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On the Stability of Orthogonal Additivity

by

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Summary. We deal with the stability of the orthogonal additivity equation, presenting a new approach to the proof of a 1995 result of R, Ger and the second author. We sharpen the estimate obtained there. Moreover, we work in more general settings, providing an axiomatic framework which covers much more cases than considered before by other authors.

1. Introduction. The concept of orthogonal additivity appeared at the beginning of the 20th century. For a comprehensive survey the reader is referred to the paper of L. Paganoni and J. Rätz [10]. Let us mention here the most important steps (from our point of view). G. Birkhoff and R. C. James introduced an orthogonality relation in normed linear spaces called the Birkhoff–James orthogonality, which generalizes the orthogonality in inner product spaces (see [2, 7]). Simultaneously, other types of orthogonality on linear spaces were introduced (see [6]). In 1975, S. Gudder and D. Strawther [5] proposed an axiomatic framework for the orthogonality relation. This idea was later developed by J. Rätz and Gy. Szabó [11–14]. They proposed the following axioms. Let $\mathbb{K}$ be a field with characteristic different from 2 and let X be a linear space of dimension greater than or equal to 2. Further, let $\perp$ be a binary relation defined on X with the properties:

(i) $x \perp 0$ and $0 \perp x$ for all $x \in X$;
(ii) if $x, y \in X \setminus \{0\}$ and $x \perp y$, then x and y are linearly independent;
(iii) if $x, y \in X$ and $x \perp y$, then for all $\alpha, \beta \in \mathbb{K}$ we have $\alpha x \perp \beta y$;
(iv) for any two-dimensional subspace P of X and for every $x \in P$, there exists a $y \in P$ such that $x \perp y$ and $x + y \perp x - y$.

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The pair $(X, \perp)$ is usually called an orthogonality space in the sense of Rätz; we will call it simply a *Rätz space.* However, in a number of papers such a space is supposed to satisfy different, usually stronger assumptions. A Rätz space covers the case of the classical orthogonality on an inner product space as well as the Birkhoff–James orthogonality ($x \perp y \Leftrightarrow \|x\| \leq \|x + \lambda y\|$ for all $\lambda \in \mathbb{R}$ with X being an arbitrary normed linear space). However, there are known orthogonality relations on normed linear spaces which do not satisfy these axioms (e.g., the isosceles, or James orthogonality: $x \perp y \Leftrightarrow \|x + y\| = \|x - y\|$ and the Pythagorean orthogonality: $x \perp y \Leftrightarrow \|x + y\|^2 = \|x\|^2 + \|y\|^2$).

In [11–14], J. Rätz and Gy. Szabó studied the functional equation of *orthogonal additivity*:

$$x \perp y \quad \text{implies} \quad f(x + y) = f(x) + f(y). \tag{1}$$

In particular, it was proved that in a Rätz space every odd orthogonally additive mapping having values in an Abelian group is additive, whereas every even orthogonally additive mapping is quadratic.

In 2002 these results were further generalized in [1] by K. Baron and P. Volkmann, who proposed the following (much weaker) axioms of orthogonality. Let X be a uniquely 2-divisible Abelian group. Further, let $\perp$ be a binary relation defined on X with the properties:

(a) $0 \perp 0$;
(b) if $x, y \in X$ and $x \perp y$, then $-x \perp -y$ and $\frac{x}{2} \perp \frac{y}{2}$;
(c) every odd orthogonally additive mapping having values in an Abelian group is additive and every even orthogonally additive mapping is quadratic.

They have obtained the following representation of an orthogonally additive mapping $f\colon X \to G$ having values in an arbitrary Abelian group:

$$f(x) = a(x) + b(x, x), \qquad x \in X,$$

with a being additive and b being biadditive, symmetric and such that $b(x, y) = 0$ whenever $x \perp y$.

In [4] R. Ger and the second author proved the stability of (1), showing that if $f\colon X \to Y$, where $(X, \perp)$ is a Rätz space and $(Y, \|\cdot\|)$ is a real Banach space, satisfies the conditional inequality

$$x \perp y \quad \text{implies} \quad \|f(x + y) - f(x) - f(y)\| \leq \varepsilon$$

for some nonnegative ε, then there exists a unique orthogonally additive function $g\colon X \to Y$ such that $\|f(x) - g(x)\| \leq \frac{16}{3}\varepsilon$ for each $x \in X$. It was also observed that the target space can be easily generalized to be a real sequentially complete Hausdorff linear topological space (see also [15]).

Even though a normed linear space with the James orthogonality does not fulfill the conditions of a Rätz space, similar results are valid in that case (see [15–18]).

In the present paper we improve the result from [4] in two directions. Firstly, we work in a more general framework, so that we cover several cases which have not been discussed so far. Moreover, we diminish the constant $\frac{16}{3}$. We apply a new approach to this stability problem—instead of discussing the odd and even case separately, as in [4], we use a single approximating sequence. However, we do not know whether the new approximating constant is optimal.

Some related results for various stability problems which were obtained thanks to the use of a single approximating sequence instead of splitting the unknown mapping into odd and even parts can be found in [3] and [8].

2. Main result. Let X be an Abelian group and let $\perp$ be a binary relation defined on X with the properties:

(α) if $x, y \in X$ and $x \perp y$, then $x \perp -y$, $-x \perp y$ and $2x \perp 2y$;
(β) for every $x \in X$, there exists a $y \in X$ such that $x \perp y$ and $x + y \perp x - y$.

Further, let $(Y, \|\cdot\|)$ be a (real or complex) Banach space.

Our main result is the following.

THEOREM. *Given an $\varepsilon \geq 0$, let $f\colon X \to Y$ be a mapping such that for all $x, y \in X$ one has*

$$x \perp y \quad \textit{implies} \quad \|f(x+y) - f(x) - f(y)\| \leq \varepsilon. \tag{2}$$

Then there exists a mapping $g\colon X \to Y$ such that

$$x \perp y \quad \textit{implies} \quad g(x+y) = g(x) + g(y), \tag{3}$$

and

$$\|f(x) - g(x)\| \leq 5\varepsilon \tag{4}$$

for all $x \in 2X = \{2x : x \in X\}$. Moreover, the mapping g is unique on the set $2X$.

Proof. Fix $x \in X$. By (β) there exists a $y \in X$ such that $x \perp y$ and $x + y \perp x - y$. By (α) we also have $\pm 2x \perp \pm 2y$, $\pm(x+y) \perp \pm(x-y)$ and we can write

$$
\begin{aligned}
\|3f(4x)-8f(2x)-f(-4x))\| &\le 3\|f(4x)-f(2x+2y)-f(2x-2y)\| \\
&\quad + \|f(-2x+2y)+f(-2x-2y)-f(-4x)\| \\
&\quad + 3\|f(2x+2y)-f(2x)-f(2y)\| \\
&\quad + 3\|f(2x-2y)-f(2x)-f(-2y)\| \\
&\quad + \|f(-2x)+f(2y)-f(-2x+2y)\| \\
&\quad + \|f(-2x)+f(-2y)-f(-2x-2y)\| \\
&\quad + 2\|f(2y)-f(y-x)-f(y+x)\| \\
&\quad + 2\|f(-2y)-f(-y-x)-f(-y+x)\| \\
&\quad + 2\|f(y-x)+f(-y-x)-f(-2x)\| \\
&\quad + 2\|f(y+x)+f(-y+x)-f(2x)\| \le 20\varepsilon,
\end{aligned}
$$

whence

$$
\left\| f(2x) - \frac{3}{8} f(4x) + \frac{1}{8} f(-4x) \right\| \le \frac{5}{2}\varepsilon, \qquad x \in X. \tag{5}
$$

By induction we will prove that for all $n \in \mathbb{N}$,

$$
\begin{aligned}
\left\| f(2x) - \frac{2^n+1}{2\cdot 4^n} f(2^{n+1}x) + \frac{2^n-1}{2\cdot 4^n} f(-2^{n+1}x) \right\| & \\
\le \left(5 - \frac{5}{2^n}\right)\varepsilon, \qquad x \in X. &
\end{aligned} \tag{6}
$$

Indeed, for $n = 1$ we have (5), and by the induction hypothesis and (5) applied for $2^n x$ and $-2^n x$ we obtain

$$
\begin{aligned}
&\left\| f(2x) - \frac{2^{n+1}+1}{2\cdot 4^{n+1}} f(2^{n+2}x) + \frac{2^{n+1}-1}{2\cdot 4^{n+1}} f(-2^{n+2}x) \right\| \\
&\quad \le \left\| f(2x) - \frac{2^n+1}{2\cdot 4^n} f(2^{n+1}x) + \frac{2^n-1}{2\cdot 4^n} f(-2^{n+1}x) \right\| \\
&\qquad + \frac{2^n+1}{2\cdot 4^n} \left\| f(2^{n+1}x) - \frac{3}{8} f(2^{n+2}x) + \frac{1}{8} f(-2^{n+2}x) \right\| \\
&\qquad + \frac{2^n-1}{2\cdot 4^n} \left\| f(-2^{n+1}x) - \frac{3}{8} f(-2^{n+2}x) + \frac{1}{8} f(2^{n+2}x) \right\| \\
&\quad \le \left(5 - \frac{5}{2^n} + \frac{2^n+1}{2\cdot 4^n}\cdot\frac{5}{2} + \frac{2^n-1}{2\cdot 4^n}\cdot\frac{5}{2}\right)\varepsilon = \left(5 - \frac{5}{2^{n+1}}\right)\varepsilon.
\end{aligned}
$$

The next step is to prove that for each $x \in X$ the sequence

$$
g_n(x) := \frac{2^n+1}{2\cdot 4^n} f(2^n x) - \frac{2^n-1}{2\cdot 4^n} f(-2^n x), \qquad n \in \mathbb{N},
$$

is convergent in Y. Since Y is a Banach space, it suffices to show that $(g_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence for every $x \in X$. Employing estimate (5) twice we obtain

$$\begin{aligned}\|g_n(x) - g_{n+1}(x)\| &= \left\| \frac{2^n+1}{2\cdot 4^n}\left(f(2^n x) - \frac{3}{8}\, f(2^{n+1}x) + \frac{1}{8}\, f(-2^{n+1}x)\right)\right. \\ &\qquad \left. - \frac{2^n-1}{2\cdot 4^n}\left(f(-2^n x) - \frac{3}{8}\, f(-2^{n+1}x) + \frac{1}{8}\, f(2^{n+1}x)\right)\right\| \\ &\le \frac{2^n+1}{2\cdot 4^n}\cdot\frac{5}{2}\,\varepsilon + \frac{2^n-1}{2\cdot 4^n}\cdot\frac{5}{2}\,\varepsilon = \frac{5}{2^{n+1}}\,\varepsilon\end{aligned}$$

for each $n \in \mathbb{N}$. This easily implies that $(g_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence. Therefore we may define

$$g(x) := \lim_{n\to\infty} g_n(x), \qquad x \in X.$$

On account of (6) we have

$$\|f(2x) - g(2x)\| \le 5\varepsilon, \qquad x \in X.$$

In order to prove that g is orthogonally additive observe first that for $x, y \in X$ such that $x \perp y$ and $n \in \mathbb{N}$, $n > 1$ we have

$$\begin{aligned}&\|g_n(x+y) - g_n(x) - g_n(y)\| \\ &= \left\| \frac{2^n+1}{2\cdot 4^n}\, f(2^n(x+y)) - \frac{2^n-1}{2\cdot 4^n}\, f(-2^n(x+y)) \right. \\ &\qquad \left. - \frac{2^n+1}{2\cdot 4^n}\, f(2^n x) + \frac{2^n-1}{2\cdot 4^n}\, f(-2^n x) - \frac{2^n+1}{2\cdot 4^n}\, f(2^n y) + \frac{2^n-1}{2\cdot 4^n}\, f(-2^n y)\right\| \\ &= \left\| \frac{2^n+1}{2\cdot 4^n}\,[f(2^n(x+y)) - f(2^n x) - f(2^n y)] \right. \\ &\qquad \left. - \frac{2^n-1}{2\cdot 4^n}\,[f(2^n(-x-y)) - f(-2^n x) - f(-2^n y)]\right\| \\ &\le \frac{2^n+1}{2\cdot 4^n}\,\|f(2^n(x+y)) - f(2^n x) - f(2^n y)\| \\ &\qquad + \frac{2^n-1}{2\cdot 4^n}\,\|f(2^n(-x-y)) - f(-2^n x) - f(-2^n y)\| \\ &\le \frac{2^n+1}{2\cdot 4^n}\,\varepsilon + \frac{2^n-1}{2\cdot 4^n}\,\varepsilon = \frac{1}{2^n}\,\varepsilon.\end{aligned}$$

By letting $n \to \infty$ we get (3).

To prove the uniqueness of g assume that g and g' satisfy (3) and (4). Then obviously

$$\|g(x) - g'(x)\| \le \|g(x) - f(x)\| + \|g'(x) - f(x)\| \le 10\varepsilon$$

for $x \in 2X$. On the other hand, the mapping $g - g'$ satisfies (3) and thus, in particular, (2) with $\varepsilon = 0$. By applying (6) to $g - g'$ we see that

$$\begin{aligned} g(2x) - g'(2x) &= \frac{2^n+1}{2\cdot 4^n}\,[g(2^{n+1}x) - g'(2^{n+1}x)] \\ &\quad - \frac{2^n-1}{2\cdot 4^n}\,[g(-2^{n+1}x) - g'(-2^{n+1}x)] \end{aligned}$$

and therefore

$$\begin{aligned} \|g(2x) - g'(2x)\| &\le \frac{2^n+1}{2\cdot 4^n}\,\|g(2^{n+1}x) - g'(2^{n+1}x)\| \\ &\quad + \frac{2^n-1}{2\cdot 4^n}\,\|g(-2^{n+1}x) - g'(-2^{n+1}x)\| \\ &\le \frac{2^n+1}{2\cdot 4^n}\,10\varepsilon + \frac{2^n-1}{2\cdot 4^n}\,10\varepsilon = \frac{10}{2^n}\,\varepsilon \end{aligned}$$

for $x \in X$. From this we easily see that $g = g'$. ∎

In case X is uniquely 2-divisible, we get approximation (4) on the whole of X; however, there are examples of non-trivial groups with $2X = \{0\}$ for which our assertion does not bring much information.

Applying parts of the foregoing proof for $\varepsilon = 0$ we infer that each solution f of (1) satisfies

$$f(2x) = \frac{3}{8}\,f(4x) - \frac{1}{8}\,f(-4x), \qquad x \in X. \tag{7}$$

Moreover, an inspection of the proof shows that if $\varepsilon = 0$ then no additional linear or topological structure is needed. Therefore, we derive the following corollary.

Corollary. *If X is an Abelian group and $\perp$ satisfies (α) and (β), and if $(Y, +)$ is an Abelian uniquely 2-divisible group, then each solution $f\colon X \to Y$ of (1) satisfies (7). In particular, odd solutions satisfy $f(2x) = 2f(x)$ whereas even solutions satisfy $f(2x) = 4f(x)$ for each $x \in 2X$.*

Remark 1. The above results can be applied both in a Rätz space and in a normed space with the James orthogonality. However, the problem remains open in the case of the Pythagorean orthogonality.

Now, we will provide an example of a binary relation which seems to be far from any known orthogonality relations but satisfies (α) and (β).

Example 1. Take $X = \mathbb{R}$ and define $\perp_0 \subset \mathbb{R}^2$ in the following way:

$$x \perp_0 y \;\Leftrightarrow\; x \cdot y \in \mathbb{R} \setminus \mathbb{Q} \text{ or } x \cdot y = 0.$$

We will check that conditions (α) and (β) are fulfilled. Indeed, (α) is quite obvious; to check (β) observe that $x + y \perp_0 x - y$ is equivalent to

$$x^2 - y^2 \in \mathbb{R} \setminus \mathbb{Q} \cup \{0\}.$$

Fix an $x \in \mathbb{R}$ and observe that if $x = 0$ it is enough to take $y = 0$, whereas if $x \neq 0$ then we have two cases. If x^2 is rational then we may take e.g. $y = \pi$, otherwise x cannot be rational and it suffices to put $y = 0$.

The foregoing example can be easily generalized to more general structures.

Example 2. Consider an arbitrary ring R and a subring $S \subset R$ for which there exists an element $\gamma \in R \setminus S$ such that $\gamma^2 \notin S$ and define

$$x \perp_1 y \;\Leftrightarrow\; x \cdot y \in R \setminus S \text{ or } x \cdot y = 0.$$

As before, one can verify that $\perp_1$ satisfies (α) and (β).

One can check that neither $\perp_0$ nor $\perp_1$ satisfy the axioms of a Rätz space, whence (α) and (β) are considerably weaker than axioms (i)–(iv) defining a Rätz space.

Remark 2. The norm structure of the target space Y in the Theorem is not necessary. A careful inspection of the proof yields the following more abstract version of the Theorem:

Let Y be a real sequentially complete Hausdorff linear topological space. Assume that a bounded convex set $V \subset Y$, symmetric with respect to zero, and a mapping $f\colon X \to Y$ are given such that for every $x, y \in X$,

$$x \perp y \quad \textit{implies} \quad f(x+y) - f(x) - f(y) \in V.$$

Then on $2X$ there is exactly one mapping $g\colon X \to Y$ satisfying (3) *and such that*

$$f(x) - g(x) \in 5\,\mathrm{seq\,cl}\,V$$

for all $x \in 2X$.

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