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Two Inequalities for the First Moments of a Martingale, its Square Function and its Maximal Function

by

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Summary. Given a Hilbert space valued martingale (M_n), let (M_n^*) and $(S_n(M))$ denote its maximal function and square function, respectively. We prove that

$$\mathbb{E}|M_n| \le 2\mathbb{E}S_n(M), \qquad n = 0,1,2,\ldots,$$
$$\mathbb{E}M_n^* \le \mathbb{E}|M_n| + 2\mathbb{E}S_n(M), \qquad n = 0,1,2,\ldots.$$

The first inequality is sharp, and it is strict in all nontrivial cases.

1. Introduction. In [1] Burkholder proposed a method for showing martingale maximal inequalities and in [2] he introduced a new approach to study the behaviour of the maximal function and square function simultaneously. In the present paper we use this method to obtain a sharp inequality between the first moments of a martingale and its square function, as well as some other inequalities involving the maximal function.

Let us fix the notation. In the following, $(\Omega, \mathcal{F}, (\mathcal{F}_n), \mathbb{P})$ will be a probability space equipped with some discrete filtration. Let $\mathcal{H}$ be a Hilbert space with norm $|\cdot|$ and scalar product $(\cdot\,,\cdot)$. Let (M_n) be an $(\mathcal{F}_n)$-martingale taking values in some separable subspace of $\mathcal{H}$. The difference sequence (d_n) of the martingale (M_n) is defined by $d_0 = M_0$ a.s., $d_n = M_n - M_{n-1}$ a.s., $n = 1,2,\ldots$. Let $(S_n(M))$ be the square function and (M_n^*) the maximal

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function of the martingale (M_n), which are processes defined by

$$S_n(M) = \Big[\sum_{k=0}^{n} |d_k|^2\Big]^{1/2}, \quad M_n^* = \sup_{0\le k\le n} |M_k|, \quad n = 0,\ 1,2,\ldots.$$

Inequalities between moments of a martingale, its square function and maximal function have been deeply studied in the literature. Such inequalities are of fundamental importance to martingale theory and harmonic analysis. We just mention two basic results:

(*Doob's inequality*) For $1 < p < \infty$,

$$\mathbb{E}|M_n^*|^p \le \left(\frac{p}{p-1}\right)^p \mathbb{E}|M_n|^p, \quad n = 0,1,2,\ldots,$$

and the constant $(p/(p-1))^p$ is best possible.

(*Burkholder–Davis–Gundy inequalities*) For $1 < p < \infty$,

$$c_p\mathbb{E}(S_n(M))^p \le \mathbb{E}|M_n|^p \le C_p\mathbb{E}(S_n(M))^p, \quad n = 0,1,2,\ldots,$$

where $C_p = c_p^{-1} = (p^* - 1)^p$, $p^* = \max\{p, p/(p-1)\}$. The constant C_p is best possible for $p \ge 2$ and the constant c_p is best possible for $p \le 2$. In the remaining cases the best constants are not known.

In this paper we continue the study on the comparison of moments of a martingale, its square function and its maximal function. We will be particularly interested in first moments. The inequality

$$c\mathbb{E}M_n^* \le \mathbb{E}S_n(M) \le C\mathbb{E}M_n^*$$

for general martingales was first proved by Davis [3]. Later, Garsia [4], [5] proved that the left inequality holds with $c = \sqrt{10}$ and the right one with $C = 2+\sqrt{5}$. Both these constants are not optimal. Quite recently, Burkholder [2] proved that the best constant in the right inequality is $\sqrt{3}$. We will exploit his methods to investigate some other inequalities of this type.

Precisely, we will prove the following two results.

THEOREM 1. *The following inequality holds*:

$$\mathbb{E}|M_n| \le 2\mathbb{E}S_n(M), \quad n = 0,1,2,\ldots \tag{1.1}$$

and the constant 2 *is best possible. Moreover*, *the inequality is strict in all nontrivial cases.*

THEOREM 2. *We have*

$$\mathbb{E}M_n^* \le \mathbb{E}|M_n| + 2\mathbb{E}S_n(M), \quad n = 0,1,2,\ldots. \tag{1.2}$$

As an immediate consequence of the theorems above we obtain an inequality between the first moments of the maximal function and the square function of a martingale, however, with a worse constant.

COROLLARY 1. *We have*

$$\mathbb{E}M_n^* \le 4\mathbb{E}S_n(M), \qquad n = 0, 1, 2, \ldots.$$

The paper is organized as follows. In the next section we present the main tools for proving martingale inequalities, which enable us to reduce the proof of a certain inequality to finding a special function with some convex-type properties. Section 3 is devoted to the proof of Theorem 1 and in the last section we deal with the proof of Theorem 2.

2. Burkholder's method. In this section we state two theorems of Burkholder. They hold for any Banach space valued martingales (M_n).

The first of them (a slight modification of Theorem 2.1 of [2]) provides the key tool to prove maximal inequalities.

THEOREM 3. *Let B be a Banach space and suppose U, V are functions from $B \times [0,\infty)^2$ to $\mathbb{R}$ satisfying*

$$U(x,y,z) \le V(x,y,z), \tag{2.1}$$

$$U(x,t,z) = U(x,t,|x| \vee z), \tag{2.2}$$

and the further condition that if $|x| \le z$ and d is any mean-zero $\mathcal{F}$-measurable random variable, then

$$\mathbb{E}U(x+d, \sqrt{y^2+|d|^2}, |x+d| \vee z) \ge U(x,y,z). \tag{2.3}$$

Then for any nonnegative integer n and any martingale (M_n), we have

$$\mathbb{E}V(M_n, S_n(M), M_n^*) \ge U(M_0, S_0(M), |M_0|). \tag{2.4}$$

Proof. We have, by (2.1),

$$\begin{aligned}\mathbb{E}V(M_n, S_n(M), M_n^*) &\ge \mathbb{E}U(M_n, S_n(M), M_n^*)\\ &= \mathbb{E}[\mathbb{E}U(M_n, S_n(M), M_n^*) \,|\, \mathcal{F}_{n-1}].\end{aligned}$$

Therefore, it suffices to prove that the process $(U(M_n, S_n(M), M_n^*))$ is a submartingale. Applying (2.3) conditionally with respect to $\mathcal{F}_{n-1}$, we obtain the inequality

$$\begin{aligned}&\mathbb{E}[U(M_n, S_n(M), M_n^*) \,|\, \mathcal{F}_{n-1}]\\ &\quad = \mathbb{E}[U(M_{n-1}+d_n, \sqrt{S_{n-1}^2(M)+d_n^2}, |M_{n-1}+d_n| \vee M_{n-1}^*) \,|\, \mathcal{F}_{n-1}]\\ &\quad\ge U(M_{n-1}, S_{n-1}(M), M_{n-1}^*)\end{aligned}$$

and the inequality (2.4) follows immediately. ∎

The second theorem (Lemma 4.1 in [2]) enables us to obtain lower bounds for the constants in martingale inequalities.

Theorem 4. *Let B be a Banach space. For a function $V : B\times[0,\infty)^2 \to \mathbb{R}$ define $U : B \times [0,\infty)^2 \to [-\infty,\infty)$ by*

$$U(x,y,z) = \inf\{\mathbb{E}V(M_n, \sqrt{y^2-|x|^2+S_n^2(M)}, M_n^*\vee z)\}, \tag{2.5}$$

where the infimum is taken over all martingales (M_n) starting from x and over all nonnegative integers n. Then the pair (U,V) satisfies (2.1)–(2.3).

We refer the reader to [2] for the proof of this result. Let us note that the above theorems may as well be used to prove inequalities which only involve a martingale and its square function, by omitting the variable z (and the condition (2.2)).

3. The proof of Theorem 1. First we prove some auxiliary inequalities, which we will need later.

Lemma 1. *Let $x,d \in \mathcal{H}$ and $y \in \mathbb{R}_+$, $y < |x|$. Then*

$$\sqrt{y^2+|d|^2} - y \geq \sqrt{|x|^2+|d|^2} - |x|. \tag{3.1}$$

If, moreover, $\sqrt{y^2+|d|^2} \geq |x+d|$, then

$$\sqrt{2y^2+2|d|^2-|x+d|^2} - 2y \geq \sqrt{2|x|^2+2|d|^2-|x+d|^2} - 2|x|. \tag{3.2}$$

Proof. The inequality (3.1) is equivalent to

$$|x| - y \geq \sqrt{|x|^2+|d|^2} - \sqrt{y^2+|d|^2} = \frac{|x|^2-y^2}{\sqrt{|x|^2+|d|^2}+\sqrt{y^2+|d|^2}},$$

or

$$\sqrt{|x|^2+|d|^2}+\sqrt{y^2+|d|^2} \geq |x|+y,$$

which is obvious.

Now we turn to (3.2). We may write it as follows:

$$\begin{aligned} 2|x|-2y &\geq \sqrt{2|x|^2+2|d|^2-|x+d|^2} - \sqrt{2y^2+2|d|^2-|x+d|^2} \\ &= \frac{2|x|^2-2y^2}{\sqrt{2|x|^2+2|d|^2-|x+d|^2}+\sqrt{2y^2+2|d|^2-|x+d|^2}}, \end{aligned}$$

which can be written as

$$\sqrt{2|x|^2+2|d|^2-|x+d|^2}+\sqrt{2y^2+2|d|^2-|x+d|^2} \geq |x|+y.$$

The left hand side above is equal to

$$|x-d| + \sqrt{|x+d|^2+2(y^2+|d|^2-|x+d|^2)}$$

and, due to the assumption $\sqrt{y^2+|d|^2} \geq |x+d|$, it can be bound from below by

$$|x-d|+|x+d| \geq 2|x| > |x|+y. \blacksquare$$

We are now ready to use Theorem 3 of Burkholder. Let us introduce functions $\widehat{U}$, $\widehat{V} : \mathcal{H} \times [0, \infty)^2 \to \mathbb{R}$ defined by

$$\widehat{U}(x, y, z) = u(x, y) = \begin{cases} \sqrt{2y^2 - |x|^2} & \text{if } y \geq |x|, \\ 2y - |x| & \text{if } y < |x|, \end{cases} \tag{3.3}$$

$$\widehat{V}(x, y, z) = v(x, y) = 2y - |x|. \tag{3.4}$$

Then we have

LEMMA 2. *The functions* $\widehat{U}$, $\widehat{V}$ *satisfy* (2.1)–(2.3).

Proof. The condition (2.2) holds trivially. Let us deal with the majorizing condition (2.1). Note that for any $x \in \mathcal{H}$ and $y \in [0, \infty)$ satisfying $|x| \leq \sqrt{2}\, y$, we have

$$\sqrt{2y^2 - |x|^2} \leq 2y - |x|. \tag{3.5}$$

Indeed, squaring both sides, we obtain $2(y - |x|)^2 \geq 0$. Therefore (2.1) holds if $y \geq |x|$. In the opposite case both sides of (2.1) are equal.

Now we turn to (2.3). Suppose first that $y \geq |x|$. If $y = 0$, then $x = 0$ and the inequality is trivial: it reduces to the inequality $\mathbb{E}|d| \geq 0$. Suppose then that $y > 0$. We shall show that for any $d \in \mathcal{H}$,

$$u(x + d, \sqrt{y^2 + d^2}) \geq u(x, y) + \frac{(x, d)}{\sqrt{2y^2 - |x|^2}}. \tag{3.6}$$

This will immediately yield (2.3) (by taking expectations of both sides). We have

$$2(\sqrt{y^2 + |d|^2})^2 - |x + d|^2 \geq 2|x|^2 + 2|d|^2 - |x|^2 - 2(x, d) - |d|^2 = |x - d|^2 \geq 0$$

and, due to (3.3) and (3.5),

$$u(x + d, \sqrt{y^2 + |d|^2}) \geq \sqrt{2(y^2 + |d|^2) - |x + d|^2}.$$

Hence it suffices to check the inequality

$$\sqrt{2(y^2 + |d|^2) - |x + d|^2} \geq \sqrt{2y^2 - |x|^2} - \frac{(x, d)}{\sqrt{2y^2 - |x|^2}}, \tag{3.7}$$

or

$$\sqrt{2y^2 - |x|^2}\sqrt{2y^2 - |x|^2 - 2(x, d) + |d|^2} \geq 2y^2 - |x|^2 - (x, d).$$

But we have

$$\begin{aligned}(2y^2 - |x|^2)&(2y^2 - |x|^2 - 2(x, d) + |d|^2) \\ &\geq (2y^2 - |x|^2)^2 - 2(2y^2 - |x|^2)(x, d) + |x|^2|d|^2 \geq (2y^2 - |x|^2 - (x, d))^2\end{aligned}$$

and the inequality follows.

Now suppose that $y < |x|$ and let $d \in \mathcal{H}$. Again, the inequality (2.3) will follow immediately by taking expectations, if we show that

$$u(x + d, \sqrt{y^2 + |d|^2}) \geq u(x, y) + (x/|x|, d).$$

If $\sqrt{y^2+d^2} < |x+d|$, then we must show that

$$2\sqrt{y^2+d^2} - |x+d| \geq 2y - |x| - (x/|x|, d),$$

or, equivalently,

$$2\sqrt{y^2+|d|^2} - 2y \geq |x+d| - |x| - (x/|x|, d). \tag{3.8}$$

By inequality (3.1), we may bound from below the left hand side of the above inequality by

$$2\sqrt{|x|^2+|d|^2} - 2|x|$$

and, therefore, it suffices to prove that

$$2\sqrt{|x|^2+|d|^2} - |x+d| \geq |x| - (x/|x|, d). \tag{3.9}$$

Now we will use the inequalities we have just proven. Setting $y = |x|$ in (3.7), we get

$$\sqrt{2(|x|^2+|d|^2) - |x+d|^2} \geq |x| - (x/|x|, d) \tag{3.10}$$

and using (3.5) with $x := x+d,\ y := \sqrt{|x|^2+|d|^2}$, we obtain

$$2\sqrt{|x|^2+|d|^2} - |x+d| \geq \sqrt{2(|x|^2+|d|^2) - |x+d|^2},$$

which establishes (3.9).

Finally, let us consider the case $\sqrt{y^2+|d|^2} \geq |x+d|$. We must prove that

$$\sqrt{2(y^2+|d|^2) - |x+d|^2} \geq 2y - |x| - (x/|x|, d),$$

or

$$\sqrt{2(y^2+|d|^2) - |x+d|^2} - 2y \geq -|x| - (x/|x|, d).$$

By inequality (3.2), the left hand side is not smaller than

$$\sqrt{2(|x|^2+|d|^2) - |x+d|^2} - 2|x|,$$

which, with the aid of (3.10), yields the desired inequality. The proof is complete. ∎

Proof of Theorem 1. It suffices to combine Lemma 2 with Theorem 3; indeed, for any fixed martingale (M_n) and any nonnegative integer n,

$$\begin{aligned} 2\mathbb{E}S_n(M) - \mathbb{E}|M_n| &= \mathbb{E}\widehat{V}(M_n, S_n(M), 0) \\ &\geq \widehat{U}(M_0, S_0(M), |M_0|) = \widehat{U}(M_0, |M_0|, |M_0|) \geq 0, \end{aligned} \tag{3.11}$$

which completes the proof of the inequality (1.1).

Now we will show that the inequality in Theorem 1 is sharp, even if $\mathcal{H} = \mathbb{R}$. Suppose that the inequality holds with a constant $C \in [1, \infty)$ and let $V(x,y,z) = v(x,y) = Cy - |x|$. Let us now apply Theorem 4. The function U defined by (2.5) does not depend on z (because V does not), therefore the pair (u, v), where $u(x,y) = U(x,y,z)$, satisfies (2.1), (2.3) and $u(0,0) > -\infty$. Let n be a fixed nonnegative integer and set $x = n,\ y = \sqrt{n}$. Applying the condition (2.3) to a mean-zero random variable taking values $s < 0$ and 1, we obtain

$$\frac{s}{s-1}\,u(n+1,\sqrt{n+1}) + \frac{1}{1-s}\,u(n+s,\sqrt{n+s^2}) \geq u(n,\sqrt{n}),$$

which, by (2.1), implies

$$\frac{s}{s-1}\,u(n+1,\sqrt{n+1}) + \frac{1}{1-s}\,v(n+s,\sqrt{n+s^2}) \geq u(n,\sqrt{n}).$$

Now we let $s \to -\infty$ to get

$$u(n+1,\sqrt{n+1}) + C - 1 \geq u(n,\sqrt{n}),$$

which, by induction, implies that for any nonnegative integer n,

$$u(n,\sqrt{n}) \geq u(0,0) - n(C-1).$$

Therefore

$$u(0,0) - n(C-1) \leq v(n,\sqrt{n}) = C\sqrt{n} - n,$$

or, equivalently,

$$C \geq \frac{2n + u(0,0)}{n+\sqrt{n}}.$$

Now letting $n \to \infty$ yields the result.

Finally, we will prove that the inequality is strict in all nontrivial cases. Let n be a fixed nonnegative integer and (M_n) be a martingale such that $\mathbb{P}(M_n \neq 0) > 0$. Let us introduce the stopping time

$$\tau = \inf\{k : M_k \neq 0\}.$$

If $\mathbb{P}(\tau = 0) > 0$, then the last inequality in (3.11) is strict and we are done. If $\mathbb{P}(\tau > 0) = 1$, then applying the optional sampling theorem to the submartingale $\widehat{U}(M_k, S_k(M), M_k^*)$, $k = 0, 1, 2, \ldots$, we have

$$\mathbb{E}U(M_n, S_n(M), M_n^*) \geq \mathbb{E}U(M_{\tau\wedge n}, S_{\tau\wedge n}(M), M_{\tau\wedge n}^*).$$

Since

$$U(M_{\tau\wedge n}, S_{\tau\wedge n}(M), M_{\tau\wedge n}^*) = U(M_{\tau\wedge n}, |M_{\tau\wedge n}|, M_{\tau\wedge n}^*) > 0$$

on the set $\{\tau \leq n\}$ (which has positive probability), the strictness follows. The proof of Theorem 1 is complete. ∎

4. The proof of Theorem 2. We start from a simple

LEMMA 3. *For $x, d \in \mathcal{H}$ and $z \in \mathbb{R}_+$ we have*

$$|x+d| \vee z - |x+d| \leq \left| -\frac{x}{|x|}\,(|x| \vee z - |x|) + d \right|. \tag{4.1}$$

Proof. We may and will assume that $z \geq |x|$. For $|x+d| \geq z$ there is nothing to prove. If $|x+d| < z$, then the left hand side is equal to $z - |x+d|$ and, squaring both sides, we obtain the equivalent inequality to prove:

$$z^2 - 2z|x+d| + |x+d|^2 \leq |x+d|^2 - 2\left(x+d, x \cdot \frac{z}{|x|}\right) + z^2,$$

or the obvious inequality

$$\left(x+d, \frac{x}{|x|}\right) \leq |x+d|. \ \blacksquare$$

As in the proof of Theorem 1, we will use Theorem 3 of Burkholder; let us introduce functions $U_1, V_1 : \mathcal{H} \times [0,\infty)^2 \to \mathbb{R}$ defined by

$$U_1(x,y,z) = \begin{cases} \sqrt{2y^2 - (|x| \vee z - |x|)^2} & \text{if } y > |x| \vee z - |x|, \\ 2y - (|x| \vee z - |x|) & \text{if } y \leq |x| \vee z - |x|, \end{cases}$$
$$V_1(x,y,z) = 2y - (|x| \vee z - |x|).$$

Note that

$$U_1(x,y,z) = u\left(\pm \frac{x}{|x|}\,(|x| \vee z - |x|), y\right), \tag{4.2}$$

$$V_1(x,y,z) = v\left(\pm \frac{x}{|x|}\,(|x| \vee z - |x|), y\right), \tag{4.3}$$

where u, v are defined by (3.3), (3.4).

We must check the assumptions of Theorem 3.

LEMMA 4. *The functions U_1, V_1 satisfy* (2.1)–(2.3).

Proof. The formulae (4.2), (4.3) will enable us to reduce the claim to Lemma 2. The condition (2.2) obviously holds; the inequality (2.1) follows immediately from (4.2), (4.3) and the condition $u \leq v$ proved in Lemma 2. Hence it suffices to show (2.3).

With fixed y, the function $x \mapsto u(x,y)$ decreases as $|x|$ increases; therefore the formula (4.2) and the inequality (4.1) imply

$$\begin{aligned} &U_1(x+d, \sqrt{y^2+d^2}, |x+d| \vee z) - U_1(x,y,|x| \vee z) \\ &\quad = u\left(\frac{x}{|x|}(|x+d| \vee z - |x+d|), \sqrt{y^2+|d|^2}\right) - u\left(-\frac{x}{|x|}(|x| \vee z - |x|), y\right) \\ &\quad \geq u\left(-\frac{x}{|x|}(|x| \vee z - |x|) + d, \sqrt{y^2+d^2})\right) - u\left(-\frac{x}{|x|}(|x| \vee z - |x|), y\right). \end{aligned}$$

Now if d is a centered $\mathcal{H}$-valued random variable, then the inequality (2.3) for $\widehat{U}$ (defined by (3.3)) and for the point

$$\left(-\frac{x}{|x|}\,(|x| \vee z - |x|), y, |x| \vee z\right) \in \mathcal{H} \times [0,\infty)^2$$

states that the expectation of the right hand side of the inequality above is nonnegative. Therefore the left hand side also has nonnegative expected value, which is the claim. ∎

Proof of Theorem 2. We repeat the arguments from the proof of Theorem 1. Fix an $\mathcal{H}$-valued martingale M and a nonnegative integer n. By Lemma 4 and Theorem 3, the inequality (1.2) is established:

$$\begin{aligned}2\mathbb{E}S_n(M) + \mathbb{E}|M_n| - \mathbb{E}M_n^* &= \mathbb{E}V_1(M_n, S_n(M), M_n^*)\\ &\geq U_1(M_0, S_0(M), |M_0|) = U_1(M_0, |M_0|, |M_0|) \geq 0.\end{aligned}$$ ∎

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