BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 58, No. 2, 2010

GLOBAL ANALYSIS, ANALYSIS ON MANIFOLDS

Weitzenböck Formula for $SL(q)$-foliations

by

Adam BARTOSZEK, Jerzy KALINA and Antoni PIERZCHALSKI

Presented by Andrzej TRAUTMAN

Summary. A Weitzenböck formula for $SL(q)$-foliations is derived. Its linear part is a relative trace of the relative curvature operator acting on vector valued forms.

1. Introduction. The aim of the paper is to get a Weitzenböck formula for a wide class of foliations, so called $SL(q)$-foliations, in the algebra of relative vector valued differential forms. For foliations and their basic properties, see [1], [2], [3] and [5]. An $SL(q)$-foliation is a foliation satisfying the condition that its transversal volume form is closed. $SL(q)$-foliations have been studied extensively in the broader, measure-theoretical, context by Sacksteder [7], Plante [4] and others. They play an important role in many contexts.

A question arises how restrictive the above condition is. The example in Section 2 shows that, locally, every foliation can be made an $SL(q)$-foliation by a suitable choice of a Riemannian metric. For $SL(q)$-foliations, one can derive many nice properties of the differential, the codifferential and the Hodge star operator. Notably, all three restrict to the leaves of the foliation and preserve all their attributes (cf. Propositions 3–7). These properties are derived thanks to our working in a very special coordinate system, called $\mathcal{F}$-normal. The description and the proof of existence of such a system can be found in Section 2.

It is interesting that the Weitzenböck formula obtained in Section 6 is similar to the classical one. The main difference is that traces are replaced by relative traces, that is, taken with respect to frames tangent to the leaves.

2010 *Mathematics Subject Classification*: Primary 58J60; Secondary 53C29, 53C12.
Key words and phrases: Weitzenböck formula, $SL(q)$-foliation, Ricci operator.

DOI: 10.4064/ba58-2-8

2. Basic notions. For the sake of simplicity, throughout our paper, we will denote the spaces of smooth sections of bundles by the same symbols as the bundles themselves, e.g., TM is, depending on context, the tangent bundle or the space of vector fields, i.e., the space of sections of this bundle. Let (M, g) be an oriented Riemannian manifold of dimension n, and let ∇^g be its Levi-Civita connection. Assume that $\mathcal{F}$ is a transversally oriented p-dimensional foliation of M and $q = n - p$. Let Q be the normal bundle: $Q = TM/T\mathcal{F}$, and (E, h) an arbitrary Riemannian vector bundle over M of rank r. Let ∇^h be any covariant derivative in E which is compatible with the metric h. The Riemannian structure and the orientations enable us to define two forms $\Omega_{\mathcal{F}}$, Ω_Q on M as follows [9]. Let $a \in M$ and let $e^1, \dots, e^n$ be an orthonormal base in T_aM^* such that $e^1, \dots, e^p$ is a base in $T_a\mathcal{F}^*$. Then, at a,

$$\Omega_{\mathcal{F}} = e^1 \wedge \cdots \wedge e^p, \qquad \Omega_Q = e^{p+1} \wedge \cdots \wedge e^n.$$

It is clear that the above formulas define global forms on M called the *tangent* and the *transversal volume forms*, respectively.

We say that $\mathcal{F}$ is an *$SL(q)$-foliation* (cf. Tondeur [8]) if the transversal volume form is closed, i.e., $d\Omega_Q = 0$.

It is easy to see that any foliation is locally (for a suitable choice of a Riemannian metric) an $SL(q)$-foliation. Indeed, let $\phi : M \to N$ be a submersion of an oriented manifold M to an oriented Riemannian manifold N. There is a smooth distribution D on M such that for, each point $x_0 \in M$, $T_{x_0}M = T_{x_0}\mathcal{F} + D_{x_0}$, where $\mathcal{F}$ is the foliation given by ϕ. Moreover, $d\phi$ gives an isomorphism between D_{x_0} and $T_{\phi(x_0)}N$. Let $(Y_1, \dots, Y_q)$ be a local orthonormal frame in a neighborhood V of $\phi(x_0)$ on N. Let $(X_1, \dots, X_n)$ be a frame in a neighborhood U of x_0 such that, for any $x \in U$, $X_1, \dots, X_p \in T\mathcal{F}$ and $d\phi_{|x_0}(X_j) = Y_j|\phi(x_0)$, $j = p + 1, \dots, n$. Let $f_{p+1}, \dots, f_n$ be any system of smooth nonvanishing functions in U such that $f_{p+1} \cdot \ldots \cdot f_n = 1$. Let g be the Riemannian metric on U such that $X_1, \dots, X_p, f_{p+1} \cdot X_{p+1}, \dots, f_n \cdot X_n$ form an orthonormal frame on U with respect to g. The foliation $\mathcal{F}$ is a Riemannian foliation [3], [8] in U if and only if all the functions $f_{p+1}, \dots, f_n$ are identically 1 on U. In any other case, we get an $SL(q)$-foliation which is not Riemannian.

We will work in a special coordinate system called an *$\mathcal{F}$-normal coordinate system centered at a, $a \in M$*, described in the following

Proposition 1. *For any point $a \in M$ there exists a neighborhood U of a and an $\mathcal{F}$-integrating map $x : U \to I^p \times I^q$ such that*

$$\begin{aligned} &g(\partial_i, \partial_j) = \delta_{ij}, && i, j = 1, \dots, n, \\ &\nabla^{\mathcal{F}}_{\partial_i}\partial_j = 0, && i, j = 1, \dots, p, \\ &\nabla^{\mathcal{F}}_{\partial_i} g^{jk} = 0, && i, j, k = 1, \dots, p, \end{aligned}$$

at a, where $\nabla^{\mathcal{F}}$ denotes the connection on leaves induced by ∇.

Proof. Let $\tilde{x} : U \to I^p \times I^q$ be any $\mathcal{F}$-integrating map such that $\tilde{x}(a) = (0,0)$. We use this map to induce a Riemannian structure on the p-dimensional manifold $N_a = \tilde{x}(L_a \cap U)$. Let $y : N_a \to I^p$ be a normal coordinate system on N_a with center at $(0,0)$. Thus, at $(0,0)$, we have

$$g(\partial_i, \partial_j) = \delta_{ij}, \qquad \nabla^{\mathcal{F}}_{\partial_i}\partial_j = 0, \qquad i,j = 1,\ldots,p.$$

The metric and the Levi-Civita connection on the manifold N_a are still denoted by g and $\nabla^{\mathcal{F}}$. Define $\check{x} : I^p \times I^q \to I^p \times I^q$ by

$$\check{x}(u,v) = (y(u), v).$$

Observe that $\bar{x} = \check{x} \circ \tilde{x}$ is an $\mathcal{F}$-integrating map such that

$$\begin{aligned} &g(\partial/\partial\bar{x}_i, \partial/\partial\bar{x}_j) = \delta_{ij}, && i,j = 1,\ldots,p,\\ &\nabla^{\mathcal{F}}_{\partial/\partial\bar{x}_i}\partial/\partial\bar{x}_j = 0, && i,j = 1,\ldots,p, \end{aligned}$$

at a.

Consider now a linear map $\hat{x}$ of $\mathbb{R}^p \times \mathbb{R}^q$ such that $\hat{x}(\cdot, 0)$ is the identity on $\mathbb{R}^p$, $\hat{x}(\{0\} \times \mathbb{R}^q)$ is orthogonal to $\mathbb{R}^p \times \{0\}$ and ∂_j, $j = p+1,\ldots,n$, are unit vectors mutually orthogonal at the origin. It is enough to define $x := \hat{x}^{-1} \circ \bar{x}$. ∎

3. Operator $d_{\mathcal{F}}$. Let

$$I^k(Q) = \{\omega \in \Lambda^k TM : \omega(X_1,\ldots,X_k) = 0 \text{ for } X_1,\ldots,X_k \in T\mathcal{F}\}$$

and

$$I(Q) = \sum_{k=0}^{n} I^k(Q).$$

Notice that $I(Q)$ is an ideal in the exterior algebra of differential forms on M. By the *algebra of relative forms* we mean the quotient algebra

$$\Lambda T\mathcal{F}^* = \Lambda TM^*/I(Q).$$

It is easy to see (by the Frobenius theorem) that $d(I(Q)) \subset I(Q)$. In a natural manner, we get the differential operator

$$d_{\mathcal{F}} : \Lambda T\mathcal{F}^* \to \Lambda T\mathcal{F}^*$$

given by

$$d_{\mathcal{F}}[\omega]_{\mathcal{F}} = [d\omega]_{\mathcal{F}}, \tag{1}$$

where $[\omega]_{\mathcal{F}}$ is the class of ω in $\Lambda T\mathcal{F}^*$.

The formula (1) suggests that, locally, in any integrating map, we can drop the brackets and treat relative forms just as usual forms on M that do not contain the differentials of transversal coordinates. More precisely, we have the following description of local sections of $\Lambda^k T\mathcal{F}^*$.

PROPOSITION 2. *Any integrating map $(U;(t,y))$ induces a trivialization of the bundle $\Lambda^kT\mathcal{F}^*$ over U. The trivialization defines the natural isomorphism of the modules of local sections:*

$$\Lambda^kT\mathcal{F}^*{}_{|U} \cong \Big\{\omega \in \Lambda^kTM^*{}_{|U} : \omega = \sum s_{i_1\ldots i_k} dt^{i_1} \wedge \cdots \wedge dt^{i_k}\Big\}.$$

The isomorphism can also be described uniquely when a Riemannian structure on M is given. Indeed, for every foliation $\mathcal{F}$, we have a short exact sequence

$$0 \to T\mathcal{F} \to TM \to Q \to 0.$$

Passing to duals, we have

$$0 \leftarrow T^*\mathcal{F} \leftarrow T^*M \leftarrow Q^* \leftarrow 0.$$

The metric g defines a natural splitting map $\nu : T^*\mathcal{F} \to T^*M$ such that

$$\nu(T^*\mathcal{F}) = Q^{*\perp}.$$

So, we can view sections of the bundle $\Lambda T^*\mathcal{F}$ as sections of ΛT^*M. For our further investigations, without ambiguity, we can omit the letter ν.

A similar construction can be made for sections of the bundle $\Lambda^kT\mathcal{F}^* \otimes E$ of relative k-forms with values in E.

Indeed, let $\nabla^{\mathcal{F}} : \Gamma(E) \to \Gamma(T\mathcal{F}^* \otimes E)$ be the partial linear connection on E induced by ∇^h, $e = (e_1, \ldots, e_r)$ a local moving frame of sections of E, and $e^* = (e^{*1,} \ldots, e^{*r})$ the frame of duals. Then $\nabla^{\mathcal{F}} e = \omega \otimes e$, where $\omega = (\omega^i_j)^q_{i,j=1}$ with ω^i_j relative scalar local 1-forms, $i,j = 1, \ldots, q$. The curvature $R^{\mathcal{F}}$ of $\nabla^{\mathcal{F}}$ is represented by a matrix $\Omega = [\Omega^i_j]^q_{i,j=1}$ with Ω^i_j relative local 2-forms, $i,j = 1, \ldots, q$, as follows: $R^{\mathcal{F}} = \sum^q_{i,j=1} \Omega^i_j \otimes e^{*j} \otimes e_i$.

The forms ω and Ω are related by the known structure equations:

$$\Omega = d_{\mathcal{F}}\omega + \omega \wedge \omega, \qquad d_{\mathcal{F}}\Omega = \Omega \wedge \omega - \omega \wedge \Omega,$$

One can extend $\nabla^{\mathcal{F}}$ to the bundle $\Lambda^kT\mathcal{F}^* \otimes E$ of relative k-forms η with values in E:

$$\begin{aligned}(\nabla^{\mathcal{F}}\eta)(X_0, \ldots, X_k) &= (\nabla^{\mathcal{F}}_{X_0}\eta)(X_1, \ldots, X_k)\\ &= \nabla^h_{X_0}(\eta(X_1, \ldots, X_k)) - \sum_{i=1}^{k} \eta(X_1, \ldots, \nabla^{\mathcal{F}}_{X_0}X_i, \ldots, X_k).\end{aligned}$$

We can define the operator of exterior differentiation for forms with values in E, $d_{\mathcal{F}} : \Lambda^kT\mathcal{F}^* \otimes E \to \Lambda^{k+1}T\mathcal{F}^* \otimes E$, by

$$\begin{aligned}(d_{\mathcal{F}}\eta)(X_1, \ldots, X_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} \nabla^h_{X_i}(\eta(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}))\\ &\quad + \sum_{1\le i<j\le k+1} (-1)^{i+j} \eta([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}).\end{aligned}$$

Using the same symbol as for differentiation of scalar forms will not lead to confusion since the operators in question have different domains.

One can check that, since ∇^g is torsion free, $d_{\mathcal{F}}$ is the antisymmetrization of $\nabla^{\mathcal{F}}$:

$$(2)\qquad (d_{\mathcal{F}}\eta)(X_1,\ldots,X_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i+1}(\nabla^{\mathcal{F}}_{X_i}\eta)(X_1,\ldots,\hat{X}_i,\ldots,X_{k+1}).$$

Notice that $d^2_{\mathcal{F}}\eta = R_{\mathcal{F}}\wedge\eta$, so $d^2_{\mathcal{F}} = 0$ if $\nabla^{\mathcal{F}}$ is flat.

4. Relative Hodge star operator. In this section, we will define the relative Hodge star operator for vector valued forms on foliated manifolds. This operator has all the properties of the regular star operator. However, to get the classical formulas for the codifferential operator $d^*_{\mathcal{F}}$, we have to assume in the next sections that our foliation $\mathcal{F}$ is an $SL(q)$-foliation, i.e., the transversal volume form Ω_Q is closed.

Definition 1. Let $a\in M$. Set

$$(3)\quad *_{\mathcal{F}}(e^{i_1}\wedge\cdots\wedge e^{i_k}\otimes s) := \operatorname{sgn}(i_1,\ldots,i_k,j_1,\ldots,j_{p-k})e^{j_1}\wedge\cdots\wedge e^{j_{p-k}}\otimes s,$$

where $(e^1,\ldots,e^p)$ is an oriented orthonormal (o.n.) base in $T_a\mathcal{F}^*$ and where $(i_1,\ldots,i_k,j_1,\ldots,j_{p-k})$ is a permutation of $(1,\ldots,p)$.

The formula (3) defines a linear operator

$$*_{\mathcal{F}} : \Lambda^kT_a\mathcal{F}^*\otimes E\to\Lambda^{p-k}T_a\mathcal{F}^*\otimes E.$$

One can easily check that it has the following properties.

Proposition 3.

(a) $*\alpha = (*_{\mathcal{F}}\alpha)\wedge\Omega_Q$,
(b) $*(\alpha\wedge\Omega_Q) = (-1)^{(p-k)(n-p)} *_{\mathcal{F}}\alpha$,
(c) $\alpha\wedge *_{\mathcal{F}}\beta = \langle\alpha,\beta\rangle\Omega_{\mathcal{F}}$,
(d) $*^2_{\mathcal{F}|\Lambda^kT\mathcal{F}^*\otimes E} = (-1)^{k(p-k)}\mathrm{Id}_{\Lambda^kT\mathcal{F}^*\otimes E}$,

where $\alpha,\beta\in\Lambda^kT\mathcal{F}^*\otimes E$.

5. Codifferential operator $d^*_{\mathcal{F}}$. In this section, we will assume that $\mathcal{F}$ is an $SL(q)$-foliation. First, we prove the following proposition.

Proposition 4. *If* $d\Omega_Q = 0$ *then, for any relative form* $\eta\in\Lambda^kT^*\mathcal{F}\otimes E$,

$$d(\eta\wedge\Omega_Q) = (d_{\mathcal{F}}\eta)\wedge\Omega_Q.$$

Proof. Let $a\in M$ and let $x=(x^1,\ldots,x^n)$ be an $\mathcal{F}$-normal coordinate system in U centered at a. Let $e^i = dx^i$ for $i=1,\ldots,n$. Without loss of generality, we can assume that $\eta = e^{i_1}\wedge\cdots\wedge e^{i_k}\otimes s$ on U. Then, by Proposition 1, $(e^1,\ldots,e^n)$ is an orthonormal base in T_aM^*. Recall that $\mathcal{F}$

is an $SL(q)$-foliation and, at the point a, $\Omega_Q = e^{p+1} \wedge \cdots \wedge e^n$, so, at this point, we have

$$d(e^{i_1} \wedge \cdots \wedge e^{i_k} \otimes s \wedge \Omega_Q) = \sum_{i=1}^{n} e^i \wedge e^{i_1} \wedge \cdots \wedge e^{i_k} \otimes \nabla^h_{\partial_{x_i}} s \wedge \Omega_Q$$
$$= \sum_{i=1}^{p} e^i \wedge e^{i_1} \wedge \cdots \wedge e^{i_k} \otimes \nabla^h_{\partial_{x_i}} s \wedge \Omega_Q = d_{\mathcal{F}}(e^{i_1} \wedge \cdots \wedge e^{i_k} \otimes s) \wedge \Omega_Q. \ \blacksquare$$

As a direct consequence of Propositions 3 and 4, we obtain

Proposition 5.

$$*d*_{|\Lambda^k T\mathcal{F}^* \otimes E} = (-1)^{(k-1)(n-p)} *_{\mathcal{F}} d_{\mathcal{F}} *_{\mathcal{F}}. \tag{4}$$

The right hand side of formula (4) defines, up to sign, the operator formally adjoint to $d_{\mathcal{F}}$ on M (not only on the leaves) in the following sense:

Proposition 6.

$$\int_M \langle d_{\mathcal{F}}\alpha, \beta\rangle \Omega_M = \int_M \langle \alpha, (-1)^{pk+1} *_{\mathcal{F}} d_{\mathcal{F}} *_{\mathcal{F}} \beta\rangle \Omega_M,$$

provided α or β is of compact support, so the operator

$$d^*_{\mathcal{F}} = (-1)^{pk+1} *_{\mathcal{F}} d_{\mathcal{F}} *_{\mathcal{F}}$$

is formally adjoint to $d_{\mathcal{F}}$.

Proof. Let $\alpha \in \Lambda^k T^*\mathcal{F} \otimes E$ and $\beta \in \Lambda^{k+1} T^*\mathcal{F} \otimes E$. Assume $\alpha = \omega \otimes s$ and $\beta = \eta \otimes t$. Let

$$\gamma = h(s,t)\omega \wedge *_{\mathcal{F}}\eta.$$

Then

$$d_{\mathcal{F}}\gamma = d_{\mathcal{F}}(h(s,t)) \wedge \omega \wedge *_{\mathcal{F}}\eta + h(s,t) \wedge d_{\mathcal{F}}\omega \wedge *_{\mathcal{F}}\eta + (-1)^k h(s,t) \wedge d_{\mathcal{F}}\omega \wedge d_{\mathcal{F}} *_{\mathcal{F}}\eta.$$

Thus, after multiplying both sides by Ω_Q we have

$$\begin{aligned}
d_{\mathcal{F}}\gamma \wedge \Omega_Q
&= d_{\mathcal{F}}(h(s,t)) \wedge \omega \wedge *_{\mathcal{F}}\eta \wedge \Omega_Q \\
&\quad + h(s,t) \wedge d_{\mathcal{F}}\omega \wedge *_{\mathcal{F}}\eta \wedge \Omega_Q + (-1)^k h(s,t) \wedge d_{\mathcal{F}}\omega \wedge d_{\mathcal{F}} *_{\mathcal{F}}\eta \wedge \Omega_Q \\
&= d(h(s,t)) \wedge \omega \wedge *\eta + h(s,t) \wedge d\omega \wedge *\eta + (-1)^k h(s,t) \wedge \omega \wedge d*\eta \\
&= h(\nabla^h s, t) \wedge \omega \wedge *\eta + h(s, \nabla^h t) \wedge \omega \wedge *\eta + h(s,t) \wedge d\omega \wedge *\eta \\
&\quad + (-1)^k h(s,t) \wedge \omega \wedge d*\eta.
\end{aligned}$$

Summing up the first and third summands, and then the second and fourth, we get

$$\begin{aligned} d_{\mathcal{F}}\gamma \wedge \Omega_Q &= [h(\nabla^h s, t) \wedge \omega + h(s,t) \wedge d\omega] \wedge *\eta \\ &\quad + (-1)^k \wedge h(s, \nabla^h t) \wedge *\eta + (-1)^{kn} h(s,t)\omega \wedge **d*\eta \\ &= h(\nabla^h s(\cdot) \wedge \omega + s \otimes d\omega, t \otimes \eta)\Omega_M \\ &\quad + (-1)^{kn}\{\omega \wedge **[h(s, \nabla^h t(\cdot)) \wedge *\eta] + h(s,t)\omega \wedge **d*\eta\} \\ &= \langle \nabla^h s(\cdot) \wedge \omega + s \otimes d\omega, t \otimes \eta\rangle \Omega_M \\ &\quad + (-1)^{kn}\langle \omega \otimes s, *[\nabla^h t \wedge *\eta + t \otimes d * \eta]\rangle \Omega_M \\ &= \langle d(\omega \otimes s), \eta \otimes t\rangle \Omega_M + (-1)^{kn}\langle \omega \otimes s, *d*(\eta \otimes t)\rangle \Omega_M. \end{aligned}$$

By Proposition 5, the last two terms are equal to

$$\langle d_{\mathcal{F}}(\omega \otimes s), \eta \otimes t\rangle \Omega_M + (-1)^{kn}\langle \omega \otimes s, *_{\mathcal{F}} d_{\mathcal{F}} *_{\mathcal{F}} (\eta \otimes t)\rangle \Omega_M.$$

Integrating over M and applying Stokes' theorem, we obtain the statement. ∎

Proposition 7. *If $\mathcal{F}$ is an $SL(q)$-foliation, $(e_i)_{i=1,\dots,p}$ a base of $T_z\mathcal{F}$, $X_j \in T\mathcal{F}$, $j = 1, \dots, k-1$, and $\rho \in \Lambda^k T\mathcal{F}^* \otimes E$, then*

$$(d^*_{\mathcal{F}}\rho)(X_1, \dots, X_{k-1}) = -\sum_{s,t=1}^{p} g^{st}(\nabla^{\mathcal{F}}_{e_t}\rho)(e_s, X_1, \dots, X_{k-1}).$$

Proof. Let $x : U \to I^p \times I^q$ be an $\mathcal{F}$-normal coordinate system centered at x_0. For $\sigma \in \Lambda^{k-1} T\mathcal{F}^* \otimes E$, we have

$$\begin{aligned} &\langle d_{\mathcal{F}}\sigma, \rho\rangle - \Big\langle \sigma, -\sum_{s=1}^{p} (\nabla^{\mathcal{F}}_s \rho)(\partial_s, \cdot)\Big\rangle \\ &\quad = \frac{1}{k!} \sum_{i_1,\dots,i_k=1}^{p} h(d_{\mathcal{F}}\sigma(\partial_{i_1}, \dots, \partial_{i_k}), \rho(\partial_{i_1}, \dots, \partial_{i_k})) \\ &\qquad + \frac{1}{(k-1)!} \sum_{s,i_1,\dots,i_{k-1}=1}^{p} h(\sigma(\partial_{i_1}, \dots, \partial_{i_{k-1}}), (\nabla^{\mathcal{F}}_s \rho)(\partial_s, \partial_{i_1}, \dots, \partial_{i_{k-1}})). \end{aligned}$$

By (2) and the compatibility of the metric and the connection,

$$\begin{aligned} &\langle d_{\mathcal{F}}\sigma, \rho\rangle - \Big\langle \sigma, -\sum_{s=1}^{p} (\nabla^{\mathcal{F}}_s \rho)(\partial_s, \cdot)\Big\rangle \\ &\quad = \frac{1}{k!} \sum_{r=1}^{k} (-1)^{r+1} \sum_{i_1,\dots,i_k=1}^{p} h((\nabla^{\mathcal{F}}_{i_r}\sigma)(\partial_{i_1}, \dots, \hat{\partial}_{i_r}, \dots, \partial_{i_k}), \rho(\partial_{i_1}, \dots, \partial_{i_k})) \\ &\qquad + \frac{1}{(k-1)!} \sum_{s,i_1,\dots,i_{k-1}=1}^{p} h(\sigma(\partial_{i_1}, \dots, \partial_{i_{k-1}}), (\nabla^{\mathcal{F}}_s \rho)(\partial_s, \partial_{i_1}, \dots, \partial_{i_{k-1}})) \end{aligned}$$

$$= \frac{1}{(k-1)!} \sum_{s,i_1,\ldots,i_{k-1}=1}^{p} [h((\nabla^{\mathcal{F}}_s \sigma)(\partial_{i_1},\ldots,\partial_{i_{k-1}}), \rho(\partial_s,\partial_{i_1},\ldots,\partial_{i_{k-1}}))$$
$$+ h(\sigma(\partial_{i_1},\ldots,\partial_{i_{k-1}}), (\nabla^{\mathcal{F}}_s \rho)(\partial_s,\partial_{i_1},\ldots,\partial_{i_{k-1}}))]$$
$$= \sum_{s=1}^{p} \partial_s \langle \sigma, \rho(\partial_s,\cdot)\rangle = \operatorname{div}^{\mathcal{F}} X,$$

where $X = \sum_{s=1}^{p} \langle \sigma, \rho(\partial_s,\cdot)\rangle \partial_s$. Observe now that since $d\Omega_Q = 0$,

$$\begin{aligned} \operatorname{div} X\, \Omega_M &= \mathcal{L}_X(\Omega_{\mathcal{F}} \wedge \Omega_Q) = \mathcal{L}_X(\Omega_{\mathcal{F}}) \wedge \Omega_Q \\ &= \operatorname{div}^{\mathcal{F}} X\, (\Omega_{\mathcal{F}} \wedge \Omega_Q) = \operatorname{div}^{\mathcal{F}} X\, \Omega_M. \end{aligned}$$

Therefore $\operatorname{div}_{\mathcal{F}} X = \operatorname{div} X$. Thus

$$\langle d_{\mathcal{F}}\sigma, \rho\rangle - \Big\langle \sigma, -\sum_{s=1}^{p} (\nabla^{\mathcal{F}}_s \rho)(\partial_s,\cdot)\Big\rangle = \operatorname{div} X. \ \blacksquare$$

6. Weitzenböck formula

THEOREM 1 (Weitzenböck formula). *Let (M,g) be a Riemannian manifold, (E,h) a Riemannian vector bundle, and $\mathcal{F}$ an $SL(q)$-foliation. Then, for any $\sigma \in \Lambda^k T\mathcal{F}^* \otimes E$,*

$$\Delta^{\mathcal{F}} \sigma = -\operatorname{trace}^{\mathcal{F}} (\nabla^{\mathcal{F}})^2 \sigma + S^{\mathcal{F}}(\sigma),$$

where

$$S^{\mathcal{F}}(\sigma)(X_1,\ldots,X_k) = \sum_{s=1}^{p}\sum_{j=1}^{k} (-1)^j (R^{\mathcal{F}}(e_s,X_j)\sigma)(e_s,X_1,\ldots,\hat{X}_j,\ldots,X_k)$$

and

$$R^{\mathcal{F}}(e_s,X_j) = -\nabla^{\mathcal{F}}_{e_s}\nabla^{\mathcal{F}}_{X_j} + \nabla^{\mathcal{F}}_{X_j}\nabla^{\mathcal{F}}_{e_s} + \nabla^{\mathcal{F}}_{[e_s,X_j]},$$

for $X_1,\ldots,X_k \in T\mathcal{F}$ and $(e_1,\ldots,e_p)$ an o.n. base of $T_x\mathcal{F}$.

Proof. Let $x_0 \in M$ and let $(x^1,\ldots,x^n)$ be an $\mathcal{F}$-normal integrating coordinate system at x_0. Observe that, by the definition of $d_{\mathcal{F}}$, Proposition 7 and the $\mathcal{F}$-normality of the coordinate system, we have

$$\begin{aligned} (d_{\mathcal{F}} d^*_{\mathcal{F}} \sigma)(\partial_{i_1},\ldots,\partial_{i_k}) &= \sum_{j=1}^{k} (-1)^{j+1} \nabla^h_{i_j} [(d^*_{\mathcal{F}}\sigma)(\partial_{i_1},\ldots,\hat{\partial}_{i_j},\ldots,\partial_{i_k})] \\ &= \sum_{j=1}^{k} (-1)^{j+1} \nabla^h_{i_j} \Big[-\sum_{s,t=1}^{p} g^{st} (\nabla^{\mathcal{F}}_t \sigma)(\partial_s,\partial_{i_1},\ldots,\hat{\partial}_{i_j},\ldots,\partial_{i_k})\Big] \\ &= \sum_{s=1}^{p}\sum_{j=1}^{k} (-1)^j [\nabla^{\mathcal{F}}_{i_j}(\nabla^{\mathcal{F}}_s \sigma)](\partial_s,\partial_{i_1},\ldots,\hat{\partial}_{i_j},\ldots,\partial_{i_k}). \end{aligned}$$

On the other hand, using Proposition 7, the $\mathcal{F}$-normality of the coordinate system and formula (2), one can calculate

$$
\begin{aligned}
(d_{\mathcal{F}}^{*\mathcal{M}} d_{\mathcal{F}}\sigma)(\partial_{i_1},\ldots,\partial_{i_k}) &= -\sum_{s,t=1}^{p} g^{st}(\nabla_t^{\mathcal{F}} d_{\mathcal{F}}\sigma)(\partial_s,\partial_{i_1},\ldots,\partial_{i_k}) \\
&= -\sum_{s=1}^{p}\nabla_s^h[(d_{\mathcal{F}}\sigma)(\partial_s,\partial_{i_1},\ldots,\partial_{i_k})] + \sum_{s=1}^{p}(d_{\mathcal{F}}\sigma)\Big(\cancelto{0}{\nabla_t^{\mathcal{F}}\partial_s},\partial_{i_1},\ldots,\partial_{i_k}\Big) \\
&\quad + \sum_{j=1}^{k}\sum_{s=1}^{p}(d_{\mathcal{F}}\sigma)\Big(\partial_s,\partial_{i_1},\ldots,\cancelto{0}{\nabla_t^{\mathcal{F}}\partial_{i_j}},\ldots,\partial_{i_k}\Big) \\
&= -\sum_{s=1}^{p}\nabla_s^h[(\nabla_s^{\mathcal{F}}\sigma)(\partial_{i_1},\ldots,\partial_{i_k})] \\
&\quad - \sum_{s=1}^{p}\nabla_{\partial_s}^{\mathcal{F}}\Big[\sum_{j=1}^{k}(-1)^j(\nabla_{i_j}^{\mathcal{F}}\sigma)(\partial_s,\partial_{i_1},\ldots,\hat{\partial}_{i_j}\ldots,\partial_{i_k})\Big] \\
&= -\sum_{s=1}^{p}[\nabla_s^{\mathcal{F}}(\nabla_s^{\mathcal{F}}\sigma)](\partial_{i_1},\ldots,\partial_{i_k}) \\
&\quad - \sum_{s=1}^{p}\sum_{j=1}^{k}(-1)^j[\nabla_{\partial_s}^{\mathcal{F}}(\nabla_{i_j}^{\mathcal{F}}\sigma)](\partial_s,\partial_{i_1},\ldots,\hat{\partial}_{i_j}\ldots,\partial_{i_k}).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&[(d_{\mathcal{F}}d_{\mathcal{F}}^* + d_{\mathcal{F}}^* d_{\mathcal{F}}\sigma)](\partial_{i_1},\ldots,\partial_{i_k}) \\
&\qquad = \sum_{j=1}^{k}(-1)^j\sum_{s=1}^{p}(\nabla_{i_j}^{\mathcal{F}}\nabla_s^{\mathcal{F}}\sigma)(\partial_s,\partial_{i_1},\ldots,\hat{\partial}_{i_j},\ldots,\partial_{i_k}) \\
&\qquad\quad - \sum_{s=1}^{p}(\nabla_s^{\mathcal{F}}\nabla_s^{\mathcal{F}}\sigma)(\partial_{i_1},\ldots,\partial_{i_k}) \\
&\qquad\quad - \sum_{j=1}^{k}(-1)^j\sum_{s=1}^{p}(\nabla_s^{\mathcal{F}}\nabla_{i_j}^{\mathcal{F}}\sigma)(\partial_s,\partial_{i_1},\ldots,\hat{\partial}_{i_j},\ldots,\partial_{i_k}).
\end{aligned}
$$

This completes the proof of the theorem. ∎

It is interesting to note that also a global (not foliated) version of the Weitzenböck formula can be used to get some geometric properties of a foliated manifold. E.g., Rummler [6] obtained that way some criteria for a foliation to be geodesic or parallel.

References

[1] S. Hurder, *Exotic classes for measured foliations*, Bull. Amer. Math. Soc. (N.S.) 7 (1982), 398–391.
[2] H. B. Lawson, Jr., *Foliations*, ibid. 80 (1974), 369–418.
[3] P. Molino, *Riemannian Foliations*, Progr Math. 73, Birkhäuser, Boston, 1988.
[4] J. Plante, *Foliations with measure preserving holonomy*, Ann. of Math. 102 (1975), 327–361.
[5] B. L. Reinhart, *Foliated manifolds with bundle-like metric*, ibid. 69 (1959), 119–132.
[6] H. Rummler, *Differential forms, Weitzenböck formula and foliations*, Publ.. Mat. 33 (1989), 543–554.
[7] R. Sacksteder, *Foliations and pseudo-groups*, Amer. J. Math. 87 (1965), 79–102.
[8] Ph. Tondeur, *Foliations on Riemannian Manifolds*, Springer, Berlin, 1988.
[9] —, *Geometry of Foliations*, Birkhäuser, Basel, 1997.

Adam Bartoszek, Antoni Pierzchalski
Faculty of Mathematics and Computer Science
Lodz University
Banacha 22
90-238 Łódź, Poland
E-mail: mak@math.uni.lodz.pl
antoni@math.uni.lodz.pl

Jerzy Kalina
Institute of Mathematics
Technical University of Lodz
Al. Politechniki 11
93-590 Łódź, Poland
E-mail: jerzy.kalina@p.lodz.pl

Received March 19, 2010;
received in final form July 30, 2010 (7752)