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Remarks on the Stone Spaces of the Integers and the Reals without **AC**

by

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Summary. In **ZF**, i.e., the Zermelo–Fraenkel set theory minus the Axiom of Choice **AC**, we investigate the relationship between the Tychonoff product $\mathbf{2}^{\mathcal{P}(X)}$, where $\mathbf{2}$ is $2 = \{0,1\}$ with the discrete topology, and the Stone space $S(X)$ of the Boolean algebra of all subsets of X, where $X = \omega, \mathbb{R}$. We also study the possible placement of well-known topological statements which concern the cited spaces in the hierarchy of weak choice principles.

1. Notation and terminology. Let $\mathbf{X} = (X, T)$ be a topological space. Throughout the paper, we shall denote topological spaces by bold letters and underlying sets by non-bold letters.

A space $\mathbf{X}$ is said to be *compact* iff every open cover $\mathcal{U}$ of X has a finite subcover $\mathcal{V}$. Equivalently, $\mathbf{X}$ is compact iff every family $\mathcal{G}$ of closed subsets of X with the *finite intersection property*, fip for abbreviation, has a non-empty intersection.

Furthermore, $\mathbf{X}$ is said to be a *Loeb space* iff $\mathcal{K}(\mathbf{X}) \setminus \{\emptyset\}$, where $\mathcal{K}(X)$ is the family of all closed subsets of $\mathbf{X}$, has a choice function. A choice function f of $\mathcal{K}(\mathbf{X}) \setminus \{\emptyset\}$ is called a *Loeb function.*

Given a set X, $\mathbf{2}^X$ will denote the Tychonoff product of the discrete space $\mathbf{2}$ ($2 = \{0,1\}$), and

$$\mathcal{B}_X = \{[p] : p \in \mathrm{Fn}(X, 2)\},$$

where $\mathrm{Fn}(X,2)$ is the set of all finite partial functions from X into 2 and $[p] = \{f \in 2^X : p \subset f\}$, will denote the standard base for the product topology on 2^X.

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If $X \neq \emptyset$ then $S(X)$ will denote the *Stone space* of the Boolean algebra of all subsets of X, i.e., the set of all ultrafilters on X together with the topology having as a base the collection of all (clopen) sets of the form

$$[Z] = \{\mathcal{F} \in S(X) : Z \in \mathcal{F}\}, \quad Z \subseteq X.$$

A family $\mathcal{F}$ of subsets of X is *independent* if for any two finite, disjoint sets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{F}$ the set $(\bigcap \mathcal{A}) \cap (\bigcap \{B^c : B \in \mathcal{B}\})$ is infinite.

Next we list the choice principles we shall be using in the paper.

1. **CAC** (Form 8 in [4]): **AC** *restricted to countable families of non-empty sets.*
2. **DC** (**Principle of Dependent Choices** and form 43 in [4]): *For every set $X \neq \emptyset$, for every binary relation R on X such that* Dom(R) $= X$, *there is a sequence $(x_n)_{n\in\omega} \subseteq X$ such that $\forall n \in \omega$, $x_n R x_{n+1}$.*
3. **SPFB**(X): *For every family $\{\mathcal{H}_i : i \in I\}$ of filterbases of X there exists a family $\{\mathcal{F}_i : i \in I\}$ of ultrafilters of X satisfying $\mathcal{H}_i \subseteq \mathcal{F}_i$ for all $i \in I$.*
4. **WSPFB**(X): *For every family $\{\mathcal{H}_i : i \in I\}$ of filterbases of X such that for every $i \in I$, there exists an ultrafilter $\mathcal{F}$ of X extending $\mathcal{H}_i$, there exists a family $\{\mathcal{F}_i : i \in I\}$ of ultrafilters of X satisfying $\mathcal{H}_i \subseteq \mathcal{F}_i$ for all $i \in I$.*
5. **BPI**(X): *Every filterbase of X is included in an ultrafilter of X.*
6. **BPI** (**Boolean Prime Ideal Theorem** and form 14 in [4]): *Every Boolean algebra has a prime ideal.* Equivalently, for every set X, **BPI**(X).
7. **UF**(X): *There is a free ultrafilter on X.*

Note that **BPI** $\to$ **BPI**$(\mathbb{R}) \to$ **BPI**$(\omega) \to$ **UF**(ω). In [1] it is shown that **UF**(ω) is equivalent to **UF**$(\mathbb{R})$ and in [6] it is shown that **BPI**(ω) does not imply **BPI**$(\mathbb{R})$ in **ZF**. Whether **UF**$(\omega) \to$ **BPI**(ω) is an *open problem.*

Throughout the paper $\aleph$ will always denote a well-ordered infinite cardinal number. As usual, ω denotes the set of natural numbers and $\mathbb{N}$ denotes the set of positive integers.

2. Introduction and some preliminary results. In this paper we study the relationship between the spaces $\mathbf{2}^{\mathcal{P}(X)}$ and $S(X)$, where $X = \omega, \mathbb{R}$, with respect to compactness, the Loeb property, embeddings, and cardinality of $S(X)$ and of infinite closed subsets of $S(X)$. Moreover, we are interested in the placement of well-known topological results concerning $\mathbf{2}^{\mathcal{P}(X)}$ and $S(X)$ in the hierarchy of weak choice principles.

Some of the goals we intend to meet in the current investigation are listed below:

(1) In **ZF** and for $X = \omega, \mathbb{R}$, the principle $\mathbf{BPI}(X)$ implies "$\mathbf{2}^{\mathcal{P}(X)}$ is a continuous image of $S(X)$" (Theorem 6(i)).
(2) In **ZF** and for $X = \omega, \mathbb{R}$, if $S(X)$ is compact and Loeb then $|S(X)| = |2^{\mathcal{P}(X)}|$, which in turn implies $\mathbf{UF}(X)$ (Theorem 6(iii)).
(3) In **ZF**, for every infinite set X, $S(X)$ embeds as a closed subspace of $\mathbf{2}^{\mathcal{P}(X)}$ (Theorem 7(i)).
(4) In **ZFC** (= **ZF** + **AC**), $\mathbf{2}^{\mathcal{P}(X)}$ does not embed as a subspace of $S(X)$, $X = \omega, \mathbb{R}$ (Corollary 11(ii)).
(5) **DC** implies that every infinite closed subset of $S(\omega)$ contains a topological copy of $S(\omega)$ (Theorem 10(i)).
(6) **DC** and "$S(\omega)$ is compact and Loeb" together imply that every infinite closed subset of $S(\omega)$ has size $|2^{\mathbb{R}}|$ (Theorem 10)(ii)).
(7) $\mathbf{BPI}(\omega)$ implies that $S(\omega) \setminus \omega$ contains a topological copy of $S(\omega)$, which in turn implies $\mathbf{UF}(\omega)$ (Theorem 10(iii)).
(8) **CAC** implies that for every infinite set X and for every countably infinite relatively discrete subspace G of $S(X)$, $\overline{G}$ is homeomorphic to $S(\omega)$ (Theorem 10(iv)).

Before launching into the proofs of the main results we present some preliminary facts. The first one, Proposition 1 below, is a good reason for studying Loeb spaces. In addition, this kind of space is useful because of Proposition 2 which is a **ZF** result concerning Tychonoff products of compact spaces.

Proposition 1.

(i) *For every set X, $S(X)$ is Loeb iff* $\mathbf{WSPFB}(X)$.
(ii) *For every set X, $S(X)$ is compact and Loeb iff* $\mathbf{SPFB}(X)$.
(iii) *For every set X,* $\mathbf{WSPFB}(X)$ *and* $\mathbf{BPI}(X)$ *iff* $\mathbf{SPFB}(X)$.
(iv) $\mathbf{WSPFB}(\omega)$ *does not imply* $\mathbf{SPFB}(\omega)$. *Equivalently, "$S(\omega)$ is Loeb" does not imply "$S(\omega)$ is compact". In particular,* $\mathbf{WSPFB}(\omega)$ *does not imply* $\mathbf{BPI}(\omega)$.

Proof. (i)($\rightarrow$) Fix a family $\{\mathcal{H}_i : i \in I\}$ of filterbases of X as in $\mathbf{WSPFB}(X)$ and let f be a Loeb function of $S(X)$. Clearly, $G_i = \bigcap\{[H] : H \in \mathcal{H}_i\}$ is a non-empty closed subset of $S(X)$. It is straightforward to see that $\{\mathcal{F}_i = f(G_i) : i \in I\}$ satisfies the conclusion of $\mathbf{WSPFB}(X)$ for the family $\{\mathcal{H}_i : i \in I\}$.

(i)($\leftarrow$) Since, for every $K \in \mathcal{K}(S(X)) \setminus \{\emptyset\}$, $K = \bigcap\{[A] : A \in \mathcal{P}(X)$ and $K \subset [A]\}$, it follows that $\mathcal{H}_K = \{A \in \mathcal{P}(X) : K \subset [A]\}$ is a filterbase of X included in every element of K. Hence, $\{\mathcal{H}_K : K \in \mathcal{K}(S(X)) \setminus \{\emptyset\}\}$ satisfies the hypotheses of $\mathbf{WSPFB}(X)$. Let $\{\mathcal{F}_K : K \in \mathcal{K}(S(X)) \setminus \{\emptyset\}\}$ satisfy the conclusion of $\mathbf{WSPFB}(X)$ for the collection $\{\mathcal{H}_K : K \in \mathcal{K}(S(X)) \setminus \{\emptyset\}\}$. It is straightforward to verify that the function $f : \mathcal{K}(S(X)) \setminus \{\emptyset\} \to S(X)$, $f(K) = \mathcal{F}_K$, is a Loeb function of $S(X)$.

(ii) is straightforward in view of (i) and the observation that **SPFB**(X) implies that every filterbase of X can be extended to an ultrafilter (equivalently, $S(X)$ is compact).

(iii) is obvious.

(iv) Any **ZF** model, such as Solovay's Model $\mathcal{M}5(\aleph)$ in [4], satisfying the negation of **UF**(ω) satisfies **WSPFB**(ω) and the negation of **SPFB**(ω) and of **BPI**(ω). ∎

Proposition 2 ([3], [10]). (**ZF**) *Let* $(\mathbf{X}_i)_{i\in\aleph}$ *be a family of compact* T_1 *spaces. Then the product* $\mathbf{X} = \prod_{i\in\aleph}\mathbf{X}_i$ *is compact and Loeb iff there exists a family* $(f_i)_{i\in\aleph}$ *such that for all* $i \in \aleph$, f_i *is a Loeb function for* $\mathbf{X}_i$. *In particular:*

(i) $\mathbf{2}^{\aleph}$ (*resp.* $[\mathbf{0},\mathbf{1}]^{\aleph}$) *is compact and Loeb.*
(ii) **AC** *restricted to families of non-empty sets of reals* (*equivalently, "*$\mathbb{R}$ *is well-orderable"*) *implies "*$\mathbf{2}^{\mathbb{R}}$ *is compact and Loeb".*

In view of (ii) of Proposition 2, a number of questions arise at this point.

Question 1.

(i) *Is any of the statements "*$\mathbf{2}^{\mathbb{R}}$ *is Loeb", "*$\mathbf{2}^{\mathbb{R}}$ *is compact" provable in* **ZF***?*
(ii) *Does any of the statements "*$\mathbf{2}^{\mathbb{R}}$ *is Loeb","*$\mathbf{2}^{\mathbb{R}}$ *is compact" imply* **AC**$(\mathbb{R})$*?*
(iii) *Does the conjunction "*$\mathbf{2}^{\mathbb{R}}$ *is Loeb" and "*$\mathbf{2}^{\mathbb{R}}$ *is compact" imply* **AC**$(\mathbb{R})$*?*
(iv) *Does "*$\mathbf{2}^{\mathbb{R}}$ *is Loeb" imply "*$\mathbf{2}^{\mathbb{R}}$ *is compact"?*
(v) *Does "*$\mathbf{2}^{\mathbb{R}}$ *is compact" imply "*$\mathbf{2}^{\mathbb{R}}$ *is Loeb"?*

Regarding Question 1(i), that "$\mathbf{2}^{\mathbb{R}}$ is compact" is not provable in **ZF** has been established in [5], and that "$\mathbf{2}^{\mathbb{R}}$ is Loeb" is not provable in **ZF** has been established in [8] (both fail in Cohen's Second Model $\mathcal{M}7$ in [4]).

Regarding (ii) and (iii) the answer is in the negative. Indeed, **BPI** implies "$\mathbf{2}^{\mathbb{R}}$ is Loeb" and "$\mathbf{2}^{\mathbb{R}}$ is compact" and it is known that in Cohen's Basic Model $\mathcal{M}1$ in [4], **BPI** holds but **AC**$(\mathbb{R})$ fails.

Taking into account the following result from [6], we get a partial answer to Question 1(v):

Theorem 3 ([6]). *The following statements are pairwise equivalent:*

(i) $\mathbf{2}^{\mathbb{R}}$ *is compact.*
(ii) **BPI**(ω).
(iii) *For every separable compact* T_2 *space* $\mathbf{X}$, $\mathbf{X}^{\mathbb{R}}$ *is compact.*
(iv) *In a Boolean algebra* $\mathcal{B}$ *of size* $\leq |\mathbb{R}|$ *every filter can be extended to an ultrafilter.*
(v) *Tychonoff products of finite subspaces of* $\mathbb{R}$ *are compact.*

Theorem 3 also justifies the introduction of the principle "$\mathbf{2}^{\mathcal{P}(\mathbb{R})}$ is compact" in the next lemma.

LEMMA 4.

(i) *"$\mathbf{2}^{\mathbb{R}}$ is compact" implies "for every separable compact T_2 space $\mathbf{X}$, for every family $\mathcal{G} = \{G_i : i \in I \subseteq \mathbb{R}\}$ of non-empty closed subsets of $\mathbf{X}$, there exists a choice function of $\mathcal{G}$". In particular, "$\mathbf{2}^{\mathbb{R}}$ is compact" implies "every family $\mathcal{G} = \{G_i : i \in \mathbb{R}\}$ of non-empty closed subsets of $\mathbf{2}^{\mathbb{R}}$ has a choice function", and $\mathbf{BPI}(\omega)$ implies "for every family $\mathcal{G} = \{\mathcal{G}_i : i \in \mathbb{R}\}$ of filterbases of ω there exists a family $\{\mathcal{F}_i : i \in \mathbb{R}\} \subset S(\omega)$ such that for every $i \in \mathbb{R}$, $\mathcal{G}_i \subset \mathcal{F}_i$".*

(ii) *"$\mathbf{2}^{\mathcal{P}(\mathbb{R})}$ is compact" implies "$\mathbf{2}^{\mathbb{R}}$ is compact" and "$\mathbf{2}^{\mathbb{R}}$ is Loeb". In particular, $\mathbf{BPI}(\mathbb{R})$ implies $\mathbf{SPFB}(\omega)$ (for every family $\{\mathcal{H}_i : i \in I\}$ of filterbases of ω there exists a family $\{\mathcal{F}_i : i \in I\} \subset S(\omega)$ satisfying $\mathcal{H}_i \subset \mathcal{F}_i$ for all $i \in I$). Moreover, $\mathbf{BPI}(\mathbb{R})$ implies $|S(\omega)| = |2^{\mathbb{R}}|$.*

Proof. (i) By Theorem 3, "$\mathbf{2}^{\mathbb{R}}$ is compact" implies "$\mathbf{X}^{\mathbb{R}}$ is compact". Let $\mathcal{G} = \{G_i : i \in I \subseteq \mathbb{R}\}$ be a family of non-empty closed subsets of $\mathbf{X}$. Then $\mathcal{S} = \{\pi_i^{-1}(G_i) : i \in I\}$ is a family of closed subsets of $\mathbf{X}^{\mathbb{R}}$ with the fip. Thus, $\bigcap \mathcal{S} \neq \emptyset$. Clearly, any $f \in \bigcap \mathcal{S}$ is a choice function of $\mathcal{G}$.

The assertion about $\mathbf{BPI}(\omega)$ follows from Theorem 7(i) below, the proof of (i)($\rightarrow$) of Proposition 1 and the first (or the second) assertion of (i) of the present lemma.

(ii) We have $\mathbf{2}^{\mathbb{R}} \simeq (\prod_{x\in\mathbb{R}} 2^{\{x\}}) \times (\prod_{x\in\mathcal{P}(\mathbb{R})\setminus\mathbb{R}}\{0\})$ ($\simeq$ means homeomorphic) and the latter set is a closed subset of $\mathbf{2}^{\mathcal{P}(\mathbb{R})}$. Thus, by our assumption, $\mathbf{2}^{\mathbb{R}}$ is compact.

On the other hand, since $(\mathbf{2}^{\mathbb{R}})^{\mathcal{P}(\mathbb{R})} \simeq \mathbf{2}^{\mathbb{R}\times\mathcal{P}(\mathbb{R})}$ and $\mathbf{2}^{\mathbb{R}\times\mathcal{P}(\mathbb{R})} \simeq \mathbf{2}^{\mathcal{P}(\mathbb{R})}$ (we have $|\mathbb{R} \times \mathcal{P}(\mathbb{R})| = |\mathcal{P}(\mathbb{R})|$ because $|\mathcal{P}(\mathbb{R})| \leq |\mathbb{R} \times \mathcal{P}(\mathbb{R})|$ and $|\mathcal{P}(\mathbb{R})| = |\mathcal{P}(\mathbb{R} \times \mathbb{R})| = |\mathcal{P}(\bigcup\{\{x\} \times \mathbb{R} : x \in \mathbb{R}\})| \geq |\bigcup\{\mathcal{P}(\{x\} \times \mathbb{R}) : x \in \mathbb{R}\}| = |\mathbb{R} \times \mathcal{P}(\mathbb{R})|$), it follows, by our assumption, that $(\mathbf{2}^{\mathbb{R}})^{\mathcal{P}(\mathbb{R})}$ is compact. Taking into account that the size of $\mathcal{K}(\mathbf{2}^{\mathbb{R}})$ is $|\mathcal{P}(\mathbb{R})|$, we can finish off the reasoning as in the proof of (i).

The first assertion about $\mathbf{BPI}(\mathbb{R})$ follows from the proof of Proposition 1 and Theorem 7(i). The second assertion follows from the original assertion of (ii) of the present lemma and Theorems 6(iii) and 7(i). ∎

We would like to point out here that in view of Proposition 2 and the fact that $\mathbb{R}$ is well-orderable in every Fraenkel–Mostowski permutation model (see [4]), every permutation model satisfies "$\mathbf{2}^{\mathbb{R}}$ is compact and Loeb".

Clearly, the set $A = \{\chi_{\{x\}} : x \in \mathbb{R}\}$, where for $U \subset \mathbb{R}$, χ_U is the characteristic function of U, is a relatively discrete subset of $\mathbf{2}^{\mathbb{R}}$ and $\chi_\emptyset$ is an accumulation point of A such that every neighborhood of $\chi_\emptyset$ leaves out finitely many members of A. If $\mathbf{UF}(\omega)$ fails, then $|S(\omega)| = \aleph_0$ and

$S(\omega)$ cannot have uncountable relatively discrete sets. However, if we assume $\mathbf{BPI}(\omega)$, we find, as a corollary to Lemma 4(i), that $S(\omega)$ has uncountable relatively discrete subsets and, in particular, $|\mathbb{R}| \le |S(\omega)|$.

Corollary 5. $\mathbf{BPI}(\omega)$ *implies "$S(\omega)$ has a relatively discrete subset of size $|\mathbb{R}|$". Hence, $|\mathbb{R}| \le |S(\omega)|$.*

Proof. Fix an almost disjoint family $\mathcal{A} = \{A_i : i \in \mathbb{R}\}$ of subsets of ω (for all $i, j \in \mathbb{R}$, $i \neq j$, $|A_i \cap A_j| < \aleph_0$) and choose, by our assumption and Lemma 4, for every $i \in \mathbb{R}$ an ultrafilter $\mathcal{F}_i \in S(\omega)$ which extends the family $\mathcal{H}_i$ of all cofinite subsets of A_i. It can be readily verified that $F = \{\mathcal{F}_i : i \in \mathbb{R}\}$ is a relatively discrete subset of $S(\omega)$. ∎

3. Main results. It is known that in **ZFC** the product $\mathbf{2}^{\mathbb{R}}$ is a continuous image of $S(\omega)$. We show in the next theorem that, in **ZF**, $\mathbf{BPI}(X)$ suffices to make $\mathbf{2}^{\mathcal{P}(X)}$ a continuous image of $S(X)$, $X = \omega, \mathbb{R}$.

Theorem 6.

(i) *In* **ZF**, *for $X = \omega, \mathbb{R}$,* $\mathbf{BPI}(X)$ *implies "$\mathbf{2}^{\mathcal{P}(X)}$ is a continuous image of $S(X)$".*
(ii) *It is relatively consistent with* **ZF** *that $S(\omega)$ is Loeb, but $\mathbf{2}^{\mathbb{R}}$ is not Loeb.*
(iii) *In* **ZF**, *for $X = \omega, \mathbb{R}$, "$S(X)$ is compact and Loeb" implies "$|S(X)| = |2^{\mathcal{P}(X)}|$", which in turn implies* $\mathbf{UF}(X)$.

Proof. (i) We prove the assertion for $X = \mathbb{R}$. The case $X = \omega$ can be treated similarly. Fix an independent family $\mathcal{A}$ in $\mathbb{R}$ of size $|\mathcal{P}(\mathbb{R})|$. Such a family is easily seen to exist in **ZF**. (If $D \subset \mathbf{2}^{\mathcal{P}(\mathbb{R})}$ is a dense set of size $|\mathbb{R}|$ (use the Hewitt–Marczewski–Pondiczery theorem [2, Theorem 2.3.15]), then the family $\mathcal{A} = \{A_x : x \in \mathcal{P}(\mathbb{R})\}$, where $A_x = \{d \in D : d(x) = 1\}$, is clearly independent.) It suffices, in view of [9, Proposition 3: if $|X| = |Y|$, i.e., there is a bijection $f : X \to Y$, then $\mathbf{2}^X$ and $\mathbf{2}^Y$ are topologically homeomorphic], to show that the product $\mathbf{2}^{\mathcal{A}}$ is a continuous image of $S(\mathbb{R})$. For every $\mathcal{F} \in S(\mathbb{R})$ let $f_{\mathcal{F}} = \chi_{\mathcal{F} \cap \mathcal{A}}$. Let $T : S(\mathbb{R}) \to \mathbf{2}^{\mathcal{A}}$ be the function $T(\mathcal{F}) = f_{\mathcal{F}}$. Since $\mathcal{A}$ is independent, it follows that for every $f \in 2^{\mathcal{A}}$, $\mathcal{W}_f = f^{-1}(\{1\}) \cup \{A^c : f(A) = 0\}$ has the fip. Hence, by $\mathbf{BPI}(\mathbb{R})$, $\mathcal{W}_f$ can be extended to an ultrafilter $\mathcal{F}_f$. Thus, $T(\mathcal{F}_f) = f$ and T is onto. Furthermore, for every $A \in \mathcal{A}$ and $i \in \{0, 1\}$, the set

$$T^{-1}([\{(A, i)\}]) = \begin{cases} [A] & \text{if } i = 1, \\ [A^c] & \text{if } i = 0, \end{cases}$$

is clearly open in $S(\mathbb{R})$. Thus, T is continuous and onto as required.

(ii) It is known that in Feferman's forcing model (Model $\mathcal{M}2$ in [4]) every ultrafilter on ω is principal. Hence $S(\omega)$ is a countable discrete space, meaning that $S(\omega)$ is Loeb. On the other hand, in $\mathcal{M}2$ there is a family of two-element subsets of $\mathcal{P}(\mathbb{R})$ having no choice functions (see [4]), hence by Theorem 12(ii) below, $\mathbf{2}^{\mathbb{R}}$ fails to be Loeb in this model.

(iii) For $X = \omega$ and for the first implication, it suffices to show that $|2^{\mathbb{R}}| \leq |S(\omega)|$. Let $\mathcal{A}$ be an independent family of ω of size $|\mathbb{R}|$. Clearly, for each $h \in 2^{\mathcal{A}}$, $\mathcal{H}_h = h^{-1}(\{1\}) \cup \{A^c : h(A) = 0\}$ is a subbase for a filter of ω. By our assumption and Proposition 1(ii), pick for each $h \in 2^{\mathcal{A}}$ an ultrafilter $\mathcal{U}_h$ which includes $\mathcal{H}_h$. Then the mapping $h \mapsto \mathcal{U}_h$, $h \in 2^{\mathcal{A}}$, is one-to-one.

For the second implication, note that if every ultrafilter on ω is fixed, then $|S(\omega)| = \aleph_0$, which is impossible in view of our assumption.

The assertions regarding $S(\mathbb{R})$ and $2^{\mathcal{P}(\mathbb{R})}$ are proved similarly upon noting also that $\mathbf{UF}(\omega) = \mathbf{UF}(\mathbb{R})$ (see [1]). ∎

THEOREM 7. *The following are provable in* **ZF**:

(i) *For every infinite set X, $S(X)$ embeds as a closed subspace of the product $\mathbf{2}^{\mathcal{P}(X)}$. Hence, if $\mathbf{2}^{\mathcal{P}(X)}$ is compact (or Loeb), then $S(X)$ is compact (resp. Loeb).*

(ii) *"$S(\mathbb{R})$ is Loeb" implies* $\mathbf{UF}(\omega)$. *Hence, by* (i), *"$\mathbf{2}^{\mathcal{P}(\mathbb{R})}$ is Loeb" implies* $\mathbf{UF}(\omega)$.

Proof. (i) Let $T : S(X) \to \mathbf{2}^{\mathcal{P}(X)}$ be the function defined by $T(\mathcal{F}) = \chi_{\mathcal{F}}$ for all $\mathcal{F} \in S(X)$. Clearly, T is one-to-one, continuous and open (we have $T([A]) = \{\chi_{\mathcal{F}} : \mathcal{F} \in [A]\} = [\{(A, 1)\}] \cap T(S(X))$). Put $F = \{T(\mathcal{F}_x) : x \in X\}$, where for every $x \in X$, $\mathcal{F}_x$ is the principal ultrafilter generated by x. As in the proof of Theorem 3.5 in [12] one verifies that for every $f \in \overline{F} \setminus F$, $f^{-1}(\{1\})$ is a free ultrafilter on X. Hence, $T(f^{-1}(\{1\})) = f$ and $\overline{F} \subseteq T(S(X))$. To complete the proof, it suffices to show that $T(S(X)) \subseteq \overline{F}$. We leave this as an easy exercise for the reader.

(ii) Basing on the fact that $\mathbf{UF}(\omega) = \mathbf{UF}(\mathbb{R})$, assume toward a contradiction that every ultrafilter on $\mathbb{R}$ is principal. This implies that $S(\mathbb{R})$ is homeomorphic to the discrete space $\mathbb{R}$. By our assumption ($S(\mathbb{R})$ is Loeb), $\mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ has a choice function. This means that $\mathcal{P}(\omega)$ is well-orderable, which in turn implies that every filter on ω can be extended to an ultrafilter. But then there is a free ultrafilter on ω, hence on $\mathbb{R}$, a contradiction. This completes the proof of (ii) and of the theorem. ∎

In view of Theorem 7 it is natural to ask whether $\mathbf{2}^{\mathbb{R}}$ embeds in $S(\omega)$, or whether $\mathbf{2}^{\mathcal{P}(\mathbb{R})}$ embeds in $S(\mathbb{R})$. The answer is *in the negative* even in **ZFC** set theory and it is derived from [2, Theorem 3.6.14, Corollary 3.6.15] (Theorem 3.6.14 is due to Novák [11]) and the fact that, in **ZFC** (in particular, in $\mathbf{ZF} + \mathbf{BPI}$), for every set X, $S(X)$ and $\beta(X)$ (the Čech–Stone extension of the discrete space X; see [2]) are homeomorphic.

We also obtain the above result as a by-product of our subsequent Theorem 8. We would like to draw the reader's attention to the fact that Theorem 8 (and the result in Remark 9) is established *in the absence* of the axiom $\mathbf{BPI}(\omega)$ (resp. $\mathbf{BPI}(X)$), or equivalently of "$S(\omega)$ is compact" (resp. of "$S(X)$ is compact").

THEOREM 8. *Let $\mathcal{A} = \{A_n : n \in \omega\}$ be a partition of ω. If $\{\mathcal{F}_n : n \in \omega\}$ is a family such that $\forall n \in \omega$, $\mathcal{F}_n \in S(A_n)$ and $\mathcal{F}$ is an ultrafilter of ω, then:*

- (i) *$\mathcal{W}_{\mathcal{F}} = \{F_w : F \in \mathcal{F},\ w \in W_F\}$, where $F_w = \bigcup\{w(n) : n \in F\}$ and $W_F = \prod_{n \in F} \mathcal{F}_n$, is a filterbase of ω. In addition, if $\mathcal{F}$ is free then $\bigcap \mathcal{W}_{\mathcal{F}} = \emptyset$.*
- (ii) *The filter $\mathcal{H}_{\mathcal{F}} = \{H \in \mathcal{P}(\omega) : W \subset H$ for some $W \in \mathcal{W}_{\mathcal{F}}\}$ generated by $\mathcal{W}_{\mathcal{F}}$ is an ultrafilter of ω. In addition, if $\mathcal{F}$ is free then $\mathcal{H}_{\mathcal{F}}$ is free.*
- (iii) *$\mathcal{H}_{\mathcal{F}} \in \overline{G}$, where $G = \{\mathcal{G}_n : n \in \omega\}$ and $\mathcal{G}_n$ is the (unique) ultrafilter of ω generated by $\mathcal{F}_n$.*
- (iv) *The mapping $T : S(\omega) \to \overline{G}$, $T(\mathcal{F}) = \mathcal{H}_{\mathcal{F}}$, is a homeomorphism. In particular, $|S(\omega)| = |\overline{G}|$.*

Proof. (i) Fix $F_w, H_u \in \mathcal{W}_{\mathcal{F}}$ and let $Q = F \cap H$. Clearly, $Q \in \mathcal{F}$, $v \in W_Q$, where $v(q) = w(q) \cap u(q), q \in Q$ and $Q_v = \bigcup\{v(s) : s \in Q\} \subseteq F_w \cap H_u$. Thus, $\mathcal{W}_{\mathcal{F}}$ is a filterbase. The second assertion is straightforward.

(ii) Fix $K \subset \omega$. If $K \notin \mathcal{H}_{\mathcal{F}}$ then $\{n \in \omega : K \cap A_n \in \mathcal{F}_n\} \notin \mathcal{F}$. Since $\mathcal{F}$ is maximal, it follows that $\{n \in \omega : K^c \cap A_n \in \mathcal{F}_n\} \in \mathcal{F}$. Hence, $K^c \in \mathcal{H}_{\mathcal{F}}$ and $\mathcal{H}_{\mathcal{F}}$ is an ultrafilter.

The second assertion follows from the second assertion of (i).

(iii) Clearly, if $\mathcal{F}$ is a principal ultrafilter then $\mathcal{H}_{\mathcal{F}} \in G$. So, we assume that $\mathcal{F}$ is a free ultrafilter. Since the family $\mathcal{V}_{\mathcal{F}} = \{[F_w] : F \in \mathcal{F}, w \in W_F\}$ is a neighborhood base of $\mathcal{H}_{\mathcal{F}}$ and $|V \cap G| = \aleph_0$ for every $V \in \mathcal{V}_{\mathcal{F}}$, it follows that $\mathcal{H}_{\mathcal{F}} \in \overline{G}$ as required.

(iv) Since for every $\mathcal{F}, \mathcal{S} \in S(\omega)$, $\mathcal{F} \neq \mathcal{S}$ implies $\mathcal{H}_{\mathcal{F}} \neq \mathcal{H}_{\mathcal{S}}$, we see that the mapping T is one-to-one. Since, for every $F \in \mathcal{P}(\omega)$,

$$\mathcal{F} \in [F] \leftrightarrow F \in \mathcal{F} \leftrightarrow \bigcup\{A_n : n \in F\} \in \mathcal{H}_{\mathcal{F}} \leftrightarrow \mathcal{H}_{\mathcal{F}} \in \Big[\bigcup\{A_n : n \in F\}\Big],$$

we see that T maps basic open sets of $S(\omega)$ to basic open sets of $T(S(\omega))$.

To complete the proof of (iv) it suffices to show that T is onto. Fix $\mathcal{H} \in \overline{G}$. It is easy to verify that $\mathcal{W} = \{W_H : H \in \mathcal{H}\}$ is a filterbase of ω, where $W_H = \{n \in \omega : H \in \mathcal{G}_n\}$.

We show next that the filter $\mathcal{F}_{\mathcal{W}}$ of ω generated by $\mathcal{W}$ is maximal. Fix $M \subseteq \omega$ and let $F_M = \bigcup\{A_n : n \in M\}$. We consider the following two cases:

- (a) $F_M \in \mathcal{H}$. In this case it is easily seen that $W_{F_M} = M$ and consequently $M \in \mathcal{F}_{\mathcal{W}}$.
- (b) $F_M^c \in \mathcal{H}$. This means that $W_{F_M^c} = \{n \in \omega : F_M^c \in \mathcal{G}_n\} = M^c \in \mathcal{F}_{\mathcal{W}}$.

Thus, $\mathcal{F}_{\mathcal{W}}$ is an ultrafilter of ω as required. Then $T(\mathcal{F}_{\mathcal{W}}) = \mathcal{H}_{\mathcal{F}_{\mathcal{W}}} = \mathcal{H}$. (Let $H \in \mathcal{H}$; then $W_H = \{n \in \omega : H \in \mathcal{G}_n\} \in \mathcal{F}_{\mathcal{W}}$. For every $n \in W_H$, $H \cap A_n \in \mathcal{F}_n$, therefore $H = \bigcup\{H \cap A_n : n \in W_H\} \in \mathcal{W}_{\mathcal{F}_{\mathcal{W}}} \subseteq \mathcal{H}_{\mathcal{F}_{\mathcal{W}}}$. Hence $\mathcal{H} \subseteq \mathcal{H}_{\mathcal{F}_{\mathcal{W}}}$ and, since $\mathcal{H}$ is an ultrafilter, it follows that $\mathcal{H} = \mathcal{H}_{\mathcal{F}_{\mathcal{W}}}$.) So, T is onto $\overline{G}$ and T is a homeomorphism, finishing the proof of the theorem. ∎

Remark 9. Analogously we can prove a generalization of Theorem 8, obtained by replacing ω by any infinite set X and replacing a countable partition by a partition indexed by any set I, changing only "In particular, $|S(\omega)| = |\overline{G}|$" in (iv) to "In particular, $|S(I)| \leq |S(X)|$".

The statement "every infinite closed subset of $S(\omega)$ includes a topological copy of $S(\omega)$" is of course a well-known **ZFC** result (see [2, Theorem 3.6.14]). However, we show next that by Theorem 8 the above statement is a theorem of a *strictly weaker* axiomatic system than **ZFC**, namely **ZF**+**DC**. In addition, although the statement "every infinite closed subset of $S(\omega)$ includes a topological copy of $S(\omega)$" implies, in **ZFC**, the statement "$S(\omega) \setminus \omega$ includes a topological copy of $S(\omega)$", this implication ceases to be valid in **ZF** set theory.

Theorem 10.

(i) **DC** *implies "every infinite closed subset of $S(\omega)$ includes a topological copy of $S(\omega)$".*

(ii) **DC** *and "$S(\omega)$ is compact and Loeb" together imply "every infinite closed subset of $S(\omega)$ has size $|2^{\mathbb{R}}|$", hence $S(\omega)$ has no countably infinite closed subspaces.*

(iii) **BPI**(ω) *implies "$S(\omega)\setminus\omega$ includes a topological copy of $S(\omega)$", which in turn implies* **UF**(ω). *Hence, "every infinite closed subset of $S(\omega)$ includes a topological copy of $S(\omega)$" does not imply "$S(\omega)\setminus\omega$ includes a topological copy of $S(\omega)$" in* **ZF**.

(iv) **CAC** *implies "for every infinite set X, and every relatively discrete subspace $G = \{\mathcal{G}_n : n \in \omega\}$ of $S(X)$, $S(\omega)$ is homeomorphic to $\overline{G}$". In particular,* **CAC** *restricted to countable families of non-empty sets of reals implies "for every countably infinite relatively discrete subset G of $S(\omega)$, $\overline{G}$ is homeomorphic to $S(\omega)$".*

Proof. (i) First we show that **DC** implies that every infinite closed subset of $S(\omega)$ includes a countably infinite relatively discrete subset. Fix an infinite closed subset F of $S(\omega)$. If F has no accumulation points, then the conclusion follows immediately from the fact that **DC** implies that every infinite set has a countably infinite subset. So assume that F has an accumulation point, say x_F. We shall construct a set $A = \{a_n : n \in \mathbb{N}\} \subseteq F$ and a set $\{V_n : n \in \mathbb{N}\}$ of open sets such that $a_i \in V_i$ and $V_i \cap V_j = \emptyset$ for $i \neq j$.

We commence by defining

$$W = \Big\{(V_1,\dots,V_n) \in \mathcal{B}^n : n \in \mathbb{N},\ V_i \cap F \neq \emptyset,\ V_i \cap V_j = \emptyset \text{ for } i \neq j, \\ \text{and } x_F \notin \bigcup\{V_i : i = 1,\dots,n\}\Big\},$$

where $\mathcal{B}$ is the clopen base $\{[U] : U \subseteq \omega\}$ of $S(\omega)$. Since $|\mathcal{B}| = |\mathbb{R}| = |\mathbb{R}^\omega|$, it follows that $|W| = |\mathbb{R}|$. For all $x, y \in W$, we define a binary relation R on W by stating xRy if and only if $x \subseteq y$. We assert that $\mathrm{Dom}(R) = W$. Indeed, let $(V_1,\dots,V_n) \in W$ for some $n \in \mathbb{N}$. Since $x_F \notin \bigcup\{V_i : i = 1,\dots,n\}$ and $\bigcup\{V_i : i = 1,\dots,n\}$ is closed, there exists $V \in \mathcal{B}$ such that $x_F \in V$ and $V \cap \bigcup\{V_i : i = 1,\dots,n\} = \emptyset$. Since x_F is an accumulation point of F, let $y \in (V \cap F) \setminus \{x_F\}$. Then there exist disjoint basic neighborhoods U_1 and U_2 of x_F and y, respectively, such that both U_1 and U_2 are contained in V. Put $V_{n+1} = U_2$. Then $(V_1,\dots,V_n,V_{n+1}) \in W$ and $(V_1,\dots,V_n)R(V_1,\dots,V_n,V_{n+1})$, so $\mathrm{Dom}(R) = W$ as asserted.

By **DC**, there exists a sequence $(V_n)_{n\in\mathbb{N}}$ of basic open sets such that $V_n \cap F \neq \emptyset$, $V_n \cap V_m = \emptyset$ for $n \neq m$ (and $x_F \notin \bigcup\{V_n : n \in \mathbb{N}\}$). Since for every $n \in \mathbb{N}$, $V_n \cap F$ is a non-empty (closed) subset of $S(\omega)$, we may let, by **DC**, $a_n \in V_n \cap F$, $n \in \mathbb{N}$. Put $A = \{a_n : n \in \mathbb{N}\}$. Then A is a countably infinite relatively discrete subset of $S(\omega)$.

Now we prove the original assertion. Let F be an infinite closed subset of $S(\omega)$ and let I_F be the set of all isolated points of F. It follows from the first part of the proof that I_F is infinite (otherwise, $H = F \setminus I_F$ is an infinite, dense-in-itself, hence closed, subset of F, hence of $S(\omega)$; thus, H contains a countably infinite relatively discrete subset, a contradiction). By **DC**, I_F has a countably infinite subset, say $G = \{\mathcal{G}_n : n \in \omega\}$. By **DC** again, pick, for every $n \in \omega$, $G_n \in \mathcal{G}_n$ such that $[G_n] \cap G = \{\mathcal{G}_n\}$. Clearly, for all $n \in \omega$ and $m \in n, G_m \notin \mathcal{G}_n$ and consequently $G_m^c \in \mathcal{G}_n$. Thus, for all $n \in \omega$, $G_n \setminus \bigcup\{G_m^c : m \in n\} = G_n \cap \bigcap\{G_m^c : m \in n\} \in \mathcal{G}_n$. Hence, we may assume that $\{G_n : n \in \omega\}$ is a family of pairwise disjoint subsets of ω. Let $\mathcal{A} = \{A_n : n \in \omega\}$ be a partition of ω such that $G_n \subseteq A_n$ for all $n \in \omega$. (Hence, $A_n \in \mathcal{G}_n$ for all $n \in \omega$.) Clearly, for every $n \in \omega$, $\mathcal{F}_n = \{U \cap A_n : U \in \mathcal{G}_n\}$ is an ultrafilter of A_n, and $\mathcal{G}_n$ is the unique ultrafilter of ω generated by $\mathcal{F}_n$. By Theorem 8, $S(\omega)$ is homeomorphic to $\overline{G} \subseteq F$, finishing the proof of (i).

(ii) This follows from part (i) and from Theorem 6(iii).

(iii) Let $\mathcal{A} = \{A_n : n \in \omega\}$ be a partition of ω into infinite sets. For each $n \in \omega$, let $\mathcal{H}_n$ be the filterbase of ω consisting of all subsets of A_n which are cofinite in A_n. By **BPI**(ω) let, for each $n \in \omega$, $\mathcal{G}_n \in S(\omega)$ be such that $\mathcal{H}_n \subset \mathcal{G}_n$ (see Lemma 4(i)). Clearly, $G = \{\mathcal{G}_n : n \in \omega\}$ is a countably infinite relatively discrete subset of $S(\omega)$. Furthermore, $G \subset S(\omega) \setminus \omega$ and $\overline{G} \subset \overline{S(\omega) \setminus \omega} = S(\omega) \setminus \omega$ (as $S(\omega) \setminus \omega$ is closed in $S(\omega)$). Letting, for each

$n \in \omega$, $\mathcal{F}_n = \{U \cap A_n : U \in \mathcal{G}_n\}$, an application of Theorem 8 at this point shows that $S(\omega)$ is homeomorphic to $\overline{G} \subset S(\omega) \setminus \omega$.

That "$S(\omega) \setminus \omega$ includes a topological copy of $S(\omega)$" implies $\mathbf{UF}(\omega)$ is straightforward.

The last assertion of (iii) follows from the fact that **DC**, hence by (i) "every infinite closed subset of $S(\omega)$ includes a topological copy of $S(\omega)$", holds in Feferman's forcing model $\mathcal{M}2$ in [4], whereas $\mathbf{UF}(\omega)$ fails in that model (see [4]).

(iv) By **CAC** let, for every $n \in \omega$, $G_n \subseteq X$ be such that $[G_n] \cap G = \{\mathcal{G}_n\}$. Without loss of generality assume that the G_n's are pairwise disjoint (see the proof of part (i)) and that $Y = X \setminus \bigcup\{G_n : n \in \omega\}$ is infinte. By **CAC**, Y has a countably infinite subset, hence Y has a partition $\{U_n : n \in \omega\}$. For each $n \in \omega$, let $A_n = G_n \cup U_n$. Then $\{A_n : n \in \omega\}$ is a partition of X and letting, for each $n \in \omega$, $\mathcal{F}_n$ be as in the proof of (iii), we may conclude by Remark 9 that $S(\omega)$ is homeomorphic to $\overline{G}^{S(X)}$, finishing the proof of (iv) and of the theorem. ∎

Corollary 11.

(i) **DC** *and "$S(\mathbb{R})$ is compact and Loeb" together imply "every infinite closed subset of $S(\mathbb{R})$ has size $|2^{\mathcal{P}(\mathbb{R})}|$", hence $S(\mathbb{R})$ has no countably infinite closed subspaces.*

(ii) *In* **ZFC**, $\mathbf{2}^{\mathcal{P}(X)}$ *does not embed as a subspace of $S(X)$, where $X = \omega, \mathbb{R}$.*

Proof. (ii) We argue only for $X = \omega$ and assume toward a contradiction that $h : \mathbf{2}^{\mathcal{P}(\omega)} \to S(\omega)$ is an embedding. Let $G = \{\chi_{\{n\}} : n \in \omega\} \subseteq 2^{\mathcal{P}(\omega)}$. Clearly, G is a relatively discrete subset of $\mathbf{2}^{\mathcal{P}(\omega)}$, $\overline{G} = G \cup \{\mathbf{0}\}$ in $\mathbf{2}^{\mathcal{P}(\omega)}$, where $\mathbf{0} = \chi_\emptyset$, and every neighborhood of $\mathbf{0}$ includes all but finitely many members of G. Thus, $\overline{G}$ is a compact subset of $2^{\mathcal{P}(\omega)}$ and consequently we may identify $\overline{G}$ with a countable closed subset of $S(\omega)$ homeomorphic to the one-point compactification of ω with the discrete topology. This contradicts the conclusion of part (ii) of Theorem 10 and completes the proof. ∎

4. Further results. In this section we generalize Theorem 3 by replacing ω with $\mathcal{P}(\omega)$. We observe, as expected, that all statements concerning $\mathbf{BPI}(\omega)$ given in Theorem 3 generalize without any difficulty. In particular, we note that Theorem 13 below is an analogue of Theorem 6 in [6].

Theorem 12.

(i) *"$\mathbf{2}^{\mathbb{R}}$ is a Loeb space" iff "every product of finite subspaces of $\mathbb{R}$ is Loeb".*

(ii) *"$\mathbf{2}^{\mathbb{R}}$ is a Loeb space" implies that $[\mathcal{P}(\mathbb{R})]^{<\omega} \setminus \{\emptyset\}$ has a choice function* *(see also* [8]*). Hence, it implies that* $|[\mathcal{P}(\mathbb{R})]^{<\omega}| = |\mathcal{P}(\mathbb{R})|$ *and a well-ordering on each* $A \in [\mathcal{P}(\mathbb{R})]^{<\omega} \setminus \{\emptyset\}$ *can be defined.*

Proof. (i) We only prove $(\rightarrow)$ as the reverse implication is obvious. Fix a family $(X_i)_{i\in I}$ of finite subsets of $\mathbb{R}$. Since $|[\mathbb{R}]^{<\omega}| = |\mathbb{R}|$ and $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$, we may assume that the sets X_i are pairwise disjoint. Thus, $\mathbf{X} = \prod_{i\in I} \mathbf{X}_i$ embeds as a closed subspace in $\mathbf{2}^{\bigcup\{X_i : i\in I\}}$ (see [8]) and the latter space can be viewed as a closed subspace of $\mathbf{2}^{\mathbb{R}}$. Hence, by our assumption, $\mathbf{X}$ is Loeb.

(ii) Since $|\mathcal{P}(\mathbb{R})| = |2^{\mathbb{R}}|$, we may view each finite subset of $\mathcal{P}(\mathbb{R})$ as a finite subset of $2^{\mathbb{R}}$. Furthermore, since $\mathbf{2}^{\mathbb{R}}$ is a T_2 space, every finite subset of $2^{\mathbb{R}}$ is a closed set. Therefore, by our assumption, the family $[\mathcal{P}(\mathbb{R})]^{<\omega} \setminus \{\emptyset\}$ has a choice function. By the fact that for every $n \in \mathbb{N}$, $|\mathcal{P}(\mathbb{R})^n| = |(2^{\mathbb{R}})^n| = |2^{\mathbb{R}\times n}| = |\mathcal{P}(\mathbb{R})|$ and our assumption we can define for every $A \in [\mathcal{P}(\mathbb{R})]^{<\omega}$ an enumeration $\{a_j^A : j \leq |A|\}$ of A. We have on the one hand $|\mathcal{P}(\mathbb{R})^\omega| = |(2^{\mathbb{R}})^\omega| = |2^{\mathbb{R}\times\omega}| = |\mathcal{P}(\mathbb{R})|$ and for every $n \in \mathbb{N}$, $|\mathcal{P}(\mathbb{R})^n| = |\mathcal{P}(\mathbb{R})|$, and on the other hand, by our assumption, $|[\mathcal{P}(\mathbb{R})]^n| \leq |\mathcal{P}(\mathbb{R})^n|$ via the map $F_n(A)(j) = a_j^A$, $j \leq n$. Hence, $|\mathcal{P}(\mathbb{R})| \leq |[\mathcal{P}(\mathbb{R})]^{<\omega}| = |\bigcup\{[\mathcal{P}(\mathbb{R})]^n : n \in \mathbb{N}\}| \leq |\bigcup\{\mathcal{P}(\mathbb{R})^n : n \in \mathbb{N}\}| \leq |\mathcal{P}(\mathbb{R})^\omega| \leq |\mathcal{P}(\mathbb{R})|$. ∎

THEOREM 13. *The following statements are pairwise equivalent:*

(i) *In a Boolean algebra* $\mathcal{B}$ *of size* $\leq |2^{\mathbb{R}}|$ *every filter can be extended to an ultrafilter.*
(ii) $\mathbf{BPI}(\mathbb{R})$.
(iii) $S(\mathbb{R})$ *is compact.*
(iv) $\mathbf{2}^{\mathcal{P}(\mathbb{R})}$ *is compact.*
(v) *For every compact* T_2 *space* $\mathbf{X}$ *having a dense subset of size* $\leq |\mathbb{R}|$, *the product* $\mathbf{X}^{\mathcal{P}(\mathbb{R})}$ *is compact.*
(vi) *Every product of non-empty finite discrete subsets of* $\mathcal{P}(\mathbb{R})$ *is compact.*

Proof. (i)→(ii). This is clear.

(ii)↔(iii). Follow the well-known proof that **BPI** is equivalent to "for every set X, the Stone space $S(X)$ of the powerset algebra $\mathcal{P}(X)$ is compact".

(iii)→(iv). This follows at once from Theorem 6(i).

(iv)→(v). Fix a compact T_2 space $\mathbf{X}$ having a dense subset D of size $\leq |\mathbb{R}|$. By [2, Theorem 2.3.15], $\mathbf{X}^{\mathcal{P}(\mathbb{R})}$ has a dense subset of size $|\mathbb{R}|$. Since our assumption implies $\mathbf{BPI}(\mathbb{R})$ (by Theorem 7(i), $S(\mathbb{R})$ is compact, and it is easy to see that the latter is true iff $\mathbf{BPI}(\mathbb{R})$ is true), we may follow the proof of (ii)→(iii) of Theorem 6 in [6] in order to verify that $\mathbf{X}^{\mathcal{P}(\mathbb{R})}$ is compact.

(v)→(vi). First notice that our assumption clearly implies that $\mathbf{2}^{\mathcal{P}(\mathbb{R})}$ is compact. Fix a family $\mathcal{A} = \{A_i : i \in I\}$ of non-empty finite subsets of $\mathbf{2}^{\mathbb{R}}$. By Lemma 4 and Theorem 12, it follows that $|I| \leq |\mathcal{P}(\mathbb{R})|$. As we observed in the proof of the first assertion of (ii) of Lemma 4, $\mathbf{2}^{\mathbb{R}}$ is homeomorphic to a closed subset of $\mathbf{2}^{\mathcal{P}(\mathbb{R})}$. It follows that, for every $i \in I$, we may view A_i as a finite subset of the compact T_2 space $\mathbf{2}^{\mathcal{P}(\mathbb{R})}$. Hence, A_i is a (discrete) closed subspace of $\mathbf{2}^{\mathcal{P}(\mathbb{R})}$. Thus, $\prod_{i\in I} A_i$ is a closed subspace of $(\mathbf{2}^{\mathcal{P}(\mathbb{R})})^{\mathcal{P}(\mathbb{R})}$. Since $(\mathbf{2}^{\mathcal{P}(\mathbb{R})})^{\mathcal{P}(\mathbb{R})} \simeq \mathbf{2}^{\mathcal{P}(\mathbb{R})\times\mathcal{P}(\mathbb{R})} \simeq \mathbf{2}^{\mathcal{P}(\mathbb{R})}$ (indeed, notice that $|\mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R})| = |\mathcal{P}(\mathbb{R})|$ and use [9, Proposition 3]), it follows that $(\mathbf{2}^{\mathcal{P}(\mathbb{R})})^{\mathcal{P}(\mathbb{R})}$ is compact, hence $\prod_{i\in I} A_i$ is compact as required.

(vi)→(i). Our assumption implies that $\mathbf{2}^{\mathcal{P}(\mathbb{R})}$ is compact. In order to verify that (i) holds, mimic the proofs of (vi)→(vii)→(i) of Theorem 6 in [6] with $\mathcal{P}(\mathbb{R})$ in place of $\mathbb{R}$. ∎

Corollary 14.

(i) *For $X = \omega, \mathbb{R}$, "$S(X)$ is compact and Loeb" iff "$\mathbf{2}^{\mathcal{P}(X)}$ is compact and Loeb".*

(ii) $\mathbf{BPI}(\mathbb{R})$ *implies "$\mathbf{2}^{\mathbb{R}}$ is compact" and "$\mathbf{2}^{\mathbb{R}}$ is Loeb". In particular, under* $\mathbf{BPI}(\mathbb{R})$*, "$\mathbf{2}^{\mathbb{R}}$ is compact" iff "$\mathbf{2}^{\mathbb{R}}$ is Loeb", and "$S(\omega)$ is compact" iff "$S(\omega)$ is Loeb".*

(iii) *"$\mathbf{2}^{\mathbb{R}}$ is compact and Loeb" iff "for every separable compact T_2 space* $\mathbf{X}$*, the product $\mathbf{X}^{\mathbb{R}}$ is compact and Loeb".*

(iv) $\mathbf{BPI}(\mathbb{R})$ *implies "$S(\mathbb{R})$ has a relatively discrete subset of size $|\mathcal{P}(\mathbb{R})|$".*

Proof. (i) follows easily from Theorems 6(i), 7(i) and the fact that $\mathbf{BPI}(X)$ iff $S(X)$ is compact.

(ii) follows from Lemma 4 and Theorem 13. (iii)(←) is obvious.

(iii)(→) Fix a separable compact T_2 space $\mathbf{X}$. By [2, Theorem 2.3.15], $\mathbf{X}^{\mathbb{R}}$ is separable. By our assumption and Theorem 3, $\mathbf{X}^{\mathbb{R}}$ is compact. Let $\mathrm{RO}(\mathbf{X}^{\mathbb{R}})$ be the family of all regular open sets of $\mathbf{X}^{\mathbb{R}}$ and let G be a countable dense subset of $\mathbf{X}^{\mathbb{R}}$. Since for any $O, Q \in \mathrm{RO}(\mathbf{X}^{\mathbb{R}})$, $O \neq Q$ implies $O \cap G \neq Q \cap G$, it follows that $|\mathrm{RO}(\mathbf{X}^{\mathbb{R}})| \leq |\mathbb{R}|$ and consequently $\mathbf{X}^{\mathbb{R}}$ has a base $\mathcal{B}$ of size $\leq |\mathbb{R}|$. It follows, by the embedding lemma, that $\mathbf{X}^{\mathbb{R}}$ embeds in the product $[0,1]^{\mathbb{R}}$ as a closed subspace. Since "$[\mathbf{0},\mathbf{1}]^{\mathbb{R}}$ is Loeb" iff "$2^{\mathbb{R}}$ is Loeb" (see [7]), it follows by our assumption that "$\mathbf{X}^{\mathbb{R}}$ is Loeb" as required.

(iv) Fix an independent family $\mathcal{A} = \{A_i : i \in \mathcal{P}(\mathbb{R})\}$ in $\mathbb{R}$ as in the proof of Theorem 6(i). For every $i \in \mathcal{P}(\mathbb{R})$ let $\mathcal{W}_i = \{A_i\} \cup \{(A_j)^c : j \in \mathcal{P}(\mathbb{R}) \setminus \{i\}\}$ and $K_i = \{\mathcal{F} \in S(\mathbb{R}) : \mathcal{W}_i \subset \mathcal{F}\}$. Clearly, K_i is a non-empty closed subset of the compact space $S(\mathbb{R})$. By Theorem 13(v), the product $S(\mathbb{R})^{\mathcal{P}(\mathbb{R})}$ is compact, hence $\bigcap\{\pi_i^{-1}(K_i) : i \in \mathcal{P}(\mathbb{R})\} \neq \emptyset$. Fixing f in the latter intersection, we easily conclude that $F = \{f(i) : i \in \mathcal{P}(\mathbb{R})\}$ is a relatively discrete subset of $S(\mathbb{R})$. ∎

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References

[1] M. R. Burke, *Liftings for noncomplete probability spaces*, in: Papers on General Topology and Applications (Madison, WI, 1991), Ann. New York Acad. Sci. 704, New York Acad. Sci., New York, 1993, 34–37.

[2] R. Engelking, *General Topology*, PWN–Polish Sci. Publ., Warszawa, 1977.

[3] J. Fossy and M. Morillon, *The Baire category property and some notions of compactness*, J. London Math. Soc. 57 (1998), 1–19.

[4] P. Howard and J. E. Rubin, *Consequences of the Axiom of Choice*, Math. Surveys Monogr. 59, Amer. Math. Soc., Providence, RI, 1998 (http://consequences.emich.edu/conseq.htm).

[5] K. Keremedis, *The compactness of* $2^{\mathbb{R}}$ *and the axiom of choice*, Math. Logic Quart. 46 (2000), 569–571.

[6] —, *Tychonoff products of two-element sets and some weakenings of the Boolean prime ideal theorem*, Bull. Polish Acad. Sci. Math. 53 (2005), 349–359.

[7] —, *The Boolean prime ideal theorem and products of cofinite topologies*, submitted.

[8] —, *Compact and Loeb Hausdorff spaces in ZF and the axiom of choice for families of finite sets*, submitted.

[9] K. Keremedis, E. Felouzis and E. Tachtsis, *On the compactness and countable compactness of* $2^{\mathbb{R}}$ *in* ZF, Bull. Polish Acad. Sci. Math. 55 (2007), 293–302.

[10] P. A. Loeb, *A new proof of the Tychonoff theorem*, Amer. Math. Monthly 72 (1965), 711–717.

[11] J. Novák, *On the Cartesian product of two compact spaces*, Fund. Math. 40 (1953), 106–112.

[12] E. Tachtsis, *On the set-theoretic strength of countable compactness of the Tychonoff product* $2^{\mathbb{R}}$, Bull. Polish Acad. Sci. Math. 58 (2010), 91–107.

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