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OPERATOR THEORY

Uniformly Continuous Set-Valued Composition Operators in the Spaces of Functions of Bounded Variation in the Sense of Riesz

by

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Summary. We show that any uniformly continuous and convex compact valued Nemytskiĭ composition operator acting in the spaces of functions of bounded φ-variation in the sense of Riesz is generated by an affine function.

Introduction. Let $(X, |\cdot|)$ and $(Y, |\cdot|)$ be two real normed spaces, C be a convex cone in X and $I \subset \mathbb{R}$ an interval. Let $cc(Y)$ be the family of all non-empty convex compact subsets of Y. For a given function $h : I \times C \to cc(Y)$, the Nemytskiĭ operator H is defined by $(HF)(t) = h(t, F(t))$ where $F : I \to cc(X)$ is a given set-valued function. It is shown that if the operator H maps the space $RV_\varphi(I; C)$ of function of bounded φ-variation in the sense of Riesz into the space $BV_\psi(I; cc(Y))$ of convex compact valued functions of bounded ψ-variation in the sense of Riesz, and is uniformly continuous, then h is affine with respect to the second variable. In particular,

$$h(t, x) = A(t)x + B(t) \quad \text{for } t \in I,\ x \in C,$$

for some function $A : I \to \mathcal{L}(C, cc(Y))$ and $B \in BV_\psi(I; cc(Y))$, where $\mathcal{L}(C, cc(Y))$ stands for the space of all linear mappings of C into $cc(Y)$. Some references are given in Remark 2.3.

1. Preliminaries. In this section we present some definitions and recall some known results concerning the Riesz φ-variation.

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Let $\mathcal{F}$ be the set of all convex functions $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = \varphi(0+) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. Then we have

REMARK 1.1. If $\varphi \in \mathcal{F}$, then φ is continuous and strictly increasing. Indeed, the continuity of φ at each point $t > 0$ follows from its convexity, and the continuity at 0 from the assumption $\varphi(0) = \varphi(0+) = 0$. Suppose that $\varphi(t_1) \geq \varphi(t_2)$ for some $0 < t_1 < t_2$. Then

$$\frac{\varphi(t_1) - \varphi(0)}{t_1 - 0} = \frac{\varphi(t_1)}{t_1} > \frac{\varphi(t_2)}{t_2} = \frac{\varphi(t_2) - \varphi(0)}{t_2 - 0},$$

contradicting the convexity of φ.

Let $I \subset \mathbb{R}$ be an interval. For a set X we denote by X^I the set of all functions $f : I \to X$.

DEFINITION 1.2. Let $\varphi \in \mathcal{F}$ and $(X, |\cdot|)$ be a real normed space. A function $f \in X^I$ is said to be of *bounded φ-variation in the sense of Riesz* in I if

$$RV_\varphi(f) := \sup_\xi \sum_{i=1}^{m} \varphi\left(\frac{|f(t_i) - f(t_{i-1})|}{t_i - t_{i-1}}\right)(t_i - t_{i-1}) < \infty,$$

where the supremum is taken over all finite increasing sequences $\xi = (t_i)_{i=0}^m$, $t_i \in I$, $m \in \mathbb{N}$.

DEFINITION 1.3. Let $\varphi \in \mathcal{F}$. We say that φ satisfies the ∞_1 *condition* if

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty.$$

It is easy that if $\varphi \in \mathcal{F}$ satisfies the ∞_1 condition then

$$\lim_{r\to 0} r\varphi^{-1}(1/r) = \lim_{\rho\to\infty} \rho/\varphi(\rho) = 0.$$

Denote by $RV_\varphi(I, X)$ the set of all functions $f \in X^I$ such that $RV_\varphi(\lambda f) < \infty$ for some $\lambda > 0$. It is a normed space endowed with the norm

$$\|f\|_\varphi := |f(a)| + p_\varphi(f), \qquad f \in RV_\varphi(I, X),$$

where $I = [a, b]$ and $p_\varphi(f) = \inf\{\epsilon > 0 : RV_\varphi(f/\epsilon) \leq 1\}$.

Let $cc(X)$ be the family of all non-empty convex compact subsets of X, and let D be the *Pompeiu–Hausdorff metric* in $cc(X)$, i.e.

$$D(A, B) := \max\{e(A, B), e(B, A)\}, \qquad A, B \in cc(X),$$

where

$$e(A, B) = \sup\{d(x, B) : x \in A\}, \qquad d(x, B) = \inf\{d(x, y) : y \in B\}.$$

Definition 1.4. Let $\varphi \in \mathcal{F}$ and $F : I \to cc(X)$. We say that F *has bounded φ-variation in the Riesz sense* if

$$W_\varphi(F) := \sup_\xi \sum_{i=1}^m \varphi\left(\frac{D(F(t_i), F(t_{i-1}))}{t_i - t_{i-1}}\right)(t_i - t_{i-1}) < \infty,$$

where the supremum is taken over all finite increasing sequences $\xi = (t_i)_{i=0}^m$, $t_i \in I$, $m \in \mathbb{N}$.

Define

$$RV_\varphi(I, cc(X)) = \{F \in cc(X)^I : W_\varphi(\lambda F) < \infty \text{ for some } \lambda > 0\},$$

and equip this set with the metric

$$D_\varphi(F_1, F_2) := D(F_1(a), F_2(a)) + p_\varphi(F_1, F_2)$$

where

$$p_\varphi(F_1, F_2) := \inf\{r > 0 : W_r(F_1, F_2) \le 1\}$$

and

$$W_r(F_1, F_2) := \sup_\zeta \sum_{i=1}^m \varphi\left(\frac{D(F_1(t_i) + F_2(t_{i-1}); F_2(t_i) + F_1(t_{i-1}))}{(t_i - t_{i-1})r}\right)(t_i - t_{i-1}),$$

where the supremum is taken over all finite increasing sequences $\zeta = (t_i)_{i=0}^m$, $t_i \in I$, $m \in \mathbb{N}$.

Lemma 1.5 (cf. Chistyakov [2, Lemma 5.2]). *Let $F_1, F_2 \in RW_\varphi(I, cc(X))$ and $\varphi \in \mathcal{F}$. Then, for $\lambda > 0$,*

$$W_\lambda(F_1, F_2) \le 1 \quad \text{if and only if} \quad p_\varphi(F_1, F_2) \le \lambda. \ \blacksquare$$

Now, let $(X, |\cdot|)$, $(Y, |\cdot|)$ be two real normed spaces and C be a convex cone in X. Given a set-valued function $h : I \times C \to cc(Y)$ we consider the composition operator $H : C^I \to Y^I$ generated by h, i.e.

$$(Hf)(t) := h(t, f(t)), \qquad f \in C^I,\ t \in I.$$

Let $\mathcal{A}(C, cc(Y))$ denote the space of all additive functions and $\mathcal{L}(C, cc(Y))$ the space of all set-valued linear functions, i.e., all $A \in \mathcal{A}(C, cc(Y))$ which are positively homogeneous.

Lemma 1.6 (cf. H. Radström [11, Lemma 3]). *Let $(X, |\cdot|)$ be a normed space and let A, B, C be subsets of X. If A, B are convex and C is non-empty and bounded, then*

$$D(A + C, B + C) = D(A, B).$$

Lemma 1.7 (cf. K. Nikodem [10, Th. 5.6]). *Let $(X, |\cdot|)$, $(Y, |\cdot|)$ be normed spaces and C a convex cone in X. A set-valued function $F : C \to cc(Y)$ satisfies the Jensen equation*

$$F\left(\frac{x_1 + x_2}{2}\right) = \frac{1}{2}\,(F(x_1) + F(x_2)), \qquad x_1, x_2 \in C,$$

if and only if there exist an additive set-valued function $A : C \to cc(Y)$ and a set $B \in cc(Y)$ such that $F(x) = A(x) + B$ for all $x \in C$. ∎

2. The composition operator. Our main result reads as follows:

THEOREM 2.1. *Let $(X, |\cdot|)$ be a real normed space, $(Y, |\cdot|)$ a real Banach space, C a convex cone in X, and $I = [a, b]$ an interval. Suppose that $\varphi, \psi \in \mathcal{F}$ and that ψ satisfies the ∞_1 condition. If the composition operator H generated by a set-valued function $h : I \times C \to cc(Y)$ maps $RV_\varphi(I, C)$ into $RW_\psi(I, cc(Y))$, and is uniformly continuous, then the set-valued function h satisfies the following condition:*

$$h(t, x) = A(t)x + B(t), \qquad t \in I,\ x \in C,$$

for some $A : I \to \mathcal{A}(X, cc(Y))$ and $B : I \to cc(Y)$. Moreover, if $0 \in C$ and $\operatorname{int} C \neq \emptyset$, then $B \in RW_\psi(I, cc(Y))$ and the linear set-valued function $A(t)$ is continuous.

Proof. For every $x \in C$, the constant function $I \ni t \mapsto x$ belongs to $RV_\varphi(I, C)$. Since H maps $BV_\varphi(I, C)$ into $RW_\psi(I, cc(Y))$, for every $x \in C$ the function $I \ni t \mapsto h(t, x)$ belongs to $RW_\psi(I, cc(Y))$.

The uniform continuity of H on $RV_\varphi(I, C)$ implies that

$$D_\psi\big(H(f_1), H(f_2)\big) \le \omega(\|f_1 - f_2\|_\varphi) \quad \text{for } f_1, f_2 \in RV_\varphi(I, C), \tag{1}$$

where $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is the *modulus of continuity* of H, that is,

$$\omega(\rho) := \sup\{D_\psi(H(f_1), H(f_2)) : \|f_1 - f_2\|_\varphi \le \rho;\ f_1, f_2 \in RV_\varphi(I, C)\},\ \rho > 0.$$

From the definition of the metric D_ψ and (1), we obtain

$$p_\psi(H(f_1), H(f_2)) \le \omega(\|f_1 - f_2\|_\varphi) \quad \text{for } f_1, f_2 \in RV_\varphi(I, C).$$

From Lemma 1.5, if $\omega(\|f_1 - f_2\|_\varphi) > 0$, the inequality (2) is equivalent to

$$W_{\omega(\|f_1 - f_2\|_\varphi)}(H(f_1), H(f_2)) \le 1, \qquad f_1, f_2 \in RV_\varphi(I, C).$$

Therefore, for any $\alpha, \beta \in I$, $\alpha < \beta$, the definitions of the operator H and the functional W_r imply

$$\psi\left(\frac{D(h(\beta, f_1(\beta)) + h(\alpha, f_2(\alpha)), h(\beta, f_2(\beta)) + h(\alpha, f_1(\alpha)))}{\omega(\|f_1 - f_2\|_\varphi)(\beta - \alpha)}\right)(\beta - \alpha) \le 1. \tag{2}$$

For $\alpha', \beta' \in \mathbb{R}$, $a \le \alpha' < \beta' \le b$, we define the function $\eta_{\alpha',\beta'} : \mathbb{R} \to [0, 1]$ by

$$\eta_{\alpha',\beta'}(t) := \begin{cases} 0 & \text{if } t \le \alpha', \\ \dfrac{t - \alpha'}{\beta' - \alpha'} & \text{if } \alpha' \le t \le \beta', \\ 1 & \text{if } \beta' \le t. \end{cases}$$

Fix $t \in I$. For an arbitrary finite sequence $a < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m < t$ and $x_1, x_2 \in C$, $x_1 \neq x_2$, the function $f_j : I \to X$ defined by

$$f_j(\tau) := \frac{1}{2}\,[\eta_{\alpha,\beta}(\tau)(x_1 - x_2) + x_j + x_2], \qquad \tau \in I,\ j = 1, 2,$$

belongs to the space $RV_\varphi(I, C)$. It is easy to verify that

$$f_1(\tau) - f_2(\tau) = \frac{x_1 - x_2}{2}, \qquad \tau \in I,$$

whence

$$\|f_1 - f_2\|_\varphi = |x_1 - x_2|/2,$$

and moreover

$$\begin{aligned} f_1(\beta_i) &= x_1, \quad f_1(\alpha_i) = \frac{x_1 + x_2}{2}, \\ f_2(\alpha_i) &= x_2, \quad f_2(\beta_i) = \frac{x_1 + x_2}{2}. \end{aligned}$$

Applying (2) for the functions f_1 and f_2 we get

$$\psi\left(\frac{D\big(h(\beta_i, x_1) + h(\alpha_i, x_2), h\big(\beta_i, \frac{x_1+x_2}{2}\big) + h\big(\alpha_i, \frac{x_1+x_2}{2}\big)\big)}{\omega(\|f_1 - f_2\|_\varphi)(\beta_i - \alpha_i)}\right)(\beta_i - \alpha_i) \le 1,$$

or

$$\begin{aligned} D\left(h(\beta_i, x_1) + h(\alpha_i, x_2), h\left(\alpha_i, \frac{x_1 + x_2}{2}\right) + h\left(\beta_i, \frac{x_1 + x_2}{2}\right)\right) \\ \le \omega(|x_1 - x_2|/2)(\beta_i - \alpha_i)\psi^{-1}(1/(\beta_i - \alpha_i)). \end{aligned}$$

Letting here $\beta_i - \alpha_i \to 0$ in such a way that $[\alpha_i, \beta_i] \ni t$, where $t \in I$, we get

$$D\left(h(t, x_1) + h(t, x_2), 2h\left(t, \frac{x_1 + x_2}{2}\right)\right) = 0,$$

and, as D is a metric and h takes convex values,

$$h\left(t, \frac{x_1 + x_2}{2}\right) = \frac{h(t, x_1) + h(t, x_2)}{2}$$

for all $t \in I$ and all $x_1, x_2 \in C$.

Thus, for each $t \in I$, the set-valued function $h(t, \cdot) : C \to cc(Y)$ satisfies the Jensen functional equation.

Consequently, by Lemma 1.7, for every $t \in I$ there exist an additive set-valued function $A(t) : C \to cc(Y)$ and a set $B(t) \in cc(Y)$ such that

$$h(t, x) = A(t)x + B(t) \quad \text{for } x \in C,\ t \in I, \tag{3}$$

which proves the first part of our result.

The uniform continuity of the operator $H : BV_\varphi(I, C) \to RW_\psi(I, cc(Y))$ implies the continuity of the function $A(t)$, so that $A(t) \in \mathcal{L}(C, cc(Y))$ (see

[10, Th. 5.3]). Putting $x = 0$ in (3), and taking into account that $A(t)0 = \{0\}$ for $t \in I$, we get

$$h(t, 0) = B(t), \qquad t \in I,$$

which implies that $B \in RW_\psi(I, cc(Y))$. ∎

Remark 2.2. The condition $\operatorname{int} C \neq \emptyset$ is assumed to guarantee the continuity of the linear functions $A(t)$.

Remark 2.3. The uniformly continuous composition operators in the single-valued case for some function spaces were considered in [3], [5], [6] and [7]. The globally Lipschitzian composition operators in some special function spaces were considered in [4], [8], [9], [10] (cf. also [1] for other references). The first paper in which the set-valued Lipschitzian composition operators were considered is due to A. Smajdor and W. Smajdor [12]. Note also that G. Zawadzka [13] considered the set-valued Lipschitzian composition operators in the space of set-valued functions of bounded variation.

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References

[1] J. Appell and P. P. Zabrejko, *Nonlinear Superposition Operators*, Cambridge Univ. Press, New York, 1990.

[2] V. V. Chistyakov, *Generalized variation of mappings with applications to composition operators and multifunctions*, Positivity 5 (2001), 323–358.

[3] A. Acosta, W. Aziz, J. Matkowski, and N. Merentes, *Uniformly continuous composition operator in the space of φ-variation functions in the sense of Riesz*, Fasc. Math. 43 (2010), 5–11.

[4] J. Matkowski, *Lipschitzian composition operators in some function spaces*, Nonlinear Anal. 3 (1997), 719–726.

[5] —, *Uniformly continuous superposition operators in the space of differentiable functions and absolutely continuous functions*, in: Int. Ser. Numer. Math. 157, Birkhäuser, 2008, 155–166.

[6] —, *Uniformly continuous superposition operators in the space of Hölder functions*, J. Math. Anal. Appl. 359 (2009), 56–61.

[7] —, *Uniformly continuous superposition operators in the space of bounded variation functions*, Math. Nachr., to appear.

[8] J. Matkowski and J. Miś, *On a characterization of Lipschitzian operators of substitution in the space $BV\langle a, b\rangle$*, ibid. 117 (1984), 155–159.

[9] N. Merentes, *Composition of functions of bounded φ-variation*, Pure Math. Appl. Ser. B 1 (1991), 39–45.

[10] K. Nikodem, *K-convex and K-concave set-valued functions*, Politech. Łódz. Zeszyty Nauk. 559 (1989).

[11] H. Rådström, *An embedding theorem for spaces of convex sets*, Proc. Amer. Math. Soc. 3 (1952), 165–169.

[12] A. Smajdor and W. Smajdor, *Jensen equation and Nemytskiĭ operator for set-valued functions*, Rad. Mat. 5 (1989), 311–320.

[13] G. Zawadzka, *On Lipschitzian operators of substitution in the space of set-valued functions of bounded variation*, ibid. 6 (1990), 279–293.

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