M(C; p_1, \infty)$ with the equivalent norm

$$\sup \Big\{ \Big(\tau \Big(\sum x_n^* x_n Q \Big) \Big)^{1/2} \tau(Q)^{-\alpha_1} \;\Big|\; Q \in \mathcal{P} \Big\}.$$

Using these norms we find $k_t(a) \le K_t(a)$ by the same reasoning as for Theorem 3.1. To prove the converse, we use duality again and mimic the proof of Theorem 3.1 using as a model the results presented in [25] for the commutative case. ∎

Corollary 3.6. *Consider* $2 < p_0, p_1 \le \infty$ *and* $0 < \theta < 1$. *Let* $a = (a_n)$ *be a sequence in* $L_{p_\theta,\infty}(\tau)$ *where* $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. *Then* $a = (a_n)$ *belongs to the space* $(M(R; p_0, \infty), M(C; p_1, \infty))_{\theta,\infty}$ *iff the operator* T_a *is bounded from* $L_{r,1}(\tau)$ *to* $L_{s,\infty}(\tau)$ *where* r, s *are determined by* $\frac{1-\theta}{\infty} + \frac{\theta}{p_1'} = \frac{1}{r}$ *(i.e.* $r = p_1'/\theta$*) and* $\frac{1-\theta}{p_0} + \frac{\theta}{1} = \frac{1}{s}$ *(i.e.* $s = p_0(1 - \theta + \theta p_0)^{-1}$*). Moreover the norm of* a *in that space is equivalent to* $\|T_a : L_{r,1}(\tau) \to L_{s,\infty}(\tau)\|^{1/2}$.

Proof. Using the equivalent of K_t found in Theorem 3.5, we obtain

$$\sup t^{-\theta} K_t(a) \simeq \sup \Big\{ \Big(\sum \|P a_n Q\|_2^2 \Big)^{1/2} \tau(P)^{-\alpha_0(1-\theta)} \tau(Q)^{-\alpha_1 \theta} \Big\}.$$

The unit ball for this last norm is characterized by

$$\sum \tau(P a_n Q a_n^*) \le \tau(P)^{2\alpha_0(1-\theta)} \tau(Q)^{2\alpha_1 \theta} \le \tau(P)^{(1-\theta)/p_0'} \tau(Q)^{\theta/p_1'}$$

or equivalently

$$\langle T_a(Q), P \rangle \le \tau(Q)^{1/r} \tau(P)^{1/s'}.$$

As before, this implies that for all $x, y \ge 0$,

$$|\langle T_a(x), y \rangle| \le \|x\|_{r,1} \|y\|_{s',1}$$

and then using $x = x_1 - x_2 + i(x_3 - x_4)$ we can extend it to arbitrary elements up to an extra factor 4. Thus we conclude by homogeneity

$$\sup_{t>0} t^{-\theta} K_t(a) \simeq \|T_a : L_{r,1}(\tau) \to L_{s,\infty}(\tau)\|^{1/2}. \;\blacksquare$$

REMARK 3.7. By [16] for any interpolation pair (B_0, B_1) we have

$$(B_0 \cap B_1, B_0 + B_1)_{\theta,q} = \begin{cases} B_{\theta,q} \cap B_{1-\theta,q} & \text{if } \theta \le 1/2, \\ B_{\theta,q} + B_{1-\theta,q} & \text{if } \theta \ge 1/2, \end{cases}$$

where $B_{\theta,q} = (B_0, B_1)_{\theta,q}$. If we apply this to the specific pair

$$B_0 = S_\infty[R], \quad B_1 = S_\infty[C],$$

the result can be interpreted in terms of operator space interpolation in Xu's sense (see [30]). This shows that we have *completely isomorphically*

$$(R \cap C, R + C)_{\theta,\infty} = \begin{cases} C_{\theta,\infty} \cap R_{\theta,\infty}, & \theta \le 1/2, \\ C_{\theta,\infty} + R_{\theta,\infty}, & \theta \ge 1/2, \end{cases}$$

where $C_{\theta,q} = (R, C)_{\theta,q}$ and $R_{\theta,q} = (C, R)_{\theta,q} = C_{1-\theta,q}$. Note that in particular, we have

$$(R \cap C, R + C)_{1/2,\infty} \simeq R_{1/2,\infty} \simeq C_{1/2,\infty}$$

as operator spaces, and similarly, by duality (see [30, §4]), for $(1/2, 1)$.

4. Non-commutative Khintchine inequality in $L_{2,q}$ $(1 \le q < \infty)$. This section is motivated by [9]. In [9], martingale inequalities extending those of [26] are proved for the non-commutative Lorentz spaces $L_{p,q}(\tau)$ associated to a semi-finite trace with $p \ne 2$. However, the case $p = 2, q \ne 2$ cannot be treated by the interpolation arguments used in [9]. In fact, even the simpler case of the Khintchine inequality is open. The problem is to find a "nice" (similarly to the case of $L_p(\tau)$ presumably involving row and column norms) equivalent of the average over all signs $\varepsilon_n = \pm 1$ of

$$\Big\| \sum \varepsilon_n x_n \Big\|_{2,q} \tag{4.1}$$

when $x_n \in L_{2,q}(\tau)$. In this section we present a partial solution, which has the advantage of being indeed a deterministic equivalent of (4.1). We call it partial because there may be a more explicitly computable equivalent for (4.1).

NOTATION. Recall that S_p denotes the Schatten p-class $(1 \le p < \infty)$, S_∞ the space of compact operators on Hilbert space, and S_1 the trace class. We will denote by $S_\infty(R)$ (resp. $S_\infty(C)$) the space of all sequences $x = (x_n)$ with $x_n \in S_\infty$ such that the series $\sum_{n=1}^\infty x_n x_n^*$ (resp. $\sum_{n=1}^\infty x_n^* x_n$) converges in norm, and we equip it with the norm $\|x\|_R = \|(\sum x_j x_j^* r)^{1/2}\|$ (resp. $\|x\|_C = \|(\sum x_j^* x_j)^{1/2}\|$). We then define

$$A_0 = S_\infty(R) \cap S_\infty(C)$$

and we equip it with the norm $\|x\|_{A_0} = \max\{\|x\|_R, \|x\|_C\}$.

We denote by $S_1(R)$ (resp. $S_1(C)$) the space that was already introduced as $M_*(R)$ (resp. $M_*(C)$) when $M_* = S_1, M = B(\ell_2)$ (see (2.1)).

We then define

$$A_1 = S_1(R) + S_1(C)$$

and we equip it with the norm $\|x\|_{A_1} = \|x\|_{S_1(R)+S_1(C)} = \inf_{x=y+z}(\|y\|_{S_1(R)} + \|z\|_{S_1(C)})$.

Clearly, the couple (A_0, A_1) forms a compatible couple in the sense of interpolation. We denote, for any $1 < p < \infty$, $1 \le q \le \infty$,

$$|||x|||_{p,q} = \|x\|_{(A_0,A_1)_{\theta,q}} \quad \text{where} \quad \theta = \frac{1-\theta}{\infty} + \frac{\theta}{1} = \frac{1}{p}.$$

Note that $A_1 = A_0^*$ isometrically (in the usual duality defined by $\langle x, y\rangle = \sum \mathrm{tr}(x_n{}^t y_n)$) and by a well known result (see [2] and references there) this implies $(A_0, A_1)_{1/2,2} = \ell_2(S_2)$ isomorphically. Moreover, the couple (A_0^*, A_1^*) can be clearly viewed as compatible and we have (see [1, p. 54]), for $0 < \theta < 1$ and $1 \le q < \infty$,

$$(A_0, A_1)^*_{\theta,q} = (A_0^*, A_1^*)_{\theta,q'} \tag{4.2}$$

with equivalent norms.

Theorem 4.1. *Let (ε_n) be the usual independent ± 1 valued random variables ("Rademacher functions"). Then for any $1 < p < \infty$ and $1 \le q < \infty$ and for any finite sequence $x = (x_n)$ of operators in $S_{p,q}$,*

$$\int \Big\|\sum \varepsilon_n x_n\Big\|_{S_{p,q}} d\mu \simeq |||x|||_{p,q} \tag{4.3}$$

and also

$$\Big\|\sum \varepsilon_n \otimes x_n\Big\|_{L_{p,q}(\mu\times \mathrm{tr})} \simeq |||x|||_{p,q} \tag{4.4}$$

where μ is the usual probability on $\{-1,1\}^{\mathbb{N}}$. Moreover, (4.4) *remains valid for $q = \infty$. Here $A \simeq B$ means there are positive constants $c_{p,q}$ and $C_{p,q}$ such that $c_{p,q}A \le B \le C_{p,q}A$.*

Remark. It is not difficult to extend this theorem to the case when the trace on $B(\ell_2)$) is replaced by any semi-finite faithful normal trace on a von Neumann algebra, but we choose for simplicity to present the details only in the case of $B(\ell_2)$. Indeed, all the ingredients for this extension now exist in the literature (see [26]).

Remark 4.2. Note that the space $M(R) \cap M(C)$ with $M = B(\ell_2)$ considered in §3 is nothing but the bidual A_0^{**} of A_0. Using this (and truncation of matrices in the most usual way), one can check that, for any $x \in A_0 + A_1$ and any $t > 0$, we have $K_t(x; A_0^{**}, A_1) = K_t(x; A_0, A_1)$ and hence the norms of the spaces $(A_0, A_1)_{\theta,q}$ and $(A_0^{**}, A_1)_{\theta,q}$ coincide on any such x for any $0 < \theta < 1$ and $1 \le q \le \infty$.

Remark 4.3. Note that by interpolation for any $1 < p < \infty$ (and $1 \le q \le \infty$) the orthogonal projection onto $\overline{\text{span}}[\varepsilon_n]$ is bounded on $L_{p,q}(\mu \times \text{tr})$. Indeed, we know that it is bounded on $L_{p_0}(\mu \times \text{tr})$ and $L_{p_1}(\mu \times \text{tr})$ for $1 < p_0 < p < p_1$, and then we can use interpolation, together with the classical reiteration theorem (see [1, p. 48]).

Remark 4.4. The equivalence between $[\varepsilon_n]$ and Sidon lacunary sequences such as $[z^{2^n}]$ proved in [17] implies that, for any $1<p<\infty$ and $1 \le q \le \infty$,

$$\Big\|\sum \varepsilon_n \otimes x_n\Big\|_{L_{p,q}(\mu\times\text{tr})} \simeq \Big\|\sum z^{2^n} \otimes x_n\Big\|_{L_{p,q}(m\times\text{tr})}$$

where m is normalized Haar measure on the unit circle $\mathbb{T}$. Again this is true by [17] on L_{p_i}, $p_0 < p < p_1$ (with simultaneous complementation), so this follows by interpolation.

Remark 4.5. Let E be any Banach space. When $1 \le p < \infty$, we denote by $L_p(E)$ the space of E-valued functions p-integrable on the unit circle $\mathbb{T}$, in Bochner's sense, with the usual norm. We denote by $H^p(E)$ the subspace of all functions f with Fourier transform $\hat{f}$ supported on the non-negative integers. The case $p=\infty$ is slightly different. We denote by $L_\infty(B(\ell_2))$ the space of essentially bounded weak-$*$ measurable functions on the unit circle with values in $B(\ell_2)$), equipped with the sup-norm. We again denote by $H^\infty(B(\ell_2))$ (resp. $\overline{H}_0^\infty(B(\ell_2))$) the subspace consisting of all functions with Fourier transform supported on the non-negative (resp. negative) integers.

By [18, Cor. 3.4] we know that the pair $(H^\infty(B(\ell_2)), H^1(S_1))$ is K-closed in $(L_\infty(B(\ell_2)), L_1(S_1))$. It follows that

$$\begin{aligned}\Big\|\sum_{n=1}^{\infty} z^{2^n} x_n\Big\|_{(H^\infty(B(\ell_2)),H^1(S_1))_{\theta,q}} &\simeq \Big\|\sum z^{2^n} x_n\Big\|_{(L_\infty(B(\ell_2)),H^1(S_1))_{\theta,q}}\\ &\simeq \Big\|\sum z^{2^n} \otimes x_n\Big\|_{L_{p,q}(m\times\text{tr})}.\end{aligned}$$

Remark 4.6. From [15] we know that the mapping $T: H^1(S_1) \to A_1$ defined by $Tf = (\hat{f}(2^n))$ is a bounded surjection from $H^1(S_1)$ to $A_1 = S_1(R) + S_1(C)$. Moreover, the (adjoint) map taking $x=(x_n)$ to $\sum z^{2^n} x_n$ is bounded from $A_1 = S_1(R)+S_1(C)$ to $H^1(S_1)$. Using the identity $H^1(S_1)^* = L_\infty(B(\ell_2))/\overline{H}_0^\infty(B(\ell_2))$, by duality, this implies that

$$T \text{ is bounded from } L_\infty(B(\ell_2))/\overline{H}_0^\infty(B(\ell_2)) = H_1(S_1)^* \text{ onto } A_1^*.$$

By interpolating, we find

$$T: (L_\infty(B(\ell_2))/\overline{H}_0^\infty(B(\ell_2)), H^1(S_1))_{\theta,q} \to (A_1^*, A_1)_{\theta,q}.$$

Note that the natural "inclusion" $H^\infty(B(\ell_2)) \to L_\infty/\overline{H}_0^\infty(B(\ell_2))$ trivially has norm ≤ 1. Therefore

$$T: (H^\infty(B(\ell_2)), H^1(S_1))_{\theta,q} \to (A_1^*, A_1)_{\theta,q}.$$

Note that $A_1^* = A_0^{**}$. Using Remark 4.2, it is now easy to check that for any *finite* sequence $x = (x_n)$ with $x_n \in S_{p,q} \subset S_\infty$, the norms of $(A_1^*, A_1)_{\theta,q}$ and $(A_0, A_1)_{\theta,q}$ coincide on x for any $1 < p < \infty, 1 \le q \le \infty$ (here $\theta = 1/p$).

Thus, invoking again [18, Cor. 3.4], by the preceding two remarks we find

LEMMA 4.7. *Let $1 < p < \infty$ and $1 \le q \le \infty$. Then T is bounded from the subspace of all analytic polynomials in $L_{p,q}(m \times \mathrm{tr})$ into $(A_0, A_1)_{\theta,q}$. In particular, for some c, for all analytic polynomials f with coefficients in $S_{p,q} = L_{p,q}(\mathrm{tr})$,*

$$|||(\hat{f}(2^n))|||_{p,q} \le c\|f\|_{L_{p,q}(m\times \mathrm{tr})}.$$

First part of the proof of Theorem 4.1. Taking $f = \sum z^{2^n} x_n$ in the preceding lemma we get

$$|||x|||_{p,q} \lesssim \Big\|\sum z^{2^n} x_n\Big\|_{L_{p,q}(m\times \mathrm{tr})}.$$

By duality we get (using Remark 4.3 and (4.2))

$$\Big\|\sum x^{2^n} x_n\Big\|_{L_{p',q'}(m\times \mathrm{tr})} \lesssim |||x|||_{p',q'},$$

and since this holds for *all* $1 < p < \infty$ and $1 \le q \le \infty$ (the case $q = \infty, q' = 1$ requires a minor adjustment, see [1, Remark, p. 55]), we deduce

$$|||x|||_{p,q} \simeq \Big\|\sum z^{2^n} x_n\Big\|_{L_{p,q}(m\times \mathrm{tr})}$$

and hence by Remark 4.4,

$$|||x|||_{p,q} \simeq \Big\|\sum \varepsilon_n x_n\Big\|_{L_{p,q}(\mu\times \mathrm{tr})}.$$

This proves (4.4). ∎

To complete the proof we use the following rather standard

LEMMA 4.8. *For any $1 < p < \infty$ and $1 \le q < \infty$, any $f \in L_{p,q}(\mu \times \mathrm{tr})$ of the form $f = \sum \varepsilon_n x_n$ with $x_n \in S_{p,q} = L_{p,q}(\mathrm{tr})$ and any $r < \infty$ we have*

$$\|f\|_{L_r(S_{p,q})} \lesssim \|f\|_{L_{p,q}(\mu\times \mathrm{tr})}.$$

Proof. We will repeatedly use Kahane's inequality for which we refer to [17] or [12, p. 100]. By K-convexity, the mapping $P : L_{p,p}(\mu\times \mathrm{tr}) \to L_r(\mu; S_p)$ defined by (projection onto span$[\varepsilon_n]$)

$$Pf = \sum \langle f, \varepsilon_n\rangle \varepsilon_n$$

is bounded for any value of $1 < p < \infty$ and (by Kahane's inequality) $1 \le r < \infty$. By interpolation, it follows that P is bounded as a map

$$(L_{p_0,p_0}(\mu \times \mathrm{tr}), L_{p_1,p_1}(\mu \times \mathrm{tr}))_{\alpha,q} \to (L_r(S_{p_0}), L_r(S_{p_1}))_{\alpha,q}$$

for any $0 < \alpha < 1$, $1 \le q \le \infty$.

We may choose $p_0 < p < p_1$, $\frac{1}{p} = \frac{1-\alpha}{p_0} + \frac{\alpha}{p_1}$ and $r \geq q$ (here we use $q < \infty$) to get

$$P : L_{p,q}(\mu \times \mathrm{tr}) \to (L_r(S_{p_0}), L_r(S_1))_{\alpha,q}$$

but now $r \geq q$ guarantees that $L_q(L_r) \subset L_r(L_q)$ and hence

$$(L_r(S_{p_0}), L_r(S_{p_1}))_{\alpha,q} \subset L_r((S_{p_0}, S_{p_1})_{\alpha,q}) = L_r(S_{p,q}).$$

Thus Lemma 4.8 follows by restricting to $f = \sum \varepsilon_n x_n$. ∎

End of proof of Theorem 4.1. By Lemma 4.8 we get

$$\Big\|\sum \varepsilon_n x_n\Big\|_{L_r(\mu;S_{p,q})} \lesssim |||x|||_{p,q}$$

and hence again by duality

$$|||x|||_{p',q'} \lesssim \Big\|\sum \varepsilon_n x_n\Big\|_{L_{r'}(\mu;S_{p',q'})}.$$

Then we conclude since Kahane's inequality allows us to replace both r and r' by, say, 2. ∎

Remark. Using Fernique's inequality in place of Kahane's (see [12]) it is easy to see that the preceding proof of (4.3) and (4.4) extends to the case when (ε_n) is replaced by a sequence of i.i.d. Gaussian normal random variables. This implies (4.3) and (4.4). Note however (see [12, p. 253]) that the Gaussian and Rademacher averages are not equivalent when $q = \infty$.

Remark 4.9. Let $A_\theta = (A_0, A_1)_\theta$. Let us denote by $\mathrm{Rad}(S_p)$ the closure in $L_p(\mu \times \mathrm{tr})$ of the set of finite sums $\sum \varepsilon_n x_n$ with $x_n \in S_p$. By the description of A_θ obtained in [24] (see also [3]), we know that if we define p by $\frac{1}{p} = \frac{1-\theta}{\infty} + \frac{\theta}{1}$, then by Theorem 4.1,

$$(A_0, A_1)_{\theta,p} = \mathrm{Rad}(S_p) = \begin{cases} A_\theta \cap A_{1-\theta} & \text{if } \theta \leq 1/2, \\ A_\theta + A_{1-\theta} & \text{if } \theta \geq 1/2, \end{cases}$$

and in particular

$$(A_0, A_1)_{\theta,p} \simeq (A_0, A_1)_\theta.$$

But this can also be seen using (6.1) and the identity $L_{p,p} = L_p$ relative to $\varphi \times \mathrm{tr}$ (and a simultaneous complementation argument) where $(\mathcal{M}, \varphi)$ is the free group II_1 factor.

From Theorem 4.1, it is natural to search for an equivalent of the K-functional for (A_0, A_1):

Problem. Find an explicit description of $K_t(x; A_0, A_1)$ (presumably in terms of x, R and C).

5. Remarks on real interpolation. We need more notation about the Lorentz space version of the Schatten classes.

NOTATION. We denote by $X^c_{p,q}$ (resp. $X^r_{p,q}$) the space of all sequences $x = (x_n)$ with $x_n \in S_{p,q}$ such that the series $\sum x_n^* x_n$ (resp. $\sum x_n x_n^*$) (assumed to be w.o.t. convergent) satisfies $(\sum x_n^* x_n)^{1/2} \in S_{p,q}$ (resp. $(\sum x_n x_n^*)^{1/2} \in S_{p,q}$). We equip these spaces with the norms

$$\|(x_n)\|_{X^c_{p,q}} := \Big\|\Big(\sum x_n^* x_n\Big)^{1/2}\Big\|_{S_{p,q}} \quad \text{and} \quad \|(x_n)\|_{X^r_{p,q}} := \|(x_n^*)\|_{X^c_{p,q}}.$$

Lastly, we set

$$X^c_p = X^c_{p,p} \quad \text{and} \quad X^r_p = X^r_{p,p}.$$

Recall the following facts and terminology from [18]. Consider a compatible couple (X_0, X_1) of Banach (or quasi-Banach) spaces. Assume $\mathcal{S} \subset X_0 + X_1$ is a closed subspace and let

$$\mathcal{S}_0 = \mathcal{S} \cap X_0, \qquad \mathcal{S}_1 = \mathcal{S} \cap X_1.$$

Let $Q_0 = X_0/\mathcal{S}_0$ and $Q_1 = X_1/\mathcal{S}_1$ be the associated quotient spaces. Clearly (Q_0, Q_1) is a compatible couple since there are natural inclusion maps

$$Q_0 \to (X_0 + X_1)/\mathcal{S} \quad \text{and} \quad Q_1 \to (X_0 + X_1)/\mathcal{S}.$$

We say that the couple $(\mathcal{S}_0, \mathcal{S}_1)$ is K-*closed* (relative to (X_0, X_1)) if there is a constant c such that

$$\forall t > 0 \ \forall x \in \mathcal{S}_0 + \mathcal{S}_1 \quad K_t(x; \mathcal{S}_0, \mathcal{S}_1) \le c K_t(x; X_0, X_1). \tag{5.1}$$

In the terminology of [18], (Q_0, Q_1) is called J-*closed* if, for some constant c, any element $x \in Q_0 \cap Q_1$ admits a simultaneous lifting $\hat{x}$ satisfying

$$\forall t > 0 \quad J_t(\hat{x}; X_0, X_1) \le c J_t(x; Q_0, Q_1).$$

In the present paper we say that $(\mathcal{S}_0, \mathcal{S}_1)$ is J-*closed* when (Q_0, Q_1) is J-closed in the above sense.

Equivalently, we will say that $(\mathcal{S}_0, \mathcal{S}_1)$ is J-closed if there is a constant c such that for any $x \in X_0 \cap X_1$ there is a single $\widehat{x} \in S_0 \cap S_1$ satisfying $\mathrm{dist}_{X_j}(x, \widehat{x}) \le c\, \mathrm{dist}_{X_j}(x, \mathcal{S}_j)$ for both $j = 0$ and $j = 1$. The basic (simple but useful) fact on which [18] rests is that, with our new terminology, K-closed is equivalent to J-closed (this statement should not be confused with the more obvious fact associated to the duality between the K and J norms or between subspaces and quotients). Note also that this is valid for quasi-normed spaces.

EXAMPLE. Consider $1 \le p_0, p_1 \le \infty$. Let $X^c_j = X^c_{p_j}$, $X^r_j = X^r_{p_j}$, for $j = 0, 1$. Let $X_j = X^c_j \oplus X^r_j$ and $\mathcal{S}_j \subset X_j$, $\mathcal{S}_j := \{(x, -x)\}$. Then

$$\mathcal{S}_j \simeq X^c_j \cap X^r_j \quad \text{and} \quad X_j/\mathcal{S}_j \simeq X^c_j + X^r_j.$$

Note also the special case: if $p_1 = 2$, then $X_1/\mathcal{S}_1 \simeq \mathcal{S}_1 \simeq \ell_2(S_2)$.

Proposition 5.1. *The above pair* $(\mathcal{S}_0, \mathcal{S}_1)$ *is* J*-closed (and hence* K*-closed) in the following cases:*

(i) $0 < p_0 < 2$ *and* $p_1 = 2$.
(ii) $2 < p_0 \le \infty$ *and* $p_1 = 2$.

Consequently, in both cases, for any p *strictly between* p_0 *and* $p_1 = 2$ *and any* $1 \le q \le \infty$, *with* θ *defined by* $1/p = (1-\theta)/p_0 + \theta/2$, *we have (with equivalent norms)*

$$(\mathcal{S}_0, \mathcal{S}_1)_{\theta,q} = X^c_{p,q} \cap X^r_{p,q} \quad \text{and} \quad (X_0/\mathcal{S}_0, X_1/\mathcal{S}_1)_{\theta,q} = X^c_{p,q} + X^r_{p,q}.$$

Proof. To show $(\mathcal{S}_0, \mathcal{S}_1)$ is J-closed is the same as showing the following

Claim. *There exists* $c > 0$ *such that the following holds. Given* $x = (x_n)$, $x_n \in S_{p_0}$, *there are* $y = (y_n)$ *and* $z = (z_n)$ *such that* $x = y + z$ *and we have* simultaneously, *for* $j = 0$ *and* $j = 1$,

$$\|y\|_{X^r_{p_j}} + \|z\|_{X^c_{p_j}} \le c\|x\|_{X^r_{p_j} + X^c_{p_j}}.$$

To prove this, we fix $\varepsilon > 0$ and choose y^0, z^0 such that $x = y^0 + z^0$ and $\|y^0\|_{X^r_{p_0}} + \|z^0\|_{X^c_{p_0}} < \|x\|_{X^r_{p_0}+X^c_{p_0}} + \varepsilon$.

Let then $\xi = (\sum y^0(n) y^0(n)^*)^{1/2}$ and $\eta = (\sum z^0(n)^* z^0(n))^{1/2}$. We have $\|\xi\|_{p_0} = \|y^0\|_{X^r_{p_0}}$ and $\|\eta\|_{p_0} = \|z^0\|_{X^c_{p_0}}$. We can assume (by perturbation) that $\xi > 0$ and $\eta > 0$ so that ξ, η are invertible. We introduce the states

$$f = \xi^{p_0}(\mathrm{tr}(\xi^{p_0}))^{-1}, \qquad g = \eta^{p_0}(\mathrm{tr}(\eta^{p_0}))^{-1}.$$

Let a be defined by $1/a = 1/p_0 - 1/2$. We use the decomposition

$$x = (\|\xi\|_{p_0} f^{1/a})\widehat{y}^0 + \widehat{z}^0 \cdot (\|\xi\|_{p_0} g^{1/a})$$

where $\widehat{y}^0 = (\|\xi\|_{p_0} f^{1/a})^{-1} y^0$ and $\widehat{z}^0 = z^0 \cdot (\|\xi\|_{p_0} g^{1/a})^{-1}$. Noting that $f = (\xi\|\xi\|_{p_0}^{-1})^{p_0}$ we have

$$\begin{aligned}\sum \|\widehat{y}^0(n)\|_2^2 &= \mathrm{tr} \sum \widehat{y}^0(n)\widehat{y}^0(n)^* = \mathrm{tr}\Big(\|\xi\|_{p_0}^{-2} f^{-1/a} \sum y^0(n) y^0(n)^* f^{-1/a}\Big)\\ &= \mathrm{tr}(f^{-1/a}(\xi/\|\xi\|_{p_0})^2 f^{-1/a}) = \mathrm{tr}(f^{2/p_0 - 2/a}) = \mathrm{tr}(f) = 1.\end{aligned}$$

Similarly $\sum \|\widehat{z}^0(n)\|_2^2 = 1$.

To simplify notation let $\alpha = \|\xi\|_{p_0} f^{1/a}$ and $\beta = \|\xi\|_{p_0} g^{1/a}$. We have $x = \alpha \cdot \widehat{y}^0 + \widehat{z}^0 \cdot \beta$. Let T be the map (Schur multiplier with respect to the bases in which α, β are diagonal) defined by $T(t) = \alpha t + t\beta$, $t \in S_\infty$. Consider the decomposition

$$x = T(T^{-1}(x)) = \alpha T^{-1}(x) + T^{-1}(x)\beta.$$

We set $y_n = \alpha T^{-1}(x_n)$ and $z_n = T^{-1}(x_n)\beta$. Clearly (since $0 \le \frac{\alpha_i}{\alpha_i + \beta_j} \le 1$) we have

$$\|y_n\|_{S_2} \le \|x_n\|_{S_2}, \quad \|z_n\|_{S_2} \le \|x_n\|_{S_2}$$

and hence $\|y\|_{\ell_2(S_2)} \le \|x\|_{\ell_2(S_2)}, \|z\|_{\ell_2(S_2)} \le \|x\|_{\ell_2(S_2)}$. But also $x_n = \alpha\widehat{y}^0(n) + \widehat{z}^0(n)\beta$ and

$$\|T^{-1}(x_n)\|_{S_2} = \left\| \frac{1}{\alpha_i+\beta_j}[\alpha_i\widehat{y}^0(n)_{ij} + z^0(n)_{ij}\beta_j] \right\|_{S_2} \le \|\widehat{y}^0(n)\|_{S^2} + \|\widehat{z}^0(n)\|_{S_2},$$

as

$$\Big(\sum \|T^{-1}(x_n)\|_{S_2}^2\Big)^{1/2} \le \Big(\sum \|\widehat{y}^0(n)\|_{S_2}^2\Big)^{1/2} + \Big(\sum \|\widehat{z}^0(n)\|_{S_2}^2\Big)^{1/2} \le 2.$$

Therefore we conclude that (we set $t_n = T^{-1}(x_n)$)

$$\begin{aligned}\|y\|_{X^r_{p_0}} = \|(\alpha T^{-1}(x_n))\|_{X^r_{p_0}} &= \left\|\alpha\Big(\sum t_n t_n^*\Big)^{1/2}\right\|_{S_{p_0}} \\ &\le 2\|\alpha\|_{S_a} \le 2\|\xi\|_{p_0} = 2\|y_0\|_{X^r_{p_0}}\end{aligned}$$

and similarly

$$\|z\|_{X^c_{p_0}} \le 2\|z_0\|_{X^c_{p_0}}.$$

This proves our claim, whence (i) and (ii) follow by duality. ∎

Corollary 5.2. *Recall that* $A_0 = S_\infty(R) \cap S_\infty(C)$ *and* $A_1 = S_1(R) + S_1(C)$. *We have*

$$\begin{aligned}(A_0, A_1)_{\theta,q} &= X^r_{p,q} \cap X^c_{p,q} \quad \text{if } \theta < 1/2,\\ (A_0, A_1)_{\theta,q} &= X^r_{p,q} + X^c_{p,q} \quad \text{if } \theta > 1/2.\end{aligned}$$

Proof. As already mentioned, by a well known self-duality result (note $A_1 \simeq A_0^*$) we have

$$(A_0, A_1)_{1/2,2} \simeq \ell_2(S_2) \simeq X^r_{2,2} \cap X^c_{2,2} \simeq X^r_{2,2} + X^c_{2,2}.$$

So the corollary follows immediately by reiteration (see [1, p. 48]). ∎

Remark. The complex analogue of the preceding statement was proved in [23, p. 109] (see also [3]).

Remark. It is known (see e.g. [8] or [10]) that maps of the form $t \mapsto \alpha T^{-1}(t), t \mapsto T^{-1}(t)\beta$ (corresponding to the Schur multipliers $[\alpha_i(\alpha_i+\beta_j)^{-1}]$ and $[\beta_j(\alpha_i+\beta_j)^{-1}]$) are c.b. on S_Q for any $1 < Q < \infty$. In particular, in the preceding proof, we have, for some constant c_Q,

$$\|y\|_{X^r_Q} \le c_Q\|x\|_{X^r_Q}, \qquad \|z\|_{X^c_Q} \le c_Q\|x\|_{X^c_Q},$$

and hence $\|y\|_{X^r_Q} + \|z\|_{X^c_Q} \le c_Q\|x\|_{X^r_Q \cap X^c_Q}$. Now fix $2 < Q < \infty$. Using this simultaneous selection for the pair $(1, Q)$, one can show that for any $1 < p < Q$ and $1 \le q \le \infty$,

$$\|x\|_{X^r_{p,q}+X^c_{p,q}} \lesssim \|x\|_{(X^r_1+X^c_1, X^r_Q\cap X^c_Q)_{\theta,q}}$$

and hence (reiteration again)

$$\|x\|_{X^r_{p,q}+X^c_{p,q}} \lesssim \|x\|_{A_{p,q}}$$

where
$$A_{p,q} := (A_0, A_1)_{\theta,q}, \qquad A_0 = X_1^r + X_1^c, \qquad A_1 = X_\infty^r \cap X_\infty^c.$$
By duality, we also have
$$\|x\|_{A_{p,q}} \lesssim \|x\|_{X_{p,q}^r \cap X_{p,q}^c}.$$

REMARK. In particular we recover a result claimed in [13, remark after Corollary 4.3] (see also [3]),
$$\|x\|_{X_{2,q}^r + X_{2,q}^c} \lesssim \|x\|_{A_{2,q}} \lesssim \|x\|_{X_{2,q}^r \cap X_{2,q}^c}.$$
Exchanging the rôles of 1 and ∞, we obtain

COROLLARY 5.3. *Let $B_0 = X_1^r \cap X_1^c$ and $B_1 = X_\infty^r + X_\infty^c$. Then*
$$(B_0, B_1)_{\theta,q} = \begin{cases} X_{p,q}^r \cap X_{p,q}^c & \text{if } \theta < 1/2, \\ X_{p,q}^r + X_{p,q}^c & \text{if } \theta > 1/2. \end{cases}$$

REMARK. It is not difficult to extend the results of this section to a general non-commutative L_p-space (associated to a semi-finite faithful normal trace) in place of S_p.

REMARK. I do not know whether Proposition 5.1 is valid for arbitrary $p_0 \neq p_1$.

6. Connection with free probability. We refer to [29] for all undefined notions in this section. Let $(\mathcal{M}, \varphi)$ be a "non-commutative probability space", i.e. $(\mathcal{M}, \varphi)$ is as (M, τ) before but we impose $\varphi(1) = 1$.

Let (ξ_n) be a free family in $\mathcal{M}$. In what follows we assume that (ξ_n) is either a free semi-circular (sometimes called "free Gaussian") family, or a free circular one (this corresponds to complex valued Gaussians) or a (free) family of Haar unitaries. The latter are the free analogues of Steinhaus random variables. They can be realized as the free generators of the free group $\mathbb{F}_\infty$ in the associated von Neumann algebra (the so-called "free group factor"). We could also include the free analogue of the Rademacher functions, i.e. a free family of copies of a single random choice of sign $\varepsilon = \pm 1$. See [27] for a discussion of more general free families. Consider now a finite sequence $x = (x_n)$ with $x_n \in L_{p,q}(M, \tau)$.

The free analogue of Khintchine's inequality is the following fact that was (essentially) observed in [5]. There are absolute positive constants c, C such that for any $1 \leq p, q \leq \infty$,
$$c|||x|||_{p,q} \leq \Big\| \sum \xi_n \otimes x_n \Big\|_{L_{p,q}(\varphi \otimes \tau)} \leq C|||x|||_{p,q}. \tag{6.1}$$
In [5] only the cases $p = 1$ and $p = \infty$ are considered, but it is also pointed out there that the orthogonal projection onto $\overline{\text{span}}[\xi_n]$ is completely bounded on $L_p(\varphi)$ for both $p = 1$ and $p = \infty$. From that simultaneous complementation, it is then routine to deduce (6.1).

Remark. In particular, $\|x\|_{2,q} \simeq \|\sum \xi_n \otimes x_n\|_{L_{2,q}(\varphi\times\tau)}$. Perhaps our problem to compute $\|x\|_{2,q}$ more explicitly can be tackled by a more detailed study of the distribution of the "non-commutative variable" $\sum \xi_n \otimes x_n$. But while, in Voiculescu's theory of the free Gaussian case, the "R-transform" (free analogue of the Fourier transform) can be calculated, it is not clear (even in case ξ_n, x_n and hence $\sum \xi_n \otimes x_n$ are all self-adjoint) how to use it to estimate the spectral distribution of $\sum \xi_n \otimes x_n$ or, say, its $L_{2,q}$ norm.

Combining this with Theorem 4.1 (with a general M in place of $B(H)$) we find:

Proposition. *For any $1 \le p < \infty$ and $1 \le q \le \infty$ and for any sequence $x = (x_n)$ of operators in $L_{p,q}(\tau)$ we have*

$$\Big\|\sum \varepsilon_n \otimes x_n\Big\|_{L_{p,q}(\mu\times\tau)} \simeq \Big\|\sum \xi_n \otimes x_n\Big\|_{L_{p,q}(\varphi\times\tau)}. \tag{6.2}$$

Note that this clearly fails for $p = \infty$ since $\operatorname{span}_{L_\infty(\mu)}\{\varepsilon_n\} \simeq \ell_1$ while $\operatorname{span}_{L_\infty(\varphi)}\{\xi_n\} \simeq \ell_2$.

We refer the reader to [6] for far reaching extensions of (6.1) or (6.2) involving random matrices. It is tempting to look for a direct, more conceptual proof of (6.2) but this has always eluded us (see however [3]). Note also that analogues of (6.1) and (6.2) (as well as (4.3)) are entirely open for $0 < p < 1$.

Remark 6.1. Proposition 5.1(ii) yields some information on the distribution function of sums such as $S = \sum \xi_n \otimes x_n$. Since the spans of the variables (ξ_n) are completely complemented simultaneously in $L_1(\varphi)$ and $L_\infty(\varphi)$ (a fortiori they form a K-closed pair), it is easy to check that, if we set $S = \sum c_n \otimes x_n$ or $S = \sum \lambda(g_n) \otimes x_n$, then uniformly over t,

$$K_t(x; A_1, A_0) \simeq K_t(S; L_1(\tau\times \mathrm{tr}), L_\infty(\tau\times\mathrm{tr})) = \int_0^t S^\dagger(s)\,ds,$$

where $S^\dagger(s)$ denotes the generalized s-number of S in the sense of [4]. Similarly, by a well known fact (see [1, p. 109]), we have, uniformly over t,

$$K_t(x; A_{1/2,2}, A_0) \simeq K_t(S; L_2(\tau\times\mathrm{tr}), L_\infty(\tau\times\mathrm{tr})) \simeq \Big(\int_0^{t^2} S^\dagger(s)^2\,ds\Big)^{1/2}.$$

The fact that the pair $(X^r_\infty \cap X^c_\infty, X^r_2 \cap X^c_2)$ is K-closed seems to yield some further information. Indeed, the K_t-functional for that pair can be estimated simply from the corresponding result for the (easy) pairs (X^r_∞, X^r_∞) and (X^c_∞, X^c_2). So we have

$$K_t(x; X^r_2 \cap X^c_2, X^r_\infty \cap X^c_\infty) \simeq \max\{K_t(x; X^r_2, X^r_\infty), K_t(x; X^c_2, X^c_\infty)\}, \tag{6.3}$$

and by a well known fact (see [1, p. 109])

$$K_t(x; X_2^r, X_\infty^r) \simeq \Big(\int_0^{t^2} \Big(\sum x_n x_n^*\Big)^\dagger(s)\, ds\Big)^{1/2},$$

$$K_t(x; X_2^c, X_\infty^c)\} \simeq \Big(\int_0^{t^2} \Big(\sum x_n^* x_n\Big)^\dagger(s)\, ds\Big)^{1/2}.$$

Therefore, we find both (6.3) and

$$\Big(\int_0^{t^2} S^\dagger(s)^2 ds\Big)^{1/2} \simeq \max\{K_t(x; X_2^r, X_\infty^r), K_t(x; X_2^c, X_\infty^c)\}, \tag{6.4}$$

where, of course, the equivalences are meant with constants independent of t.

However, a very short and direct proof was recently given in [3] that there is a constant c such that

$$\forall t > 0 \quad S^\dagger(ct) \le \Big(\Big(\sum x_n x_n^*\Big)^{1/2}\Big)^\dagger(t) + \Big(\Big(\sum x_n^* x_n\Big)^{1/2}\Big)^\dagger(t),$$

from which (6.4) (and hence (6.3)) follows easily.

Acknowledgments. I am grateful to Quanhua Xu and Yanqi Qiu for stimulating discussions.

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Received October 9, 2011 (7850)