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The Double Tangency Symmetries in Laguerre Planes

by

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Summary. The group generated by double tangency symmetries in a Laguerre plane is investigated. The geometric classification of involutions of a symmetric Laguerre plane is given. We introduce the notion of projective automorphisms using the double tangency and parallel perspectivities. We give the description of the groups of projective automorphisms and automorphisms generated by double tangency symmetries as subgroups of the group $\mathbf{M}(\mathbb{F},\mathbb{R})$ of automorphisms of a chain geometry $\Sigma(\mathbb{F},\mathbb{R})$ following Benz.

Introduction. In [7] we introduced the axioms (C) and (S) characterizing miquelian Laguerre planes of characteristic different from 2. Any Laguerre plane $\mathbb{L}$ satisfying (C) and (S) will be called a *symmetric Laguerre plane*. For any pair of non-tangent circles K, L and any point $p \in K$ of a symmetric Laguerre plane there is exactly one circle through p tangent to both K and L. This defines the so called *double tangency perspectivity*. It has a unique extension to a *double tangency symmetry* $\mathrm{S}_{K,L}$, i.e. an involutory automorphism exchanging K, L (cf. [7]) and preserving all the circles tangent to both K and L. A symmetry with two pointwise fixed generators is a special case of a double tangency symmetry, the same as the Laguerre inversion considered by H. Mäurer in [9]. It is natural to investigate the group of automorphisms generated by all double tangency symmetries. This gives a new approach to the results of H. Mäurer (cf. [10], [11]) and yields some new results.

In [2] W. Benz described a Laguerre plane over a field $\mathbb{F}$ as a chain geometry $\Sigma(\mathbb{F},\mathbb{R})$ where $\mathbb{R}$ is the ring of dual numbers over $\mathbb{F}$. This yields a clear representation of the group $\mathbf{Aut}(\mathbb{L})$ as the group $\mathbf{M}(\mathbb{F},\mathbb{R})$ described by homographies and automorphisms of $\mathbb{R}$ preserving $\mathbb{F}$.

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In [12] H. Zeitler proved for symmetric Minkowski planes that the products of an even number of symmetries with respect to circles are exactly normographies, i.e. the homographies represented by matrices with determinants from $\mathbb{F}^*\mathbb{R}^{*2}$. Additionally any normography is a product of at most four symmetries. In symmetric Laguerre planes the notion of normography cannot be applied because $\mathbb{F}^*\mathbb{R}^{*2} = \mathbb{R}^*$.

In this paper we describe the subgroup of $\mathbf{M}(\mathbb{F}, \mathbb{R})$ generated by double tangency symmetries (Theorem 2.2). In general this group properly contains the group generated by symmetries with pointwise fixed generators. We emphasize that homographies of symmetric Laguerre planes are exactly products of an even number of double tangency symmetries (two or four). We give a geometric description of involutions of symmetric Laguerre planes by invariant circles. We prove that a symmetry with respect to a circle is a composition of three double tangency symmetries, and other involutions are compositions of two such symmetries. In [7] we introduced the double tangency pencil $\langle K, L\rangle$ as the set of circles tangent to both K and L. We investigate the group generated by double tangency symmetries associated with pairs of circles from this pencil. We get the three-reflection theorem for double tangency symmetries (cf. Theorem 3.1(3)) similar to that for symmetries with respect to circles from an orthogonal pencil in symmetric Minkowski planes (cf. [12]).

In the last section we introduce the notion of *t-projectivity* as the composition of double tangency and parallel perspectivities. We show that the von Staudt group of a circle (associated with t-projectivities) is $\mathbf{PGL}_2(\mathbb{F})$ (Theorem 4.1). This motivates the definition of a *projective automorphism* of a symmetric Laguerre plane as an automorphism which has a t-projectivity as the restriction to any circle (Definition 4.3). Theorem 4.2 characterizes the group of all projective automorphisms. This group properly contains the group generated by double tangency symmetries.

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1. Preliminaries. A *Laguerre plane* is a structure $\mathbb{L} = (\mathcal{P}, \mathcal{C}, \|)$, where $\mathcal{P}$ is a set of *points* denoted by small Latin letters, $\mathcal{C} \subset 2^{\mathcal{P}}$ is a set of *circles* denoted by capital Latin letters, and $\| \subset \mathcal{P} \times \mathcal{P}$ is an equivalence relation known as *parallelity*. The equivalence classes of $\|$ will be called *generators* and denoted also by capital Latin letters. We suppose that the following axioms are satisfied:

(1) Any three mutually non-parallel points a, b, c are joined by a unique circle, denoted by $(a, b, c)^\circ$.

(2) For every circle K and any two non-parallel points $p \in K$, $q \notin K$ there is precisely one circle L which passes through q and satisfies $K \cap L = \{p\}$.
(3) For any point p and any circle K there exists exactly one point q such that $p \parallel q$ and $q \in K$; we write $q = pK$.
(4) There is a circle containing at least three but not all points.

We say circles K and L are *tangent* at p if $K \cap L = \{p\}$ or $K = L$. If p is a point of a circle K then we write $\langle p, K\rangle$ for the pencil of circles *tangent* to K at the point p. If $q \nparallel p$ the circle of the pencil $\langle p, K\rangle$ passing through q will be denoted by $(p, K, q)^\circ$. For any pair of non-parallel points x, y the set of circles containing them will be called the *pencil of circles with the vertices* x, y and denoted by $\langle x, y\rangle$.

For any circles K, L the map

$$[K \xrightarrow{p} L] : K \to L;\ x \mapsto xL$$

is called a *parallel perspectivity.*

The *derived plane at a point* p of a Laguerre plane $\mathbb{L}$ consists of all points not parallel to p and, as lines, all circles passing through p (excluding p) and all generators not passing through p. The derived plane is an affine plane and is denoted by $\mathbb{A}_p$.

An *automorphism* of a Laguerre plane is a permutation of the set of points which maps circles to circles (and generators to generators). An automorphism ϕ is called *central* if there exists a fixed point p such that ϕ induces a central collineation of $\overline{\mathbb{A}_p}$, the projective extension of the derived affine plane $\mathbb{A}_p$.

An $\mathbb{L}$-*homothety* is a central automorphism which induces a homothety of $\mathbb{A}_p$ for some fixed point p. An automorphism ϕ is an $\mathbb{L}$-homothety iff there exist non-parallel points p, q such that $\phi(M) = M$ for any $M \in \langle p, q\rangle$.

An $\mathbb{L}$-*strain* is a central automorphism which induces a central collineation of $\overline{\mathbb{A}_p}$ with a proper axis for some fixed point p. The $\mathbb{L}$-strain will be said to be *with respect to a generator* or *with respect to a circle* if the axis is associated with a generator or a circle respectively. An involutory $\mathbb{L}$-strain with respect to a generator fixes also the points of the other generator. It is called a *Laguerre symmetry* and denoted by $\mathrm{S}_{X,Y;M}$ where X, Y are the pointwise fixed generators and M is a fixed circle (not pointwise). An involutory $\mathbb{L}$-strain with respect to a circle is called the *symmetry* (*with respect to the circle*) and denoted by S_K where K is the fixed circle.

A Laguerre plane $\mathbb{L} = (\mathcal{P}, \mathcal{C}, \|)$ is called *symmetric* if the following axioms are satisfied:

(C) For any circles K, L and any point $p \in K \setminus L$ there exists exactly one circle M such that $M \in \langle p, K\rangle$ and $|M \cap L| = 1$.

(S) If K, L, M, N are circles and a, b, c, d are points with $K \cap L = \{a\}$, $L \cap M = \{b\}$, $M \cap N = \{c\}$, $N \cap K = \{d\}$ and $a \nparallel c$, then there exists a circle passing through a, b, c, d.

We have ([7, Theorem 2.2, p. 241]):

THEOREM 1.1. *A Laguerre plane* $\mathbb{L}$ *is symmetric iff it is a plane over a field of characteristic different from* 2.

In the following we assume $\mathbb{L}$ to be a symmetric Laguerre plane.

For any distinct circles K, L and a point $p \in K \setminus L$ the unique circle of the pencil $\langle p, K\rangle$ tangent to L will be denoted by $(p, K, L)^\circ$, and the point of tangency of the circles L and $(p, K, L)^\circ$ by pKL. Additionally we define $pKL := p$ for $p \in K \cap L$. For distinct circles K, L the set of circles tangent to K and L will be denoted by $\langle K, L\rangle$ and called a *double tangency pencil*. If K and L are tangent at p we have $\langle K, L\rangle = \langle p, K\rangle$ by [7, Proposition 2.1, p. 241]. If K, L are any non-tangent circles the map

$$[K \xrightarrow{t} L] : K \to L;\ x \mapsto xKL$$

is called a *double tangency perspectivity.*

By (C) and [7, Proposition 2.2, p. 241] we have:

PROPOSITION 1.1. *Let* K, L *be non-tangent circles*, $a \in K\setminus L$, $b := aKL$, $c := aL$, $d := bK$ *and* $N := (c, L, d)^\circ$. *Then* N *is tangent to* K.

By [7, Theorem 3.1, p. 242] a double tangency perspectivity $[K \xrightarrow{t} L]$ has a unique extension to an involutory automorphism. This automorphism is called the *double tangency symmetry associated with* K, L and denoted by $\mathrm{S}_{K,L}$.

[7, Theorem 3.1, p. 242] and [7, Theorem 3.2, p. 244] imply:

PROPOSITION 1.2. *Let* K, L *be non-tangent circles. Then*:

(1) $\mathrm{S}_{K,L}(M) = M$ *for* $M \in \langle K, L\rangle$ *and conversely* $\mathrm{S}_{K,L}$ *is the only such automorphism.*
(2) *If* $\mathrm{S}_{K,L}(x) \neq x$, *then* $\mathrm{S}_{K,L}(M) = M$ *for* $M \in \langle x, \mathrm{S}_{K,L}(x)\rangle$.
(3) *If* $L' = \mathrm{S}_{K,L}(K') \neq K'$, *then* $\mathrm{S}_{K,L} = \mathrm{S}_{K',L'}$.
(4) *If* $K \cap L = \{x, y\}$, *then* $\mathrm{S}_{K,L} = \mathrm{S}_{X,Y;M}$ *where* M *is any circle fixed by* $\mathrm{S}_{K,L}$ *and* X, Y *are the generators through* x, y *respectively. If* $K \cap L = \emptyset$, *then* $\mathrm{S}_{K,L}$ *does not have fixed points.*
(5) $\mathrm{S}_{X,Y;M} = \mathrm{S}_{X,Y;M'}$ *where* M' *is any circle fixed by* $\mathrm{S}_{X,Y;M}$.
(6) *Through any pair of points* x, y *such that* $x \in X$, $y \in Y$ *there is exactly one circle fixed by* $\mathrm{S}_{X,Y;M}$.

Direct calculations (over a field of characteristic different from 2) give the following properties of the symmetry with respect to a circle:

PROPOSITION 1.3. *Let $K, L, M \in \mathcal{C}$. Then:*

(1) *There is exactly one circle P such that $\mathrm{S}_P(K) = L$, and S_P is the unique automorphism interchanging K, L and preserving all generators.*

(2) *There is exactly one circle N such that $\mathrm{S}_K \circ \mathrm{S}_L \circ \mathrm{S}_M = \mathrm{S}_N$.*

Let $\mathbb{F}$ be a field of characteristic different from 2 and let $\mathbb{R}$ be a ring extension of $\mathbb{F}$ by an element ε with $\varepsilon^2 = 0$. We denote by $\overline{\mathrm{Aut}_{\mathbb{F}}\,\mathbb{R}}$ the group of all automorphisms of $\mathbb{R}$ preserving $\mathbb{F}$ and by $\mathrm{Aut}_{\mathbb{F}}\,\mathbb{R}$ the group of all automorphisms of $\mathbb{R}$ preserving $\mathbb{F}$ pointwise (cf. [12]). We have

$$\overline{\mathrm{Aut}_{\mathbb{F}}\,\mathbb{R}} = \{\phi_\lambda^\sigma \mid \lambda \in \mathbb{F}^*,\ \sigma \in \mathrm{Aut}\,\mathbb{F}\} \tag{1.1}$$

where $\phi_\lambda^\sigma(a + b\varepsilon) := a^\sigma + \lambda b^\sigma \varepsilon$. Let $\phi_\lambda := \phi_\lambda^{\mathrm{id}}$. Then

$$\mathrm{Aut}_{\mathbb{F}}\,\mathbb{R} = \{\phi_\lambda \mid \lambda \in \mathbb{F}^*\}. \tag{1.2}$$

An automorphism ϕ_λ is an involution iff $\lambda = -1$; we write $\overline{z} := \phi_{-1}(z)$.

According to [2] any symmetric Laguerre plane is isomorphic to a chain geometry $\Sigma(\mathbb{F}, \mathbb{R}) = (\mathcal{P}_{\mathbb{F}}, \mathcal{C}_{\mathbb{F}}, \|_{\mathbb{F}})$ where $\mathcal{P}_{\mathbb{F}} = \mathbb{P}(\mathbb{R})$, $\mathcal{C}_{\mathbb{F}} = \{\mathbb{P}(\mathbb{F})^\varphi \mid \varphi \in \mathbf{\Gamma}(\mathbb{R})\}$ and $\mathbf{\Gamma}(\mathbb{R}) := \mathbf{PGL}_2(\mathbb{R})$. Any element x of $\mathbb{P}(\mathbb{R})$ (a point) with representative (x_1, x_2) is denoted by $[x_1, x_2]$. The relation of parallelity is defined by

$$[x_1, x_2] \parallel_{\mathbb{F}} [y_1, y_2] :\Leftrightarrow \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \in \mathbb{R} \setminus \mathbb{R}^*. \tag{1.3}$$

Any element M of $\mathcal{C}_{\mathbb{F}}$ (a circle) is a set of points described by the equation

$$[x_1, x_2]\ \mathrm{M} \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{bmatrix} = 0 \tag{1.4}$$

where $\overline{\mathrm{M}}^{\mathbf{T}} + \mathrm{M} = 0$. The matrix M can be written

$$\mathrm{M} := \begin{bmatrix} \alpha\varepsilon & \frac{1+\beta\varepsilon}{2} \\ -\frac{1-\beta\varepsilon}{2} & \gamma\varepsilon \end{bmatrix}$$

where $\alpha, \beta, \gamma \in \mathbb{F}$ (cf. [2, p. 27]). We say that *the circle M has matrix* M. This representation is useful because of the connection with parabolas (and lines) with equations $y + \alpha x^2 + \beta x + \gamma = 0$ in the so called isotropic model (cf. [2, p. 19]).

We have (cf. [2, Satz 3.1, p. 88] for any commutative ring $\mathbb{R}$):

PROPOSITION 1.4. *For any two triples of mutually non-parallel points (x, y, z) and (x', y', z') there exists exactly one $\varphi \in \mathbf{\Gamma}(\mathbb{R})$ such that $\varphi(x) = x'$, $\varphi(y) = y'$, $\varphi(z) = z'$.*

An easy calculation gives:

PROPOSITION 1.5. *Two circles with matrices*

$$\begin{bmatrix} \alpha_1\varepsilon & \frac{1+\beta_1\varepsilon}{2} \\ -\frac{1-\beta_1\varepsilon}{2} & \gamma_1\varepsilon \end{bmatrix}, \quad \begin{bmatrix} \alpha_2\varepsilon & \frac{1+\beta_2\varepsilon}{2} \\ -\frac{1-\beta_2\varepsilon}{2} & \gamma_2\varepsilon \end{bmatrix}$$

are tangent iff $(\beta_1-\beta_2)^2 = 4(\alpha_1-\alpha_2)(\gamma_1-\gamma_2)$.

We will denote by E the circle $\mathbb{P}(\mathbb{F})$ (with matrix $\left[\begin{smallmatrix} 0 & 1/2 \\ -1/2 & 0 \end{smallmatrix}\right]$).

Any automorphism φ of $\Sigma(\mathbb{F},\mathbb{R})$ is a map

$$[x_1,x_2] \mapsto [x_1^\tau, x_2^\tau]\begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{1.5}$$

where $\tau \in \overline{\mathrm{Aut}_{\mathbb{F}}\,\mathbb{R}}$ and $\left|\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right| \in \mathbb{R}^*$ (cf. [2, Satz 3.1, p. 176]).

In the following we suppose that $\left|\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right| \in \mathbb{F}^*$ in (1.5) (this is possible since $\mathbb{F}^*\mathbb{R}^{*2} = \mathbb{R}^*$).

The group of automorphisms of $\Sigma(\mathbb{F},\mathbb{R})$ is denoted by $\mathbf{M}(\mathbb{F},\mathbb{R})$. If $\mathbb{L}$ is a symmetric Laguerre plane, then the group $\mathbf{Aut}(\mathbb{L})$ is isomorphic to $\mathbf{M}(\mathbb{F},\mathbb{R})$.

Recall that the group $\Gamma(\mathbb{R})$ (a normal subgroup of $\mathbf{M}(\mathbb{F},\mathbb{R})$) consists of the maps defined by (1.5) with $\tau = \mathrm{id}$; we denote by $\overline{\Gamma(\mathbb{R})}$, $\widetilde{\Gamma(\mathbb{R})}$ the normal subgroups of $\mathbf{M}(\mathbb{F},\mathbb{R})$ described by (1.5) for $\tau \in \{\mathrm{id}, \phi_{-1}\}$, $\tau \in \mathrm{Aut}_{\mathbb{F}}\,\mathbb{R}$ respectively.

2. Involutions. The group generated by double tangency symmetries

THEOREM 2.1. *Let* $\varphi \in \widetilde{\Gamma(\mathbb{R})}$ *be an involution. Then exactly one of the following holds*:

(a) *For any* $x \neq \varphi(x)$ *and* $M \in \langle x, \varphi(x)\rangle$ *we have* $\varphi(M) = M$. *Then* φ *is a double tangency symmetry and it maps*

$$[x_1,x_2] \mapsto [\overline{x}_1, \overline{x}_2]\begin{bmatrix} a & r \\ s & -\overline{a} \end{bmatrix}, \qquad \text{where } r,s \in \mathbb{F}.$$

(b) *There exists exactly one circle* K *such that* $\varphi(x) = x$ *for* $x \in K$. *Then* $\varphi = \mathrm{S}_K$, φ *is a composition of three double tangency symmetries and it maps*

$$[x_1,x_2] \mapsto [\overline{x}_1, \overline{x}_2]\begin{bmatrix} a & r\varepsilon \\ s\varepsilon & \overline{a} \end{bmatrix}, \qquad \text{where } r,s \in \mathbb{F}.$$

(c) *For any* $x \nparallel \varphi(x)$ *there exists exactly one* $M \in \langle x, \varphi(x)\rangle$ *such that* $\varphi(M) = M$. *Then* φ *is a composition of two double tangency symmetries and it maps*

$$[x_1,x_2] \mapsto [x_1, x_2]\begin{bmatrix} a & b \\ c & -a \end{bmatrix}.$$

We have divided the proof into a sequence of lemmas.

LEMMA 2.1. *If a map* $[x_1, x_2] \mapsto [x_1^\tau, x_2^\tau]\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]$ *with* $\tau \in \mathrm{Aut}_{\mathbb{F}} \mathbb{R}$ *is an involution*, *then* $\tau = \mathrm{id}$ *or* $\tau(x) = \overline{x}$ *for* $x \in \mathbb{R}$. *In the first case we have* $d = -a$. *In the second case*, *either*

$$d = -\overline{a} \quad \text{and} \quad b, c \in \mathbb{F} \tag{2.1}$$

or

$$d = \overline{a} \quad \text{and} \quad b, c \in \mathbb{R} \setminus \mathbb{R}^*. \tag{2.2}$$

Proof. From $[x_1^{\tau\tau}, x_2^{\tau\tau}]\mathrm{A}^\tau \mathrm{A} = [x, y]$ it follows that $\tau^2 = \mathrm{id}$.

A standard computation in the ring $\mathbb{R}$ proves the assertion for the case $\tau = \mathrm{id}$. Therefore in what follows we assume that $\tau = \phi_{-1}$ is the conjugacy.

From the condition

$$\begin{bmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where $\lambda \in \mathbb{F}^*$ we get

$$a\overline{a} + \overline{b}c = b\overline{c} + d\overline{d}, \tag{2.3}$$

$$\overline{a}b + \overline{b}d = 0, \tag{2.4}$$

$$a\overline{c} + c\overline{d} = 0. \tag{2.5}$$

Now, (2.3) gives $a\overline{a} - d\overline{d} = b\overline{c} - \overline{b}c$. Since $a\overline{a} - d\overline{d} \in \mathbb{F}$ and $b\overline{c} - \overline{b}c \in \mathbb{R} \setminus \mathbb{R}^*$, we get

$$a\overline{a} = d\overline{d}, \tag{2.6}$$

$$b\overline{c} = \overline{b}c. \tag{2.7}$$

Write

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad B = \begin{bmatrix} -\overline{d} & \overline{b} \\ \overline{c} & -\overline{a} \end{bmatrix}.$$

Suppose first that $b \in \mathbb{R}^*$. By (2.4), (2.7) we obtain $\overline{b}A = bB$. Since $\det B = \overline{\det A}$ and $\det A \in \mathbb{F}^*$ we get $\det A = \det B$. Hence $A = \lambda B$ where $\lambda = \pm 1$. The case $\lambda = -1$ is impossible because then $b = -\overline{b}$, which contradicts $b \in \mathbb{R}^*$. Thus $d = -\overline{a}$, $b = \overline{b}$, $c = \overline{c}$, hence $b \in \mathbb{F}^*$ and $c \in \mathbb{F}$.

Let us consider the case $b \notin \mathbb{R}^*$, which gives $a, d \in \mathbb{R}^*$. By (2.4)–(2.6) we obtain $\overline{a}A = -bB$. Hence $A = \lambda B$ where $\lambda = \pm 1$. If $\lambda = 1$, then $b = \overline{b}$, hence $b = 0$. We also have $d = -\overline{a}$ and $c = \overline{c}$, hence $c \in \mathbb{F}$. This together with the previous case proves (2.1). If $\lambda = -1$, then $a = \overline{d}$ and $b = -\overline{b}$, $c = -\overline{c}$, hence $b, c \in \mathbb{R} \setminus \mathbb{R}^*$. ∎

Lemma 2.2. *An involution* $[x_1, x_2] \mapsto [\overline{x}_1, \overline{x}_2]\left[\begin{smallmatrix} a & r\varepsilon \\ s\varepsilon & \overline{a} \end{smallmatrix}\right]$, *where* $r, s \in \mathbb{F}$, *is a symmetry with respect to a circle with matrix*

$$\begin{bmatrix} \frac{r}{2a_1}\varepsilon & \frac{1}{2} - \frac{a_2}{2a_1}\varepsilon \\ -\frac{1}{2} - \frac{a_2}{2a_1}\varepsilon & -\frac{s}{2a_1}\varepsilon \end{bmatrix}$$

where $a = a_1 + a_2\varepsilon$, $a_1, a_2 \in \mathbb{F}$.

Lemma 2.3. *An involution* φ: $[x_1, x_2] \mapsto [\overline{x_1}, \overline{x}_2]\left[\begin{smallmatrix} a & r \\ s & -\overline{a} \end{smallmatrix}\right]$, *where* $r, s \in \mathbb{F}$, *is a double tangency symmetry. A circle with matrix*

$$\mathrm{M} = \begin{bmatrix} \alpha\varepsilon & \frac{1+\beta\varepsilon}{2} \\ -\frac{1-\beta\varepsilon}{2} & \gamma\varepsilon \end{bmatrix}$$

is fixed by φ *iff* $a_1\beta = a_2 + \alpha s - \gamma r$ *where* $a = a_1 + a_2\varepsilon$, $a_1, a_2 \in \mathbb{F}$.

Proof. A circle with matrix M is fixed by φ iff $\overline{\mathrm{A}}\mathrm{M}^{\mathbf{T}} = \mathrm{M}\mathrm{A}^{\mathbf{T}}$ where $\mathrm{A} = \left[\begin{smallmatrix} a & r \\ s & -\overline{a} \end{smallmatrix}\right]$. A calculation shows that this equation is equivalent to $a_1\beta = a_2 + \alpha s - \gamma r$.

Suppose first that $a \in \mathbb{F}$. Then φ fixes E (cf. [2, Lemma 1.1, p. 94]). Let K, L be the circles with matrices

$$\begin{bmatrix} \varepsilon & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}, \quad \begin{bmatrix} \frac{a^2}{a^2+rs}\varepsilon & \frac{1}{2} + \frac{as}{a^2+rs}\varepsilon \\ -\frac{1}{2} + \frac{as}{a^2+rs}\varepsilon & \frac{s^2}{a^2+rs}\varepsilon \end{bmatrix}.$$

We have $\varphi(K) = L$ and K, L are tangent to E by Proposition 1.5. We prove that $\varphi = \mathrm{S}_{K,L}$. By Proposition 1.5 a circle with matrix M is tangent to both K and L iff $\beta^2 = 4(\alpha - 1)\gamma$ and

$$\left(\beta - \frac{2as}{a^2 + rs}\right)^2 = 4\left(\alpha - \frac{a^2}{a^2 + rs}\right)\left(\gamma - \frac{s^2}{a^2 + rs}\right).$$

This clearly forces $a\beta = s\alpha - r\gamma$. Thus $\varphi(M) = M$ for $M \in \langle K, L\rangle$, hence $\varphi = \mathrm{S}_{K,L}$ by Proposition 1.2(1).

Let $a \notin \mathbb{F}$. If a circle K has matrix M where $\triangle = \beta^2 - 4\alpha\gamma \neq 0$, then the involution

$$[x_1, x_2] \mapsto [\overline{x}_1, \overline{x}_2]\begin{bmatrix} -\frac{\beta}{2\triangle} - \frac{\varepsilon}{4} & \frac{\alpha}{\triangle} \\ -\frac{\gamma}{\triangle} & \frac{\beta}{2\triangle} - \frac{\varepsilon}{4} \end{bmatrix}$$

maps E to K. We prove that this involution is $\mathrm{S}_{K,E}$. To do this, we take the circle $M = (p, E, \varphi(p))^\circ$, where $p = [p_1, 1]$, $p_1 \in \mathbb{F}$, and infer that M is also tangent to K. Any circle tangent to E at p has matrix

$$\begin{bmatrix} -m\varepsilon & \frac{1+2p_1m\varepsilon}{2} \\ -\frac{1-2p_1m\varepsilon}{2} & -mp_1^2\varepsilon \end{bmatrix}$$

for some $m \in \mathbb{F}$. From the condition $\varphi(p) \in M$ we deduce that $m = \triangle/(4(\alpha p_1^2 + \beta p_1 + \gamma))$. From this, using Proposition 1.5 we conclude that the circles K, M are tangent. ∎

LEMMA 2.4. *An involution $\varphi \in \Gamma(\mathbb{R})$ has fixed circles.*

Proof. By Lemma 2.1, φ is a map $[x_1, x_2] \mapsto [x_1, x_2]\left[\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right]$. We recall the assumption $\left|\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right| \in \mathbb{F}^*$. Write $a = a_1 + a_2\varepsilon$, $b = b_1 + b_2\varepsilon$, $c = c_1 + c_2\varepsilon$, where $a_i, b_i, c_i \in \mathbb{F}$. A circle with matrix

$$\mathrm{M} = \begin{bmatrix} \alpha\varepsilon & \frac{1+\beta\varepsilon}{2} \\ -\frac{1-\beta\varepsilon}{2} & \gamma\varepsilon \end{bmatrix}$$

is fixed by φ iff $\mathrm{BM} = d\mathrm{M}\overline{\mathrm{B}}^{\mathbf{T}}$ where $\mathrm{B} = \left[\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right]$, $d \in \mathbb{R}^*$. This equation is equivalent to the following system of linear equations:

$$\begin{cases} 2a_1\alpha + b_1\beta = b_2, \\ c_1\alpha + b_1\gamma = -a_2, \\ c_1\beta - 2a_1\gamma = -c_2. \end{cases}$$

This system has a solution since $c_1b_2 + 2a_1a_2 + b_1c_2 = 0$ is equivalent to $\left|\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right| \in \mathbb{F}^*$. ∎

LEMMA 2.5. *An involution $\varphi \in \Gamma(\mathbb{R})$ is the composition $\mathrm{S}_M \circ \mathrm{S}_{K,L}$ for some circles K, L, M. The circle M is an arbitrary circle fixed by φ.*

Proof. Let M be fixed by φ. By Proposition 1.4 there exists $\psi \in \Gamma(\mathbb{R})$ such that $\psi(M) = E$. The involution $\psi \circ \varphi \circ \psi^{-1}$ has matrix $\left[\begin{smallmatrix} q & r \\ s & -q \end{smallmatrix}\right]$ where $q, r, s \in \mathbb{F}$ since $\psi \circ \varphi \circ \psi^{-1}(E) = E$ (cf. [2, Lemma 1.1, p. 94]). This involution is the composition of the maps $[x_1, x_2] \mapsto [\overline{x}_1, \overline{x}_2]$ and $[x_1, x_2] \mapsto [\overline{x}_1, \overline{x}_2]\left[\begin{smallmatrix} q & r \\ s & -q \end{smallmatrix}\right]$. By Lemmas 2.2 and 2.3, the first automorphism is S_E and the second is $\mathrm{S}_{K'L'}$ for some circles K', L'. Hence $\varphi = \mathrm{S}_M \circ \mathrm{S}_{K,L}$ where $K = \psi(K')$, $L = \psi(L')$. ∎

LEMMA 2.6. *Let K, L, M, N be circles and a, b, c, d be points such that $K \cap M = \{a\}$, $L \cap M = \{b\}$, $L \cap N = \{c\}$, $N \cap K = \{d\}$, $a \parallel c$, $b \parallel d$ and let P be a circle such that $\mathrm{S}_P(K) = L$. Then*:

(1) $\mathrm{S}_P(M) = N$.
(2) $P = \mathrm{S}_{K,L}(P) = \mathrm{S}_{M,N}(P)$.
(3) *The circles K, L, M, N are fixed by $\mathrm{S}_{X,Y;P}$ where X, Y are the generators through a, b respectively.*
(4) $\mathrm{S}_P = \mathrm{S}_{X,Y;P} \circ \mathrm{S}_{K,L} \circ \mathrm{S}_{M,N}$.

Proof. (1) We have $\mathrm{S}_P(a) = c$, $\mathrm{S}_P(b) = d$. Hence S_P maps M to a circle tangent to K at d and to L at c. The only such circle is N.

(2) Let $P' := \mathrm{S}_{K,L}(P)$. We have $\mathrm{S}_{P'}(K) = \mathrm{S}_{K,L} \circ \mathrm{S}_P \circ \mathrm{S}_{K,L}(K) = L$, hence $P = P'$ by Proposition 1.3(1). By (1) the same reasoning applies to the equation $\mathrm{S}_{M,N}(P) = P$.

(3) The symmetry $\mathrm{S}_{X,Y;K}$ fixes the circle M as a circle tangent to K at a point of the generator X. Hence it also fixes the circle L (as a circle tangent to

M at b). From this and Proposition 1.3(1) we conclude that $\mathrm{S}_{X,Y;K}(P) = P$. Hence, by Proposition 1.2(5), $\mathrm{S}_{X,Y;K} = \mathrm{S}_{X,Y;P}$ and $\mathrm{S}_{X,Y;P}$ fixes each of the circles K, L, M, N.

(4) We first prove that the composition $\varphi = \mathrm{S}_{K,L} \circ \mathrm{S}_{M,N}$ is an involution. If $xKL \parallel \mathrm{S}_{M,N}(x)$ for any $x \in K$, then φ fixes all generators. It is clear that $\varphi(K) = L$ and $\varphi(L) = K$. Hence $\varphi = \mathrm{S}_P$ by Proposition 1.3. From Lemmas 2.1 and 2.3 we see that $\varphi \in \Gamma(\mathbb{R})$. This is a contradiction since $\mathrm{S}_P \notin \Gamma(\mathbb{R})$ by Lemma 2.2. Therefore there exists $x \in K$ such that $xKL \nparallel \mathrm{S}_{M,N}(x)$. Let $x' := \mathrm{S}_{M,N}(x)$, $Q := (x, K, L)^\circ$, $Q' := (x', K, L)^\circ$. We get $\mathrm{S}_{M,N}(Q') = Q$, hence $\varphi(Q') = Q$ since $\mathrm{S}_{K,L}(Q) = Q$. This gives $\varphi(x') = x$ and φ is an involution by [2, Satz 3.2, p. 89]. From $\varphi(K) = L$ it follows that $\varphi(P) = P$. Hence $\varphi = \mathrm{S}_P \circ \mathrm{S}_{R,S}$ for some circles R, S by Lemma 2.5. It is also clear that $\varphi(X) = X$ and $\varphi(Y) = Y$. Therefore $\mathrm{S}_{R,S} = \mathrm{S}_{X,Y;P}$ by Proposition 1.2(4), and (4) is proved. ∎

Corollary 2.1. *Any involution $\varphi \in \Gamma(\mathbb{R})$ is the composition of two double tangency symmetries.*

To complete the proof of Theorem 2.1 it is enough to establish the following extension of Lemma 2.4:

Lemma 2.7. *There is exactly one circle fixed by an involution $\varphi \in \Gamma(\mathbb{R})$ through any point x such that $\varphi(x) \neq x$.*

Proof. By Lemma 2.5, φ is the composition of a double tangency symmetry ψ and the symmetry S_P, where P is any fixed circle of φ. If $x \in P$, then $\varphi(x) \in P$ and $\varphi(x) = \psi(x)$. By Proposition 1.2(2), ψ fixes all the circles of the pencil $\langle x, \varphi(x)\rangle$ while S_P fixes only P. Hence P is the only circle through x fixed by φ.

Let us consider the case $x \notin P$. Write $y := xP$. We get $\varphi(y) = \varphi(x)P$ since $\varphi(P) = P$. Let $K' := (y, P, \varphi(x))^\circ$, $L' := (\varphi(y), P, x)^\circ$, $M := (x, L', \varphi(x))^\circ$. From Proposition 1.1 it follows that $M \in \langle K', L'\rangle$. It is evident that $\varphi(K') = L'$. Hence $\varphi(M) = M$. Set $N = \mathrm{S}_P(M)$, $x' = \mathrm{S}_P(x)$. The circle $\mathrm{S}_P(L')$ is tangent to N at x'. Let $K := (x', N, \varphi(x))^\circ$. By Proposition 1.1, K is tangent to K' at $\varphi(x)$, hence K is tangent to M. Write $L := \mathrm{S}_P(K)$. It is clear that $L \in \langle M, N\rangle$, $\varphi(K) = L$ and $\varphi(N) = N$. This gives $\psi(M) = N$, hence $\psi = \mathrm{S}_{M,N}$, by Proposition 1.2(2). From Lemma 2.6(4) we conclude that $\varphi = \mathrm{S}_{K,L} \circ \mathrm{S}_{X,\varphi(X);M}$ where X is the generator through x. By Proposition 1.2(1), $\mathrm{S}_{K,L}$ fixes all the circles of the pencil $\langle x, \varphi(x)\rangle$. By Proposition 1.2(6), $\mathrm{S}_{X,\varphi(X);M}$ fixes exactly one circle of the pencil $\langle x, \varphi(x)\rangle$. From this we obtain the assertion for the point x. ∎

Lemma 2.8. *Any automorphism $\varphi \in \Gamma(\mathbb{R})$ is the composition of two involutions from $\Gamma(\mathbb{R})$.*

Proof. An automorphism φ is the map $[x_1, x_2] \mapsto [x_1, x_2]\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ (where $\left|\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right| \in \mathbb{F}^*$). We have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{cases} \dfrac{1}{-bc}\begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix} \cdot \begin{bmatrix} bc & bd \\ -ac & -bc \end{bmatrix} & \text{for } bc \in \mathbb{R}^*, \\ \dfrac{1}{-c^2}\begin{bmatrix} c & d-a \\ 0 & -c \end{bmatrix} \cdot \begin{bmatrix} -cd & ad-d^2-bc \\ c^2 & cd \end{bmatrix} & \text{for } c \in \mathbb{R}^*,\ b \notin \mathbb{R}^*, \\ \dfrac{1}{-b^2}\begin{bmatrix} -b & 0 \\ a-d & b \end{bmatrix} \cdot \begin{bmatrix} ab & b^2 \\ ad-a^2-bc & -ab \end{bmatrix} & \text{for } b \in \mathbb{R}^*,\ c \notin \mathbb{R}^*. \end{cases}$$

Moreover, because of the assumption $\left|\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right| \in \mathbb{F}^*$ the matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is proportional (over $\mathbb{R}^*$) to one of the following three which, on the other hand, can be decomposed as follows:

$$\begin{bmatrix} t & r\varepsilon \\ s\varepsilon & 1 \end{bmatrix} = \frac{1}{rs}\begin{bmatrix} 0 & r \\ -s & 0 \end{bmatrix} \cdot \begin{bmatrix} -rs\varepsilon & -r \\ st & rs\varepsilon \end{bmatrix} \qquad \text{for } r, s, t \in \mathbb{F}^*,$$

$$\begin{bmatrix} t & r\varepsilon \\ 0 & 1 \end{bmatrix} = \frac{1}{1-t}\begin{bmatrix} -r\varepsilon & t \\ t-1 & r\varepsilon \end{bmatrix} \cdot \begin{bmatrix} r\varepsilon & -1 \\ 1-t & -r\varepsilon \end{bmatrix} \qquad \text{for } r, t \in \mathbb{F}^*, t \neq 1,$$

$$\begin{bmatrix} 1 & r\varepsilon \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -r\varepsilon \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \text{for } r \in \mathbb{F}. \ \blacksquare$$

THEOREM 2.2.

(1) $\overline{\Gamma(\mathbb{R})}$ *is the group generated by all double tangency symmetries.*
(2) *Any automorphism* $\varphi \in \Gamma(\mathbb{R})$ *is the composition of two or four double tangency symmetries.*

Proof. (2) By Lemma 2.8 and Corollary 2.1.
(1) By (2) and Theorem 2.1(b). ∎

3. The three-reflection theorem

LEMMA 3.1. *Let* $x, y \in K$, $x \parallel z$, $x \neq y, z$. *If* $\varphi(x) = \varphi'(x)$, $\varphi(y) = \varphi'(y)$, $\varphi(z) = \varphi'(z)$, $\varphi(K) = \varphi'(K) = K$ *for automorphisms* $\varphi, \varphi' \in \Gamma(\mathbb{R})$, *then* $\varphi = \varphi'$.

Proof. It is enough to prove that the identity is the only automorphism $\varphi \in \Gamma(\mathbb{R})$ fixing the points x, y, z and the circle K. Suppose that $\varphi(v) \neq v$ for some $v \in K$. By Proposition 1.4 there exists exactly one automorphism of $\Gamma(\mathbb{R})$ mapping x, y, v to $x, y, \varphi(v)$ respectively. This automorphism is the $\mathbb{L}$-homothety with vertices x, y mapping v to $\varphi(v)$. This contradicts our assumption $\varphi(z) = z$. Hence φ fixes K pointwise, and consequently it is the identity by Proposition 1.4. ∎

THEOREM 3.1. *Let K, L be non-tangent circles, $x, x' \in K \setminus L$ and $x \neq x'$. Then*:

(1) *There exists exactly one automorphism $\varphi \in \Gamma(\mathbb{R})$ such that $\varphi(x) = x'$, $\varphi(K) = K$, $\varphi(L) = L$.*
(2) *The automorphism φ is the composition of two double tangency symmetries $\mathrm{S}_{M',N'} \circ \mathrm{S}_{M,N}$ where $M, N, M', N' \in \langle K, L\rangle$ and $\mathrm{S}_{M,N}$ can be chosen arbitrarily.*
(3) *For any $M_i, N_i \in \langle K, L\rangle$ ($M_i \neq N_i$, $1 \leq i \leq 3$) there exist $M, N \in \langle K, L\rangle$ such that $\mathrm{S}_{M_1,N_1} \circ \mathrm{S}_{M_2,N_2} \circ \mathrm{S}_{M_3,N_3} = \mathrm{S}_{M,N}$.*

Proof. (1) Let us take $\varphi := \mathrm{S}_{M',N} \circ \mathrm{S}_{M,N}$ where $M = (x, K, L)^\circ$, $M' = (x', K, L)^\circ$, $N \in \langle K, L\rangle$, $N \neq M, M'$; then φ is as required in (1). Now we show that an automorphism φ required in (1) is indeed unique. Let $y := xKL$, $y' := x'KL$, $z := yK$, $z' = y'K$. We obtain $\varphi(y) = y'$, $\varphi(z) = z'$ and Lemma 3.1 completes the proof.

(2) Let M, N be arbitrary distinct circles of the pencil $\langle K, L\rangle$, $x'' := \mathrm{S}_{M,N}(x)$, $M' := (x'', K, L)^\circ$, $N' := (x', K, L)^\circ$. It is easy to check that $\varphi = \mathrm{S}_{M',N'} \circ \mathrm{S}_{M,N}$.

(3) This is clear from (1) and (2). ∎

COROLLARY 3.1. *Any $\mathbb{L}$-homothety with vertices p, q can be represented as the composition $\varphi := \mathrm{S}_{M',N} \circ \mathrm{S}_{M,N}$ where $M, M', N \in \langle K, L\rangle$ and K, L are arbitrary distinct circles of the pencil $\langle p, q\rangle$.*

4. Projectivities and projective automorphisms. A natural question arises how to define a projective automorphism of a symmetric Laguerre plane to preserve the analogy to the classical definition of a projective collineation, as a map whose restriction to any line is a projectivity, i.e. a composition of perspectivities. We propose to use double tangency and parallel perspectivities (cf. Definitions 4.1 and 4.3). This allows us to omit the notion of a derived affine plane. The restriction of any double tangency symmetry to any non-fixed circle is a double tangency perspectivity by Proposition 1.2(3). For a fixed circle the restriction is a composition of double tangency and parallel perspectivities (cf. Theorem 4.1). Hence automorphisms from $\overline{\Gamma(\mathbb{R})}$ should be called projective. The same refers to an $\mathbb{L}$-strain with respect to a circle since its restriction is a parallel perspectivity. The group $\overline{\Gamma(\mathbb{R})}$ does not contain non-involutory $\mathbb{L}$-strains with respect to circles. Indeed, any $\mathbb{L}$-strain with respect to E is a map $[x_1, x_2] \mapsto [x_1^\tau, x_2^\tau]$ where $\tau \in \mathrm{Aut}_{\mathbb{F}}\, \mathbb{R}$. Consequently, the proposed group of projective automorphisms is larger than $\overline{\Gamma(\mathbb{R})}$.

A difference between Laguerre and Minkowski planes is worth pointing out here. For the latter, the only automorphism of $\mathbb{R}$ preserving $\mathbb{F}$ pointwise is the involutory conjugacy $z \mapsto \overline{z}$. Therefore, if we present a Minkowski plane $\mathbb{M}$ in the form $\mathbb{M} = \Sigma(\mathbb{F}, \mathbb{R})$ for a suitable ring $\mathbb{R}$, then the group of projective automorphisms of $\mathbb{M}$ is $\overline{\Gamma(\mathbb{R})}$.

Let $K, L \in \mathcal{C}$.

Definition 4.1. A map $\phi: \ K \to L$ is called a *t-projectivity* if it is a composition of double tangency and parallel perspectivities.

Definition 4.2. The *von Staudt group of the circle* K is the set of all t-projectivities of K. It will be denoted by Γ_K.

Definition 4.3. An automorphism $\varphi \in \mathbf{Aut}(\mathbb{L})$ is called *projective* if $\varphi_{|K}$ is a t-projectivity for any $K \in \mathcal{C}$.

Theorem 4.1. *Let $K \in \mathcal{C}$.*

(1) $\Gamma_K \approx \mathbf{PGL}_2(\mathbb{F})$.

(2) *If $\phi \in \Gamma_K$ is an involution, then*:

  (a) *there exist $L, M \in \mathcal{C}$ such that* $\phi = \mathrm{S}_{L,M}|_K$,

  (b) *there exists $N \in \mathcal{C}$ such that*

$$\phi = [N \xrightarrow{p} K] \circ [K \xrightarrow{t} N].$$

(3) *If $\phi \in \Gamma_K$ is not an involution, then there exist $L, M \in \mathcal{C}$ such that*

$$\phi = [M \xrightarrow{p} K] \circ [L \xrightarrow{t} M] \circ [K \xrightarrow{t} L].$$

Proof. We can assume that $K := E = \mathbb{P}(\mathbb{F})$ since a parallel perspectivity establishes the isomorphism of Γ_E and Γ_K for any $K \in \mathcal{C}$.

(2a) From the analytic description presented in Theorem 2.1(a) it follows that any product $[M \xrightarrow{p} E] \circ [L \xrightarrow{t} M] \circ [E \xrightarrow{p} L] = [M \xrightarrow{p} E] \circ \mathrm{S}_{L,M}|_L \circ [E \xrightarrow{p} L]$ is an involution of $\mathbf{PGL}_2(\mathbb{F})$ for any non-tangent circles L, M. In particular if E is a fixed circle of $\mathrm{S}_{L,M}$, then $\mathrm{S}_{L,M}|_E$ is an involution of $\mathbf{PGL}_2(\mathbb{F})$ (in this case the matrix of $\mathrm{S}_{L,M}$ has coefficients in $\mathbb{F}$). From this we obtain $\Gamma_E \subseteq \mathbf{PGL}_2(\mathbb{F})$. Let $x, x', y, y' \in E$ and $x \neq y$, $x' \neq y'$. We will show that there exist circles L, M such that $\mathrm{S}_{L,M}(x) = x'$ and $\mathrm{S}_{L,M}(y) = y'$. From this the assertion follows since any involution of $\mathbf{PGL}_2(\mathbb{F})$ is determined by any two pairs of interchanging points. Suppose first $x \neq x'$ and $y \neq y'$. Let z be any point such that $z \neq x$, $z \parallel x$ and write $L := (x', E, z)^\circ$, $N = (z, y, y')^\circ$, $z' = x'N$, $M = (x, E, z')^\circ$, $P = (z, L, z')^\circ$. By Proposition 1.1, the circle P is also tangent to M. Hence $\mathrm{S}_{L,M}$ fixes the circles E, P and $\mathrm{S}_{L,M}(x) = x'$, $\mathrm{S}_{L,M}(z) = z'$. By Proposition 1.2(2), $\mathrm{S}_{L,M}(N) = N$, hence $\mathrm{S}_{L,M}(y) = y'$. In the case $y = y'$ the proof is analogous, we take $N := (z, E, y)^\circ$. If additionally $x = x'$, it is enough to consider an arbitrary double tangency symmetry $\mathrm{S}_{E,Q}$ where $Q \in \langle x, y \rangle$ and $Q \neq E$.

(b) To deduce (b) from (a) we can use any automorphism fixing the generators. Let Q be the circle such that $\mathrm{S}_Q(L) = E$, by 1.3(1). We have $[\mathrm{S}_Q(M) \xrightarrow{p} E] \circ [E \xrightarrow{t} \mathrm{S}_Q(M)] = \mathrm{S}_{L,M}|_E$.

(1) and (3) follow from the above since any element of $\mathbf{PGL}_2(\mathbb{F})$ is the composition of two involutions. ∎

Corollary 4.1. *An automorphism $\varphi \in \mathbf{Aut}(\mathbb{L})$ is projective iff $[\varphi(E) \xrightarrow{p} E] \circ \varphi|_E \in \mathbf{PGL}_2(\mathbb{F})$.*

Proof. "⇒" This follows directly from Theorem 4.1(1).

"⇐" By Theorem 4.1(1), $\varphi|_E$ is a t-projectivity and we have $\varphi|_K = [\varphi(E) \xrightarrow{p} \varphi(K)] \circ \varphi|_E \circ [K \xrightarrow{p} E]$, hence $\varphi|_K$ is a t-projectivity. ∎

Theorem 4.2. *$\widetilde{\Gamma(\mathbb{R})}$ is the group of all projective automorphisms of a symmetric Laguerre plane $\mathbb{L}$.*

Proof. Write $x_1 := x_{11} + x_{12}\varepsilon$, $x_2 := x_{21} + x_{22}\varepsilon$, $a := a_1 + a_2\varepsilon, \dots$. By (1.1), (1.5) any automorphism $\varphi \in \mathbf{M}(\mathbb{F}, \mathbb{R})$ maps

$$[x_{11} + x_{12}\varepsilon, x_{21} + x_{22}\varepsilon] \mapsto [x_{11}^\sigma + \lambda x_{12}^\sigma\varepsilon, x_{21}^\sigma + \lambda x_{22}^\sigma\varepsilon] \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $\sigma \in \operatorname{Aut}\mathbb{F}$, $\lambda \in \mathbb{F}^*$. An easy computation shows that $[\varphi(E) \xrightarrow{p} E] \circ \varphi|_E$ is the map

$$[x_{11}, x_{21}] \mapsto [x_{11}^\sigma, x_{21}^\sigma] \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}.$$

From this and Corollary 4.1 we obtain: φ is projective $\Leftrightarrow [\varphi(E) \xrightarrow{p} E] \circ \varphi|_E \in \mathbf{PGL}_2(\mathbb{F}) \Leftrightarrow \sigma = \mathrm{id} \Leftrightarrow \varphi \in \widetilde{\Gamma(\mathbb{R})}$. ∎

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