BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 59, No. 3, 2011

MATHEMATICAL FINANCE

On Backward Stochastic Differential Equations Approach to Valuation of American Options

by

Tomasz KLIMSIAK and Andrzej ROZKOSZ

Presented by Jerzy ZABCZYK

Summary. We consider the problem of valuation of American (call and put) options written on a dividend paying stock governed by the geometric Brownian motion. We show that the value function has two different but related representations: by means of a solution of some nonlinear backward stochastic differential equation, and by a weak solution to some semilinear partial differential equation.

1. Introduction. We consider a financial market model in which the price dynamics of a dividend paying stock $X^{s,x}$ evolves (under the equivalent martingale measure P) according to the stochastic differential equation (SDE) of the form

$$X_t^{s,x} = x + \int_s^t (r-d)X_\theta^{s,x}\,d\theta + \int_s^t \sigma X_\theta^{s,x}\,dW_\theta, \qquad t \in [s,T]. \tag{1.1}$$

Here $x > 0$, W is a standard Wiener process, $d \ge 0$ is the dividend rate on the stock, $r \ge 0$ is the risk-free interest rate and $\sigma > 0$ is the volatility.

It is well known (see, e.g., [8, Section 2.5]) that the arbitrage-free value of an American option with payoff function $g : \mathbb{R} \to [0,\infty)$ and expiration time T is given by

$$V(s,x) = \sup_{s\le\tau\le T} Ee^{-r(\tau-s)}g(X_\tau^{s,x}), \tag{1.2}$$

where E denotes the expectation with respect to P and the supremum is taken over all stopping times with respect to the standard augmentation

2010 *Mathematics Subject Classification*: Primary 91G20; Secondary 60H30, 60H99.
Key words and phrases: backward stochastic differential equation, obstacle problem, American option.

DOI: 10.4064/ba59-3-8

$\{\mathcal{F}_t\}$ of the filtration generated by W. From [6] we also know that the optimal stopping problem and, a fortiori, the value function V are related to the solution $(Y^{s,x}, Z^{s,x}, K^{s,x})$ of the reflected backward stochastic differential equation (RBSDE)

(1.3)
$$\begin{cases} Y_t^{s,x} = g(X_T^{s,x}) - \int_t^T rY_\theta^{s,x}\,d\theta + K_T^{s,x} - K_t^{s,x} - \int_t^T Z_\theta^{s,x}\,dW_\theta, \quad t \in [s,T], \\ Y_t^{s,x} \ge g(X_t^{s,x}), \quad t \in [s,T], \\ K^{s,x} \text{ is increasing, continuous, } K_s^{s,x} = 0, \quad \int_s^T (Y_t^{s,x} - g(X_t^{s,x}))\,dK_t^{s,x} = 0, \end{cases}$$

via the equality

$$V(s,x) = Y_s^{s,x}, \qquad (s,x) \in Q_T \equiv [0,T] \times \mathbb{R}. \tag{1.4}$$

Formula (1.4) when combined with general results on connections between RBSDEs and parabolic PDEs proved in [4] provides a probabilistic proof of the fact that $V = \{V(s,x) : (s,x) \in Q_T\}$ with $V(s,x)$ given by (1.2) is a viscosity solution of the obstacle problem (or, in another terminology, the variational inequality)

$$\begin{cases} \min(u(s,x) - g(x), -\mathcal{L}_{BS}u(s,x) + ru(s,x)) = 0, & (s,x) \in Q_T, \\ u(T,x) = g(x), & x \in \mathbb{R}, \end{cases} \tag{1.5}$$

where $\mathcal{L}_{BS}$ is the Black and Scholes differential operator defined by

$$\mathcal{L}_{BS}u = \partial_s u + (r-d)x\partial_x u + \frac{1}{2}\sigma^2 x^2 \partial_{xx}^2 u.$$

In the present paper we concentrate on the American call and put options with exercise price $K > 0$ for which the payoff function is given by

$$g(x) = \begin{cases} (x-K)^+, & \text{call option}, \\ (K-x)^+, & \text{put option}. \end{cases}$$

We prove that in that case the process $K^{s,x}$ has the form

$$K_t^{s,x} = \begin{cases} \int_s^t (dX_\theta^{s,x} - rK)^+ \mathbf{1}_{\{Y_\theta^{s,x} = g(X_\theta^{s,x})\}}\,d\theta, & \text{call option}, \\ \int_s^t (rK - dX_\theta^{s,x})^+ \mathbf{1}_{\{Y_\theta^{s,x} = g(X_\theta^{s,x})\}}\,d\theta, & \text{put option}, \end{cases} \tag{1.6}$$

for $t \in [s,T]$, i.e. the first two components $(Y^{s,x}, Z^{s,x})$ of the solution of

(1.3) solve the usual (nonreflected) BSDE

$$(1.7)\qquad Y_t^{s,x} = g(X_T^{s,x}) + \int_t^T (-rY_\theta^{s,x} + q(X_\theta^{s,x}, Y_\theta^{s,x}))\,d\theta - \int_t^T Z_\theta^{s,x}\,dW_\theta, \qquad t\in[s,T],$$

where

$$q(x,y) = \begin{cases} (dx - rK)^+ \mathbf{1}_{(-\infty, g(x)]}(y), & \text{call option},\\ (rK - dx)^+ \mathbf{1}_{(-\infty, g(x)]}(y), & \text{put option},\end{cases}$$

for $x, y \in \mathbb{R}$. The above result is in fact a reformulation of the representation for the Snell envelope of the discounted payoff process $\xi_t = e^{-r(t-s)} g(X_t^{s,x})$, $t\in[s,T]$ (see Section 3). Therefore our contribution here consists in providing a new proof of the last statement and clarifying relations between (1.3) and (1.7). We also hope that our proof of the representation for the Snell envelope for ξ will be of interest, because in contrast to other proofs known to us it avoids considering the parabolic free-boundary value problem associated with the optimal stopping problem (1.2).

Formula (1.6) has an analytical counterpart. Let $\varrho(x) = (1+|x|^2)^{-\alpha}$, $x\in\mathbb{R}$, where α is chosen so that $\int_{\mathbb{R}} \varrho^2(x) x^2\,dx < \infty$. By a solution of (1.5) we understand a pair (u,μ) consisting of a measurable function $u : Q_T \to \mathbb{R}$ with some regularity properties and a Radon measure μ on Q_T such that

$$(1.8)\qquad \begin{cases} \mathcal{L}_{BS} u = ru - \mu, \\ u(T) = g, \quad u \ge g, \quad \displaystyle\int_{Q_T} (u-g)\varrho^2\,d\mu = 0 \end{cases}$$

(see Section 2 for details). We prove that (1.8) has a unique solution (u,μ) such that μ is absolutely continuous with respect to the Lebesgue measure and

$$(1.9)\qquad d\mu(t,x) = q(x, u(t,x))\,dt\,dx.$$

Moreover, for each $(s,x)\in Q_T$ such that $x\neq 0$,

$$(1.10)\qquad (Y_t^{s,x}, Z_t^{s,x}) = (u(t, X_t^{s,x}), \sigma x \partial_x u(t, X_t^{s,x})), \qquad t\in[s,T],\ P\text{-a.s.},$$

i.e. (1.3) provides a probabilistic representation for the first component u of a solution of (1.8). In particular, $V = u$. Formula (1.9) is an analytical analogue of (1.6).

From (1.8), (1.9) it follows that V is a solution of the semilinear Cauchy problem

$$(1.11)\qquad \mathcal{L}_{BS} u = ru - q(\cdot, u), \qquad u(T,\cdot) = g.$$

The above problem was considered in [2, 3] as an alternative to the obstacle problem formulation (1.5) and the free boundary problem formulation (see, e.g., [8, Section 2.7]). In [2] it is shown that (1.11) has a unique viscosity solution (since q is discontinuous, the standard definition of a viscosity solution is modified appropriately) and $V = u$. Our approach to (1.5) via (1.8) shows that in fact (1.11) results from a better understanding of the nature of solutions of (1.5).

2. Obstacle problem for the Black and Scholes equation. In this section we prove existence, uniqueness and stochastic representation of solutions of the obstacle problem (1.8). We begin with the precise definition of solutions of (1.8).

Let $Q_{st} = [s,t]\times\mathbb{R}$, $Q_t = Q_{0t}$, and let $\mathcal{R}$ denote the space of all functions $\varrho : \mathbb{R} \to \mathbb{R}$ of the form $\varrho(x) = (1+|x|^2)^{-\alpha}$, $x \in \mathbb{R}$, for some $\alpha \ge 0$ such that $\int_{\mathbb{R}} \varrho^2(x)x^2\,dx < \infty$.

Given $\varrho \in \mathcal{R}$ we denote by $\mathbb{L}_{2,\varrho}(\mathbb{R})$ the Hilbert space of functions u on $\mathbb{R}$ such that $u\varrho \in \mathbb{L}_2(\mathbb{R})$ equipped with the inner product $\langle u, v\rangle_{2,\varrho} = \int_{\mathbb{R}} uv\varrho^2\,dx$. Similarly, by $\mathbb{L}_{2,\varrho}(Q_{st})$ we denote the Hilbert space of functions u on Q_{st} such that $u\varrho \in \mathbb{L}_2(Q_{st})$ with the inner product $\langle u,v\rangle_{2,\varrho,s,t} = \int_{Q_{st}} uv\varrho^2\,dx\,dt$. If $s = 0$ we drop the subscript s in the notation. Set $H_\varrho = \{\eta \in \mathbb{L}_{2,\varrho}(\mathbb{R}) : x\partial_x\eta(x) \in \mathbb{L}_{2,\varrho}(\mathbb{R})\}$, $W_\varrho = \{\eta \in \mathbb{L}_2(0,T;H_\varrho) : \partial_t\eta \in \mathbb{L}_2(0,T;H_\varrho^{-1})\}$, where H_ϱ^{-1} is the space dual to H_ϱ. By $\langle\cdot,\cdot\rangle_{\varrho,T}$ we denote the duality pairing between $\mathbb{L}_2(0,T;H_\varrho)$ and $\mathbb{L}_2(0,T;H_\varrho^{-1})$. Finally, $V = W_\varrho \cap C(Q_T)$.

We say that a pair (u,μ), where $u \in V$ and μ is a Radon measure on Q_T, is a solution of the obstacle problem (1.8) if

$$u(T) = g, \qquad u \ge g, \qquad \int_{Q_T} (u-g)\varrho^2\,d\mu = 0 \tag{2.1}$$

and the equation

$$\mathcal{L}_{BS}u = ru - \mu \tag{2.2}$$

is satisfied in the strong sense, i.e. for every $\eta \in C_0^\infty(Q_T)$,

$$\langle\partial_t u,\eta\rangle_{\varrho,T} + \langle\mathcal{L}_{BS}u,\eta\rangle_{\varrho,T} = r\langle u,\eta\rangle_{2,\varrho,T} - \int_{Q_T} \eta\varrho^2\,d\mu,$$

where

$$\langle\mathcal{L}_{BS}u,\eta\rangle_{\varrho,T} = \langle(r-d)x\partial_x u,\eta\rangle_{2,\varrho,T} - \frac{1}{2}\sigma^2\langle\partial_x u, \partial_x(x^2\eta\varrho^2)\rangle_{2,T}.$$

We say that a pair (u,μ) satisfies (2.2) in the weak sense if μ is a Radon measure on Q_T, $u \in \mathbb{L}_2(0,T;H_\varrho) \cap C([0,T],\mathbb{L}_{2,\varrho}(\mathbb{R}))$ and for every $\eta \in C_0^\infty(Q_T)$,

$$\langle u, \partial_t \eta\rangle_{\varrho,T} - \langle \mathcal{L}_{BS} u, \eta\rangle_{\varrho,T} = \langle h(T), \eta(T)\rangle_{2,\varrho} - \langle u(0), \eta(0)\rangle_{2,\varrho} \\ - r\langle u, \eta\rangle_{2,\varrho,T} + \int_{Q_T} \eta\varrho^2\, d\mu.$$

Let $\{\mathcal{F}_t\}$ denote the standard augmentation of the natural filtration generated by W. By a solution of RBSDE (1.3) we understand a triple $(Y^{s,x}, Z^{s,x}, K^{s,x})$ of $\{\mathcal{F}_t\}$-progressively measurable processes on $[s,T]$ such that

$$E \sup_{t\in[s,T]} |Y_t^{s,x}|^2 < \infty, \quad E\int_s^T |Z_t^{s,x}|^2\, dt < \infty, \quad E|K_T^{s,x}|^2 < \infty \tag{2.3}$$

and (1.3) is satisfied P-a.s. A pair $(Y^{s,x}, Z^{s,x})$ of $\{\mathcal{F}_t\}$-progressively measurable processes is a solution of BSDE (1.7) if (1.7) holds P-a.s. and $Y^{s,x}, Z^{s,x}$ satisfy the integrability conditions (2.3).

From the general results proved in [4] it follows that (1.3) has a unique solution. We shall prove that the third component $K^{s,x}$ of the solution is absolutely continuous.

Proposition 2.1. *If* $(Y^{s,x}, Z^{s,x}, K^{s,x})$ *is a solution of RBSDE* (1.3) *then*

$$K_t^{s,x} - K_\tau^{s,x} \le \int_\tau^t \mathbf{1}_{\{Y_\theta^{s,x} = S_\theta\}} (dX_\theta^{s,x} - rK)^+\, d\theta, \qquad s \le \tau \le t \le T. \tag{2.4}$$

Proof. We prove the theorem in the case of call option. The proof for put option is similar and therefore left to the reader.

Suppose that $(Y^{s,x}, Z^{s,x}, K^{s,x})$ is a solution of (1.3) and u is a viscosity solution of (1.5). By [4, Theorem 8.5],

$$Y_t^{s,x} = u(t, X_t^{s,x}), \qquad t \in [s,T]. \tag{2.5}$$

Set $S_t = g(X_t^{s,x})$, $t \in [s,T]$, and denote by $\{L_t^0(\xi) : t \ge 0\}$ the local time at 0 of a continuous semimartingale ξ. By the Tanaka–Meyer formula, for $t \in [s,T]$ we have

$$(X_t^{s,x} - K)^+ = \int_s^t \mathbf{1}_{(K,\infty)}(X_\theta^{s,x})(r-d)X_\theta^{s,x}\, d\theta \\ + \int_s^t \mathbf{1}_{(K,\infty)}(X_\theta^{s,x})\sigma X_\theta^{s,x}\, dW_\theta + \frac{1}{2} L_t^0(X^{s,x} - K) \tag{2.6}$$

and

$$0 = (Y_t^{s,x} - S_t)^- = -\int_s^t \mathbf{1}_{(-\infty,0]}(Y_\theta^{s,x} - S_\theta)\, dY_\theta^{s,x} \\ + \int_s^t \mathbf{1}_{(-\infty,0]}(Y_\theta^{s,x} - S_\theta)\, dS_\theta + \frac{1}{2} L_t^0(Y^{s,x} - S) \tag{2.7}$$

$$
\begin{aligned}
&= \int_s^t \mathbf{1}_{\{Y_\theta^{s,x}=S_\theta\}}(-rY_\theta^{s,x}\,d\theta + dK_\theta^{s,x} - Z_\theta^{s,x}\,dW_\theta)\\
&\quad + \int_s^t \mathbf{1}_{(K,\infty)}(X_\theta^{s,x})\mathbf{1}_{\{Y_\theta^{s,x}=S_\theta\}}((r-d)X_\theta^{s,x}\,d\theta + \sigma X_\theta^{s,x}\,dW_\theta)\\
&\quad + \frac{1}{2}\int_s^t \mathbf{1}_{\{Y_\theta^{s,x}=S_\theta\}}\,dL_\theta^0(X^{s,x}-K) + \frac{1}{2}L_t^0(Y^{s,x}-S).
\end{aligned}
$$

Write $I=\{u=g\}$ and observe that $(t,K)\notin I$ for all $t\in[0,T)$, because $u=V$ by [4, Proposition 2.3] and hence u is strictly positive. Consequently,

$$
\int_s^t \mathbf{1}_{\{Y_\theta^{s,x}=S_\theta\}}\,dL_\theta^0(X^{s,x}-K)=0.
$$

Furthermore, from (2.6) and Proposition 4.2 and Remark 4.3 in [4] it follows that $\sigma X_t^{s,x}\mathbf{1}_{(K,\infty)}(X_t^{s,x}) = Z_t^{s,x}$ P-a.s. on $\{Y_t^{s,x}=S_t\}$. From (2.7) we therefore get

$$
\begin{aligned}
&K_t^{s,x} - K_\tau^{s,x} + \frac{1}{2}L_t^0(Y^{s,x}-S) - \frac{1}{2}L_\tau^0(Y^{s,x}-S)\\
&\quad = \int_\tau^t r\mathbf{1}_{\{Y_\theta^{s,x}=S_\theta\}}S_\theta\,d\theta - \int_\tau^t \mathbf{1}_{\{Y_\theta^{s,x}=S_\theta\}}\mathbf{1}_{(K,\infty)}(X_\theta^{s,x})(r-d)X_\theta^{s,x}\,d\theta\\
&\quad = \int_\tau^t \mathbf{1}_{\{Y_\theta^{s,x}=S_\theta\}}\mathbf{1}_{(K,\infty)}(X_\theta^{s,x})((r-d)X_\theta^{s,x} - r(X_\theta^{s,x}-K)^+)^-\,d\theta.
\end{aligned}
$$

Hence

(2.8)

$$
K_t^{s,x} - K_\tau^{s,x} \le \int_\tau^t \mathbf{1}_{\{Y_\theta^{s,x}=S_\theta\}}\mathbf{1}_{(K,\infty)}(X_\theta^{s,x})((r-d)X_\theta^{s,x} - r(X_\theta^{s,x}-K)^+)^-\,d\theta.
$$

Since, by (2.5), $Y^{s,x}$ is strictly positive, $\{Y_t^{s,x}=g(X_t^{s,x})\}\subset\{X_t^{s,x}>K\}$ and hence $K^{s,x}$ increases only on the set $\{X_t^{s,x}>K\}$. Therefore (2.8) forces (2.4). ∎

Proposition 2.2. *There exists at most one solution of the problem* (1.8).

Proof. Suppose that (u_1,μ_1), (u_2,μ_2) are two solutions of (1.8). Write $u=u_1-u_2$, $\mu=\mu_1-\mu_2$. Then (u,μ) satisfies (2.2) in the strong sense. Since by standard regularization arguments we can take u as a test function in (2.2) and obviously (2.2) is satisfied on Q_{tT} for any $t\in[0,T)$, we have

$$\|u(t)\|^2_{2,\varrho}+\frac{1}{2}\sigma^2\|x\partial_x u\|^2_{2,\varrho,t,T}=\langle(\mu-d)x\partial_x u,u\rangle_{2,\varrho,t,T}+\sigma^2\langle\partial_x u,xu\rangle_{2,\varrho,t,T}$$
$$+\sigma^2\langle\partial_x u,x^2u\partial_x\varrho,\varrho\rangle_{2,t,T}+r\|u\|^2_{2,\varrho,t,T}+\int\limits_{Q_{tT}}u\,d\mu.$$

From the above, the fact that $\int_{Q_{tT}}u\,d\mu\le 0$, $|\partial_x\varrho|\le C\varrho$ and the elementary inequality $ab\le\varepsilon a^2+\varepsilon^{-1}b^2$ we get

$$\|u(t)\|^2_{2,\varrho}\le C\int\limits_t^T\|u(s)\|^2_{2,\varrho}\,ds,\qquad t\in[0,T].$$

By Gronwall's lemma, $u=0$, and in consequence, $\mu=0$. ∎

Given $\delta>0$ write $D^+_\delta=(0,T)\times(\delta,+\infty)$, $D^-_\delta=(0,T)\times(-\infty,\delta)$ and $D^+=D^+_0$, $D^-=D^-_0$, $D=D^+\cup D^-$. Note that from the well known explicit formula for $X^{s,x}$ it follows that $X^{s,x}_t\in D^+$, $t\in[s,T]$, P-a.s. if $x>0$, and $X^{s,x}_t\in D^-$, $t\in[s,T]$, P-a.s. if $x<0$. Note also that if $x\ne 0$ and $t>s$ then the distribution density of the random variable $X^{s,x}_t$ is given by the formula

$$\begin{aligned}(2.9)\qquad &p(s,x,t,y)\\ &=\frac{1}{y\sqrt{2\pi(t-s)}}\exp\biggl(\frac{-(\ln(y/x)+(\sigma^2/2-r+d)(t-s))^2}{t-s}\biggr)\mathbf{1}_{\{y/x>0\}}.\end{aligned}$$

It follows in particular that for fixed $s\in[0,T)$, $x\ne0$ and $\delta\in(0,T-s]$ the function $p(s,x,\cdot,\cdot)$ is bounded on $Q_{s+\delta,T}$.

Theorem 2.3.

(i) *There exists a unique solution* (u,μ) *of the problem* (1.8).
(ii) *Let* $x\ne0$ *and let* $(Y^{s,x},Z^{s,x},K^{s,x})$ *be a solution of RBSDE* (1.3). *Then*

$$(Y^{s,x}_t,Z^{s,x}_t)=(u(t,X^{s,x}_t),\sigma\partial_x u(t,X^{s,x}_t)),\qquad t\in[s,T],\ P\text{-a.s.}$$

and for any $\eta\in C_0(Q_{sT})$,

$$E\int\limits_s^T\eta(t,X_t)\,dK^{s,x}_t=\int\limits_{Q_{sT}}\eta(t,y)p(s,x,t,y)\,d\mu(t,y).\tag{2.10}$$

Proof. By [12, Theorem 2.2] for each $n\in\mathbb{N}$ there exists a unique viscosity solution u_n of the penalized problem

$$\frac{\partial u_n}{\partial t}+\mathcal{L}_{BS}u_n=ru_n-n(u_n-g)^-,\qquad u_n(T)=g.\tag{2.11}$$

Let $(Y^{s,x,n},Z^{s,x,n})$ denote a solution of the BSDE

$$Y^{s,x,n}_t=g(X^{s,x}_T)-\int_t^T rY^{s,x,n}_\theta\,d\theta+\int_t^T n(Y^{s,x,n}_\theta-g(X^{s,x}_\theta))^-\,d\theta-\int_t^T Z^{s,x,n}_\theta\,dW_\theta.$$

Using standard arguments one can show that $x\mapsto EY^{s,x,n}_s$ is Lipschitz continuous uniformly in s. Therefore u_n has the same regularity, because by [12, Theorem 2.2], $Y^{s,x,n}_t=u_n(t,X^{s,x}_t)$, $t\in[s,T]$, P-a.s., and hence $u_n(s,x)=EY^{s,x,n}_s$. Since the operator $\mathcal{L}_{BS}$ is uniformly elliptic on each domain D^+_δ, for each $\delta>0$ there is a unique weak solution v_δ of the terminal-boundary value problem

$$\frac{\partial v_\delta}{\partial t}+\mathcal{L}_{BS}v_\delta=rv_\delta-n(v_\delta-g)^-,\ v_\delta(T)=g,\ v_\delta(t,x)=u_n(t,x)\text{ on }[0,T]\times\{\delta\}$$

(see [10, Theorem V.6.1]). Since v_δ is a viscosity solution of the above problem as well, $v_\delta=u_{n|D^+_\delta}$ by uniqueness. Using this, Lipschitz continuity of u_n and [7, Theorem 1.5.9] we conclude that $u_n\in C^{1,2}(D)$. Hence, by Proposition 1.2.3 and Theorem 2.2.1 in [11], $Y^{s,x,n}_t\in\mathbb{D}^{1,2}$ for every $(s,x)\in Q_T$ such that $x\neq0$, where $\mathbb{D}^{1,2}$ is the domain of the derivative operator in $\mathbb{L}_2(\Omega)$ (see [11, Section 1.2] for a precise definition). Consequently, applying once again Proposition 1.2.3 and Theorem 2.2.1 in [11] and using the fact that g and $x\mapsto x^-$ are Lipschitz continuous functions we conclude that if $x\neq0$ then $g(X^{s,x}_T)$, $\int_t^T rY^{s,x,n}_\theta\,d\theta$, $\int_t^T n(Y^{s,x,n}_\theta-g(X^{s,x}_\theta))^-\,d\theta\in\mathbb{D}^{1,2}$. Moreover, by [11, Proposition 1.2.3] and [5, Lemma 5.1], there exists an adapted bounded process A such that for every $s<\tau\le t$,

$$\begin{aligned}D_\tau Y^{s,x,n}_t=Z^{s,x,n}_\tau&+\int_\tau^t D_\tau Z^{s,x,n}_\theta\,d\theta+r\int_\tau^t D_\tau Y^{s,x,n}_\theta\,d\theta\\&-n\int_\tau^t A_\theta D_\tau(Y^{s,x,n}_\theta-g(X_\theta))\,d\theta,\end{aligned}$$

where D_τ denotes the derivative operator. From this it follows in particular that

$$D_tY^{s,x,n}_t=Z^{s,x,n}_t,\qquad P\text{-a.s.}$$

for every $t\in[s,T]$. On the other hand, by the remarks following the proof of Proposition 2.2 and the remark following the proof of [11, Proposition 1.2.3],

$$D_\tau Y^{s,x,n}_t=\partial_xu_n(t,X^{s,x}_t)D_\tau X^{s,x}_t,\qquad P\text{-a.s.}$$

for every $r,t\in[s,T]$. Moreover, by [11, Theorem 2.2.1], $D_tX^{s,x}_t=\sigma X^{s,x}_t$. Thus, if $x\neq0$, then

$$Z^{s,x,n}_t=\sigma X^{s,x}_t\partial_xu_n(t,X_t),\quad P\text{-a.s.}$$

By the results of Section 6 in [4] and standard estimates for diffusions we have

$$(2.12)\qquad E\sup_{s\le t\le T}|u_n(t,X^{s,x}_t)|^2+E\int_s^T|\sigma X^{s,x}_t\partial_x u_n(t,X^{s,x}_t)|^2\,dt \le CE\sup_{s\le t\le T}|g(X^{s,x}_t)|^2\le C|x|^2.$$

By the above and [1, Proposition 5.1] it follows that $u_n\in\mathbb{L}_2(0,T;H_\varrho)$. Accordingly, u_n is a weak solution of (2.11). Furthermore, from the results proved in [4, Section 6] it follows that for every $(s,x)\in Q_T$,

$$(2.13)\qquad E\sup_{s\le t\le T}|(u_n-u_m)(t,X^{s,x}_t)|^2+E\int_s^T|\sigma X^{s,x}_t\partial_x(u_n-u_m)(t,X^{s,x}_t)|^2\,dt + E\sup_{s\le t\le T}|K^{s,x,n}_t-K^{s,x,m}_t|^2\to 0$$

as $m,n\to\infty$. From (2.12), (2.13) and [1, Proposition 5.1] we conclude that there exists $u\in C(Q_T)\cap\mathbb{L}_2(0,T;H_\varrho)$ such that $u_n\to u$ uniformly on compact subsets of Q_T, $u_n\to u$ in $\mathbb{L}_2(0,T;H_\varrho)$ and $u_n\to u$ in $C([0,T],\mathbb{L}_{2,\varrho}(\mathbb{R}))$. Moreover, using (2.12) and once again [1, Proposition 5.1] we see that $\|u_n\|_{\mathbb{L}_2(0,T;H_\varrho)}\le C$. Therefore from (2.11) it follows that the sequence $\{\mu_n\}$ of measures defined by $d\mu_n=n(u_n-g)^-\,d\lambda$, $n\in\mathbb{N}$, where λ is the 2-dimensional Lebesgue measure, is tight. If $\mu_n\to\mu$ weakly, which we may assume, then letting $n\to\infty$ in (2.11) we conclude that the pair (u,μ) satisfies equation (2.2) in the weak sense and that

$$u(t,X^{s,x}_t)=Y^{s,x}_t,\ t\in[s,T],\ P\text{-a.s.},\qquad Z^{s,x}_t=\sigma X^{s,x}_t\partial_x u(t,X^{s,x}_t),\ dt\otimes P\text{-a.s.}$$

because in [4, Section 6] it is proved that $Y^{s,x,n}_t\to Y^{s,x}_t$, $t\in[s,T]$, P-a.s. and $E\int_s^T|Z^{s,x,n}_t-Z^{s,x}_t|^2\,dt\to 0$. In particular, it follows from the above that $u\ge g$. Let $\eta\in C_0(Q_T)$. Since $u_n\to u$ uniformly,

$$\int_{Q_T}(u_n-g)\eta\,d\mu_n\to\int_{Q_T}(u-g)\eta\,d\mu\ge 0.$$

On the other hand,

$$\int_{Q_T}(u_n-g)\eta\,d\mu_n=-\int_{Q_T}n((u_n-g)^-)^2\,d\lambda\le 0.$$

From this we get (2.1). Furthermore, if $x\ne 0$ then for any $\delta\in(0,T-s)$ and $\eta\in C_0(Q_{s+\delta,T})$ we have

$$(2.14)\qquad E\int_s^T\eta(t,X^{s,x}_t)\,dK^{s,x,n}_t=\int_{Q_{sT}}\eta(t,y)p(s,x,t,y)\,d\mu_n(t,y).$$

Since it is known that $K_t^{s,x,n} \to K_t^{s,x}$ uniformly in $t \in [s,T]$ in probability (see [4, Section 6]), letting $n \to \infty$ in (2.14) and using (2.9), (2.13) we get (2.10) for $\eta \in C_0(Q_{s+\delta,T})$, and hence for any $\eta \in C_0(Q_{sT})$. In order to complete the proof we have to show that $u \in W_\varrho$. Since $p(s,x,\cdot,\cdot)$ is positive for every $(s,x) \in Q_T$ such that $x \neq 0$, it follows from (2.10) and Proposition 2.1 that $d\mu \le \mathbf{1}_{\{u=g\}}(t,x)(dx - rK)^+ \, d\lambda$, i.e. for every $\eta \in C_0^+(Q_T)$,

$$\int_{Q_T} \eta(t,x)\, d\mu(t,x) \le \int_{Q_T} \eta(t,x)\mathbf{1}_{\{u=g\}}(t,x)(dx - rK)^+ \, dx\, dt.$$

Hence there exists a measurable function α on Q_T such that $0 \le \alpha \le 1$ and

$$\frac{d\mu}{d\lambda}(t,x) = \alpha(t,x)\mathbf{1}_{\{u=g\}}(t,x)(dx - rK)^+. \tag{2.15}$$

This implies that $u \in W_\varrho$ and u satisfies (2.2) in the strong sense, i.e. (u,μ) is a solution of (1.8). ∎

Remark 2.4. It is known that $\{u = g\} = \{(t,x) \in Q_T : x \ge s(t)\}$ for some nonincreasing function s in the case of call option and $\{u = g\} = \{(t,x) \in Q_T : 0 \le x \le s(t)\}$ for some nondecreasing s in the case of put option (see, e.g., [8, Proposition 2.7.6]). It follows that in both cases the 2-dimensional Lebesgue measure of the boundary of $\{u = g\}$ equals zero.

3. Linear RBSDEs and nonlinear BSDEs.

We begin by proving the key formulas (1.6), (1.9). As a first application we will show the semimartingale representation for the Snell envelope of the discounted payoff process and the early exercise premium representation for V.

Theorem 3.1.

(i) *If* (u,μ) *is a solution of the obstacle problem* (1.8), *then* μ *is given by* (1.9).

(ii) *If* $(Y^{s,x}, Z^{s,x}, K^{s,x})$ *is a solution of* (1.3), *then* $K^{s,x}$ *is given by* (1.6).

Proof. We prove the theorem in the case of call option. The proof for put option requires only some obvious changes and is left to the reader.

Suppose that $(Y^{s,x}, Z^{s,x}, K^{s,x})$ is a solution of (1.3) and (u,μ) is a solution of (1.8). By (2.15), u solves the equation

$$\partial_t u + (r-d)x\partial_x u + \frac{1}{2}\sigma^2 x^2 \partial_{xx}^2 u = ru - \alpha(t,x)\mathbf{1}_{\{u=g\}}(t,x)(dx - rK)^+ \tag{3.1}$$

in the strong sense. Let $I = \{u = g\}$ and $I_0 = \operatorname{Int} I$. If $I_0 \neq \emptyset$ then by (3.1), for any $\eta \in C_0^\infty(I_0)$ we have

$$\begin{aligned}\int\limits_{Q_T} u(t,x)\partial_t\eta(t,x)\,dt\,dx &- \frac{1}{2}\int\limits_{Q_T} \sigma^2x^2\partial^2_{xx}u(t,x)\eta(t,x)\,dt\,dx\\ &- \int\limits_{Q_T}(r-d)x\partial_x u(t,x)\eta(t,x)\,dt\,dx\\ &= \int\limits_{Q_T}(-ru(t,x)+\alpha(t,x)\mathbf{1}_{\{u=g\}}(t,x)(dx-rK)^+)\eta(t,x)\,dt\,dx\\ &+\int\limits_{\mathbb{R}} g(x)\eta(T,x)\,dx-\int\limits_{\mathbb{R}} u(0,x)\eta(0,x)\,dx.\end{aligned}$$

Since $\operatorname{supp}\eta \subset I_0$ and g is regular on I_0, we deduce from the above that

$$\begin{aligned}\int\limits_{I_0}(r-d)x\mathbf{1}_{[K,\infty)}(x)\eta(t,x)\,dt\,dx &= \int\limits_{I_0} rg(x)\eta(t,x)\,dt\,dx\\ &- \int\limits_{I_0}\alpha(t,x)\mathbf{1}_{\{u=g\}}(t,x)(dx-rK)^+\eta(t,x)\,dt\,dx.\end{aligned}$$

Equivalently, we have

$$\begin{aligned}&\int\limits_{I_0} f(t,x)\eta(t,x)\,dt\,dx\\ &\qquad= \int\limits_{I_0}\alpha(t,x)\mathbf{1}_{\{u=g\}}(t,x)\mathbf{1}_{[K,\infty)}(x)(dx-rK)^+\eta(t,x)\,dt\,dx,\end{aligned}$$

where $f(t,x) = (r-d)x\mathbf{1}_{[K,\infty)}(x) - r(x-K)^+ = (-dx+rK)\mathbf{1}_{[K,\infty)}(x)$. Since

$$\begin{aligned}\alpha(t,x)(dx-rK)^+ &= -\alpha(t,x)((r-d)x\mathbf{1}_{[K,\infty)}(x)-r(x-K)^+)^-\\ &= -\alpha(t,x)f^-(x)\end{aligned}$$

on I_0, it follows that

$$\int\limits_{I_0} f(t,x)\eta(t,x)\,dt\,dx = -\int\limits_{I_0}\alpha(t,x)f^-(t,x)\eta(t,x)\,dt\,dx$$

for any $\eta \in C_0^\infty(I_0)$. Hence $f(t,x) = -\alpha(t,x)f^-(t,x)$ a.e. on I_0. Since $f = f^+ - f^-$, we have $f^+(t,x) = (1-\alpha(t,x))f^-(t,x)$, and consequently $(1-\alpha(t,x))f^-(t,x) = 0$ a.e. on I_0, i.e. $\alpha(t,x)(dx-rK)^+ = (dx-rK)^+$ a.e. on I_0. Since by Remark 2.4 the Lebesgue measure of ∂I equals zero, the above equality holds a.e. on I, which in view of (2.15) completes the proof of (i).

In case $x = 0$ part (ii) is trivial since in that case $X_t^{s,x} = K_t^{s,x} = 0$, $t \in [s,T]$. In case $x \neq 0$ part (ii) follows from part (i) and results proved in [9]. To see this, let us denote by X the canonical process on the space $C([0,T];\mathbb{R})$ of continuous functions on $[0,T]$, and by $P_{s,x}$ the law of $X^{s,x}$, i.e. $P_{s,x} = P \circ (X^{s,x})^{-1}$. We may and will assume that $X^{s,x}_s = x$, $t \in [0,s]$, and hence that $P_{s,x}$ is a measure on $C([0,T];\mathbb{R})$. Write

$$M_{s,t} = X_t - X_s - \int_s^t (r-d)X_\theta\, d\theta, \qquad B_{s,t} = \int_s^t \frac{1}{\sigma X_\theta}\, dM_{s,\theta}, \qquad 0 \le s \le t \le T,$$

and observe that if $x \neq 0$ then under $P_{s,x}$ the process $B_{s,\cdot}$ is a standard Wiener process on $[s,T]$ with respect to the natural filtration generated by X. Furthermore, for $0 \le s < t \le T$ set

$$K_{s,t} = u(s,X_s) - u(t,X_t) + \int_s^t ru(\theta,X_\theta)\, d\theta + \int_s^t \sigma\partial_x u(\theta,X_\theta)\, dB_{s,\theta}$$

and

$$\tilde{K}_{s,t} = \int_s^t (dX_\theta - rK)^+ \mathbf{1}_{\{u(\theta,X_\theta)=g(X_\theta)\}}\, d\theta.$$

Let $(Y^{s,x}, Z^{s,x}, K^{s,x})$ be a solution of (1.3) and let $\tilde{K}^{s,x}$ denote the process defined by the right hand side of (1.6). By Theorem 2.3, for every $(s,x) \in [0,T) \times \mathbb{R}$,

$$K^{s,x}_t - K^{s,x}_s = u(s,X^{s,x}_s) - u(t,X^{s,x}_t) + \int_s^t ru(\theta,X^{s,x}_\theta)\, d\theta$$
$$+ \int_s^t \sigma\partial_x u(\theta,X^{s,x}_\theta)\, dW_\theta, \qquad 0 \le s < t \le T,\ P\text{-a.s.}$$

From this and the fact that the law of $(X, B_{s,\cdot})$ under $P_{s,x}$ is equal to the law of $(X^{s,x}, W_\cdot - W_s)$ under P we conclude that the law of $K_{s,\cdot}$ under $P_{s,x}$ is equal to the law of $K^{s,x}$ under P. Consequently, by (2.10), for every $s \in [0,T)$, $x \neq 0$,

$$E_{s,x}\int_s^T \eta(t,X_t)\, dK_{s,t} = \int_{Q_{sT}} \eta(t,y)p(s,x,t,y)\, d\mu(t,y) \tag{3.2}$$

for all $\eta \in C_0(Q_{sT})$, where $E_{s,x}$ denotes the expectation with respect to $P_{s,x}$. Thus, the additive functional $K = \{K_{s,t} : 0 \le s \le t \le T\}$ of the Markov family $\{(X, P_{s,x}) : (s,x) \in [0,T) \times \mathbb{R}\}$ corresponds to the measure μ in the sense defined in [9]. Similarly, for every $s \in [0,T)$, $x \neq 0$ the law of $\tilde{K}_{s,\cdot}$ under $P_{s,x}$ is equal to the law of $\tilde{K}^{s,x}$ under P, and hence, by part (i), (3.2) is satisfied with K replaced by $\tilde{K}$, i.e. the additive functional $\tilde{K} = \{\tilde{K}_{s,t} : 0 \le s \le t \le T\}$ corresponds to μ, too. By [9, Corollary 6.6], $P_{s,x}(K_{s,t} = \tilde{K}_{s,t}, t \in [s,T]) = 1$ for every $s \in [0,T)$, $x \neq 0$. Hence $P(K^{s,x}_t = \tilde{K}^{s,x}_t, t \in [s,T]) = 1$ for $s \in [0,T)$, $x \neq 0$, which completes the proof. ∎

COROLLARY 3.2. *If* $(Y^{s,x}, Z^{s,x}, K^{s,x})$ *is a solution of* (1.3) *then* $(Y^{s,x}, Z^{s,x})$ *is a solution of* (1.7). *Conversely, if* $(Y^{s,x}, Z^{s,x})$ *is a solution of* (1.7) *then* $(Y^{s,x}, Z^{s,x}, K^{s,x})$ *with* $K^{s,x}$ *defined by* (1.6) *is a solution of* (1.3).

Proof. The first part follows immediately from Theorem 3.1. The second part is a consequence of the first one and the fact that the solution of (1.7) is unique, because for every $x \in \mathbb{R}$ the function $y \mapsto q(x, y)$ is decreasing. ∎

Let ξ denote the discounted payoff process for the American option, i.e.

$$\xi_t = e^{-r(t-s)} g(X_t^{s,x}), \qquad t \in [s, T].$$

By (1.7),

$$e^{r(t-s)} Y_t^{s,x} = e^{-r(T-s)} g(X_T^{s,x}) + \int_t^T e^{-r(\theta-s)} q(X_\theta^{s,x}, Y_\theta^{s,x})\, d\theta - \int_t^T e^{-r(\theta-s)} Z_\theta^{s,x}\, dW_\theta.$$

From this and the fact that $V(t, X_t^{s,x}) = u(t, X_t^{s,x}) = Y_t^{s,x}$, $t \in [s, T]$, we obtain

COROLLARY 3.3. *The Snell envelope* $\eta_t = e^{-r(t-s)} V(t, X_t^{s,x})$, $t \in [s, T]$, *of* ξ *admits the representation*

$$\eta_t = E\Big(e^{-r(T-s)} g(X_T^{s,x}) + \int_t^T e^{-r(\theta-s)} q(X_\theta^{s,x}, Y_\theta^{s,x})\, d\theta \,\Big|\, \mathcal{F}_t\Big). \tag{3.3}$$

From (3.3) we immediately get the early exercise premium representation for V. For instance, for American put option,

$$V(s, x) = E e^{-r(T-s)} g(X_T^{s,x}) + E \int_s^T e^{-r(t-s)} (rK - dX_t^{s,x})^+ \mathbf{1}_{\{V=g\}}(t, X_t^{s,x})\, dt. \tag{3.4}$$

Representations (3.3), (3.4) are known (see [8, Corollary 2.7.11]). To our knowledge our proof is new. Let us stress, however, that we were influenced by the results of [2].

Acknowledgements. This research was supported by Polish Ministry of Science and Higher Education (grant no. N N201 372 436).

References

[1] V. Bally and A. Matoussi, *Weak solutions for SPDEs and backward doubly stochastic differential equations*, J. Theoret. Probab. 14 (2001), 125–164.

[2] F. S. Benth, K. H. Karlsen and K. Reikvam, *A semilinear Black and Scholes partial differential equation for valuing American options*, Finance Stoch. 7 (2003), 277–298.
[3] —, —, —, *On a semilinear Black and Scholes partial differential equation for valuing American options. Part II: approximate solutions and convergence*, Interfaces Free Bound. 6 (2004), 379–404.
[4] N. El Karoui, C. Kapoudjian, E. Pardoux, S. Peng and M. C. Quenez, *Reflected solutions of backward SDEs, and related obstacle problems for PDE's*, Ann. Probab. 25 (1997), 702–737.
[5] N. El Karoui, S. Peng and M. C. Quenez, *Backward stochastic differential equations in finance*, Math. Finance 7 (1997), 1–77.
[6] N. El Karoui and M. C. Quenez, *Non-linear pricing theory and backward stochastic differential equations*, in: Lecture Notes in Math. 1656, Springer, 1997, 191–246.
[7] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, NJ, 1994.
[8] I. Karatzas and S. E. Shreve, *Methods of Mathematical Finance*, Springer, New York, 1998.
[9] T. Klimsiak, *On time-dependent functionals of diffusions corresponding to divergence form operators*, J. Theoret. Probab. (2011), to appear.
[10] O. A. Ladyženskaya, V. A. Solonnikov and N. N. Ural'ceva, *Linear and Quasi-Linear Equations of Parabolic Type*, Transl. Math. Monogr. 23, Amer. Math. Soc., Providence, RI, 1968.
[11] D. Nualart, *The Malliavin Calculus and Related Topics*, Springer, Berlin, 1995.
[12] É. Pardoux, *Backward stochastic differential equations and viscosity solutions of systems of semilinear parabolic and elliptic PDEs of second order*, in: Stochastic Analysis and Related Topics VI (The Geilo Workshop, 1996), L. Decreusefond et al. (eds.), Birkhäuser, Boston, 1998, 79–127.

Tomasz Klimsiak, Andrzej Rozkosz
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: tomas@mat.uni.torun.pl
rozkosz@mat.uni.torun.pl

Received April 29, 2011;
received in final form October 26, 2011 (7830)