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Representations of Reals in Reverse Mathematics

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Summary. Working in the framework of reverse mathematics, we consider representations of reals as rapidly converging Cauchy sequences, decimal expansions, and two sorts of Dedekind cuts. Converting single reals from one representation to another can always be carried out in RCA_0. However, the conversion process is not always uniform. Converting infinite sequences of reals in some representations to other representations requires the use of WKL_0 or ACA_0.

Early in the study of computable analysis, several authors noted that many representations of computable reals could be computably converted to other representations on a real by real basis [8], [7], [5]. Mostowski [4] observed that converting certain sequences of computable reals between representations was not a computable process. A more recent development of representations of sequences of reals from the viewpoint of computable analysis in the TTE (Type-2 Theory of Effectivity) framework appears in Chapter 4 of [11]. Because of the significance of sequences of reals in computable analysis (see [6] and [11]) and reverse mathematics (see [9]), this is more than an idle curiosity.

We will analyze representations of reals using the techniques of reverse mathematics. The subsystems used in this paper are RCA_0, WKL_0, and ACA_0. The systems differ in the available set comprehension axioms. RCA_0 includes the recursive comprehension axiom, which essentially asserts the existence of relatively computable sets. WKL_0 appends a weak version of König's lemma that says that infinite 0-1 trees have infinite paths. ACA_0 adds a comprehension scheme for arithmetically definable sets. Simpson's

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book [9] is an excellent resource for complete details about these subsystems.

Section 1 introduces the various representations considered here and notions of equality between reals. Section 2 includes conversion results that can be proved in RCA_0, including conversions for single reals. Section 3 presents equivalence results showing the necessity of using stronger axiom systems for some conversions. That section ends with a table summarizing the results of Sections 2 and 3. Section 4 presents related results on sequences of irrational numbers and change of basis for expansions.

1. Representations of reals. We will consider four ways of representing reals and encoding these representations in RCA_0. The first is the usual rapidly converging Cauchy sequence used in reverse mathematics. A function $\varrho : \mathbb{N} \to \mathbb{Q}$ is a *rapidly converging Cauchy sequence* if ϱ satisfies

$$\forall k\ \forall i\ |\varrho(k) - \varrho(k+i)| \le 2^{-k}.$$

For our purposes, a decimal expansion is a special sort of rapidly converging Cauchy sequence in which $\delta(j)$ gives the first j decimal places of the decimal representation of the real. Thus, $\delta : \mathbb{N} \to \mathbb{Q}$ is a *decimal expansion* if $\delta(0)$ is an integer or the special digit -0, and

$$\forall k\ \exists j \in \{0, \dots, 9\}\ (\delta(k+1) - \delta(k) = \operatorname{sign}(\delta(0)) \cdot j \cdot 10^{-k-1}).$$

In this definition, decimal expansions terminating in either repeating nines or repeating zeros are allowed. We will treat these special cases in our discussion of equality. To make the signs work correctly, we must distinguish between -0 and 0 as a digit. For example, the first digit of an element of the interval $(-1, 0)$ will be -0. The first digit in a representation of 0 could be either -0 or 0.

The remaining two representations are forms of Dedekind cuts. Since RCA_0 proves that the complement of any given subset of $\mathbb{Q}$ exists, we can encode a cut by specifying just the elements of the lower set. To be precise, a set $\lambda \subseteq \mathbb{Q}$ is a (lower) *Dedekind cut* if $\emptyset \subsetneq \lambda \subsetneq \mathbb{Q}$ and

$$\forall s \in \mathbb{Q}\ \forall s' \in \mathbb{Q}\ ((s \in \lambda \wedge s' \notin \lambda) \to s < s').$$

This definition is exactly like that in Section IV of Dedekind [1] in that cuts representing a rational number may or may not contain the rational. Many modern analysis texts specify the location of the rational in this case. We can append this requirement to the definition as follows. A set $\sigma \subset \mathbb{Q}$ is an *open cut* if it is a Dedekind cut and $\forall s \in \sigma\ \exists s' \in \sigma\ (s < s')$. This completes our list of representations of reals: rapidly converging Cauchy sequences, decimal expansions, Dedekind cuts, and open cuts.

In reverse mathematics, equality of sets is defined extensionally from equality on natural numbers. Similarly, equality of representations of reals requires definition. For example, following Simpson [9], if ϱ and τ are rapidly converging Cauchy sequences, then we say that ϱ and τ are equal (and write $\varrho = \tau$) if

$$\forall k\ (|\varrho(k) - \tau(k)| \leq 2^{-k+1}).$$

Naïvely, we are saying that $\varrho = \tau$ if the sequences converge to the same real. Technically, we are abusing notation, since we may write $\varrho = \tau$ (as reals) even when ϱ and τ are not equal as sets.

Since a decimal expansion is a special sort of rapidly converging Cauchy sequence, equality of decimal expansions is defined as in the preceding paragraph. In RCA_0 it is easy to prove that if ϱ and τ are decimal expansions, then $\varrho = \tau$ if and only if either ϱ and τ agree in every digit, or else (subject to renaming ϱ and τ) there is a j such that $\varrho(i) = \tau(i)$ for $i < j$, $|\varrho(j)| = |\tau(j)| + 10^j$, and $\varrho(k) = 0$ and $\tau(k) = 9$ for $k > j$. Of course, since decimal expansions are rapidly converging Cauchy sequences, equality between reals in these two representations is defined.

Now we may turn to equality of cuts. Two Dedekind cuts are equal (as reals) if they differ in at most one element. Since open cuts are Dedekind cuts, this definition extends to comparisons between open cuts or between open cuts and other Dedekind cuts. RCA_0 can prove that if two open cuts are equal (as reals) then they must agree on all elements, and so are equal as sets also.

Finally, suppose that λ is a Dedekind cut and ϱ is a rapidly converging Cauchy sequence. We say that λ and ϱ are equal (as reals) if

$$\forall k\ \forall s\ \forall s'\ ((s \in \lambda \wedge s' \notin \lambda) \rightarrow [s, s'] \cap [\varrho(k) - 2^{-k}, \varrho(k) + 2^{-k}] \neq \emptyset).$$

Intuitively, a rapidly converging Cauchy sequence can be viewed as specifying a real as a nested sequence of closed intervals, and similarly, a Dedekind cut can be viewed as specifying a real as the intersection of a set of closed intervals. If the intervals all overlap, then the two representations must correspond to the same real. It is also worth noting that the formula

$$[s, s'] \cap [\varrho(k) - 2^{-k}, \varrho(k) + 2^{-k}] \neq \emptyset$$

can be written as a comparison of rational endpoints,

$$\neg(\varrho(k) + 2^{-k} < s \vee s' < \varrho(k) - 2^{-k}),$$

which is a Δ^0_0 formula. Thus the formula encoding $\lambda = \varrho$ is Π^0_1, as are the formulas encoding equality between rapidly converging Cauchy sequences and equality between cuts.

We have defined four representations of real numbers, and have defined equality between any possible pair of representations. With this terminology, we can discuss conversions between representations.

2. Conversions in RCA_0. In this section, we will examine those situations where it is possible to convert a sequence of reals in one representation to a sequence in another representation while working within RCA_0. By the end of the section, we will be able to dispense with conversions of single reals. Conversions that require stronger axiom systems will be presented in the next section. In the statement of the following theorems, the notation (RCA_0) indicates that the result is provable in RCA_0.

THEOREM 1 (RCA_0). *If $\langle\lambda_i\rangle_{i\in\mathbb{N}}$ is a sequence of Dedekind cuts, then there is a sequence $\langle\delta_i\rangle_{i\in\mathbb{N}}$ of decimal expansions such that for each $i \in \mathbb{N}$, $\lambda_i = \delta_i$.*

Proof. Suppose $\langle\lambda_i\rangle_{i\in\mathbb{N}}$ is a sequence of Dedekind cuts. We will indicate how to compute $\delta_i(j)$, the jth element of the ith decimal expansion.

For $j = 0$, let z be the greatest integer in λ_i. Note that z exists because $\lambda_i \neq \mathbb{Q}$ and the complement of λ_i is closed upward. If $z \geq 0$, then $\delta_i(0) = z$. If $z < 0$, then $\delta_i(0) = z + 1$, where $-1 + 1$ is taken to be -0.

Suppose $\delta_i(j)$ has been computed. If $\delta_i(0) \geq 0$, let d be the greatest element of $K = \{k \cdot 10^{j+1} \mid k \in \{0, \ldots, 9\}\}$ such that $\delta_i(j) + d \in \lambda_i$, and set $\delta_i(j+1) = \delta_i(j) + d$. If $\delta_i(0) < 0$, let d be the greatest element of K such that $\delta_i(j) - d \notin \lambda_i$, and set $\delta_i(j+1) = \delta_i(j) - d$.

The preceding computation shows that the proof of the existence of $\langle\delta_i\rangle_{i\in\mathbb{N}}$ can be carried out in RCA_0. The claim that $\lambda_i = \delta_i$ for all $i \in \mathbb{N}$ follows immediately from the definition of equality between Dedekind cuts and rapidly converging Cauchy sequences. ∎

Since every open cut is a Dedekind cut and every decimal expansion is a rapidly converging Cauchy sequence, Theorem 1 has the following corollary.

COROLLARY 2 (RCA_0). *If $\langle\mu_i\rangle_{i\in\mathbb{N}}$ is a sequence of reals in a representation in the following list, then for any representation appearing lower in the list there is a sequence $\langle\tau_i\rangle_{i\in\mathbb{N}}$ in that representation such that for all $i \in \mathbb{N}$, $\mu_i = \tau_i$:*

- *open cuts,*
- *Dedekind cuts,*
- *decimal expansions,*
- *rapidly converging Cauchy sequences.*

With one additional result, we can resolve the conversion problem for all single reals.

THEOREM 3 (RCA_0). *Suppose ϱ is a rapidly converging Cauchy sequence. Then there is an open cut σ such that $\varrho = \sigma$.*

Proof. Let ϱ be a rapidly converging Cauchy sequence. Either ϱ represents a rational or it does not. (This assertion is not uniform.) If ϱ represents the rational r, then let $\lambda = \{q \in \mathbb{Q} \mid q < r\}$. Otherwise, ϱ is not equal to any rational. Consequently, for any $q \in \mathbb{Q}$, there is a $k \in \mathbb{N}$ such that $\varrho(k) + 2^{-k} < q$ or $\varrho(k) - 2^{-k} > q$. The open cut λ is constructed by excluding q when $\varrho(k) + 2^{-k} < q$ and including q when $\varrho(k) - 2^{-k} > q$. ∎

Combining Theorem 3 and Corollary 2 for constant sequences yields the following corollary showing that for single reals all conversions can be carried out in RCA_0. The computability-theoretic analog of this result was observed by Robinson [8], Myhill [5], and Rice [7].

Corollary 4 (RCA_0). *If μ is a single real in any of the four representations, then there is a real τ in each of the other representations such that $\mu = \tau$.*

Proof. Theorem 3 allows conversions from the bottom of the list in Corollary 2 to the top. ∎

In the next section we will see that the nonuniformity in the proof of Theorem 3 is unavoidable. Consequently, proving the analog of Corollary 4 for sequences of reals requires stronger axiom systems than RCA_0.

3. Conversions requiring WKL_0 and ACA_0. In this section we will show that conversions between some representations of reals require axioms beyond RCA_0. Our work will be simplified by the following technical lemma. This lemma extends a conservation result due to Kohlenbach [3, Proposition 3.1].

Lemma 5 (RCA_0). *The following are equivalent:*

(1) WKL_0.
(2) *If $\langle f_i\rangle_{i\in\mathbb{N}}$ and $\langle g_i\rangle_{i\in\mathbb{N}}$ are sequences of functions with pairwise disjoint ranges, that is, such that $\forall i\ \forall n\ \forall m\ (f_i(n) \neq g_i(m))$, then there is a sequence $\langle X_i\rangle_{i\in\mathbb{N}}$ of sets such that for each i, $\forall n\ (f_i(n) \in X_i \wedge g_i(n) \notin X_i)$.*
(3) *If $\langle T_i\rangle_{i\in\mathbb{N}}$ is a sequence of infinite 0-1 trees, then there is a sequence $\langle X_i\rangle_{i\in\mathbb{N}}$ such that for each i, X_i is an infinite path through T_i.*

Proof. Since the existence of a separating set for a single pair of functions implies WKL_0 [9, Lemma IV.4.4], as does the existence of an infinite path through a single infinite 0-1 tree, it suffices to show that (2) and (3) follow from WKL_0.

Suppose $\langle f_i\rangle_{i\in\mathbb{N}}$ and $\langle g_i\rangle_{i\in\mathbb{N}}$ are sequences of functions with pairwise disjoint ranges, as in (2). Fix a bijection between $\mathbb{N}\times\mathbb{N}$ and $\mathbb{N}$, and identify each ordered pair with its integer code. Define functions f and g by setting $f(i,n) = (f_i(n), i)$ and $g(i,n) = (g_i(n), i)$. Since we are viewing ordered pairs as being interchangeable with their integer codes, we may think of f and g

as functions from $\mathbb{N}$ to $\mathbb{N}$. Note that if $f(i,n) = g(j,m)$, then $j = i$ and $f_i(n) = g_j(m) = g_i(m)$, contradicting the claim that the ranges of f_i and g_i are disjoint. Thus f and g have disjoint ranges. WKL_0 suffices to prove the existence of a separating set X for f and g [9, Lemma IV.4.4]. For each i, let $X_i = \{m \mid (m,i) \in X\}$. Then for all n, $(f_i(n), i) \in X$ so $f_i(n) \in X_i$, and $(g_i(n), i) \notin X$ so $g_i(n) \notin X_i$. Thus WKL_0 proves (2) as desired.

Now we will use WKL_0 to prove (3). Let $\langle T_i \rangle_{i\in\mathbb{N}}$ be a sequence of infinite 0-1 trees. Form a tree T of finite sequences of natural numbers as follows. For $j \in \mathbb{N}$, if for each $i < j$ we are given a sequence σ_i in T_i of length $j - i$, then form the sequence

$$\sigma = (\sigma_0(0), (\sigma_0(1), \sigma_1(0)), \ldots, (\sigma_0(j-1), \ldots, \sigma_{j-1}(0))).$$

By identifying the inner finite sequences with their integer codes, σ can be viewed as a sequence of j natural numbers. Let T be the tree of all such sequences. Since each T_i is a 0-1 tree, $\sigma(n)$ can take at most 2^{n+1} possible values, so T is a bounded tree. WKL_0 suffices to prove the existence of an infinite path through T [9, Lemma IV.1.4]. Given a path $X = (p_0, p_1, p_2, \ldots)$ through T, for each i the sequence $X_i = (p_i(i), p_{i+1}(i), p_{i+2}(i), \ldots)$ is a path through T_i. This completes the proof of (3)from WKL_0. ∎

Now we can turn to the theorems on converting representations. The next three theorems will enable us to completely analyze all possible conversions.

THEOREM 6 (RCA_0). *The following are equivalent*:

(1) WKL_0.

(2) *If $\langle \varrho_i \rangle_{i\in\mathbb{N}}$ is a sequence of rapidly converging Cauchy sequences then there is a sequence $\langle \delta_i \rangle_{i\in\mathbb{N}}$ of decimal expansions such that for each $i \in \mathbb{N}$, $\varrho_i = \delta_i$.*

Proof. To prove that (1) implies (2), assume WKL_0 and let $\langle \varrho_i \rangle_{i\in\mathbb{N}}$ be a sequence of rapidly converging Cauchy sequences. For each ϱ_i, construct a tree T_i as follows. Put a sequence δ in T_i if δ is an initial segment of a decimal expansion and for each $j < \mathrm{lh}(\delta)$, $\varrho_i(j) - 2^{-j+1} \le \delta(j) \le \varrho_i(j) + 2^{-j+1}$. For each k, each initial segment of the sequence consisting of the first k digits of the decimal expansion of $\varrho_i(k)$ satisfies these conditions, so T_i is an infinite tree. If δ_i is an infinite path through T_i, then from the definition of equality for rapidly converging Cauchy sequences, $\varrho_i = \delta_i$. RCA_0 suffices to prove that the sequence $\langle T_i \rangle_{i\in\mathbb{N}}$ exists, and by Lemma 5, WKL_0 proves the sequence $\langle \delta_i \rangle_{i\in\mathbb{N}}$ exists.

To prove the reversal, it suffices to use RCA_0 and (2) to separate the ranges of disjoint functions [9, Lemma IV.4.4]. The computable analysis counterexample corresponding to this implication appears as part of Theorem 4 of [4]. Suppose f and g are injections such that $\forall n\, \forall m\, (f(n) \ne g(m))$. Define a sequence of rapidly converging Cauchy sequences as follows. For each i and j, let

$$\varrho_i(j) = \begin{cases} 1 & \text{if } \forall k < j\ (f(k) \neq i \wedge g(k) \neq i), \\ 1 + 2^{-k} & \text{if } k < j \wedge f(k) = i, \\ 1 - 2^{-k} & \text{if } k < j \wedge g(k) = i. \end{cases}$$

By the recursive comprehension axiom, $\langle \varrho_i \rangle_{i \in \mathbb{N}}$ exists. Apply (2) to obtain a sequence $\langle \delta_i \rangle_{i\in\mathbb{N}}$ of decimal expansions such that for all $i \in \mathbb{N}$, $\varrho_i = \delta_i$. Note that if $f(k) = i$ then $\delta_i(0) = 1$, and if $g(k) = i$ then $\delta_i(0) = 0$. Thus the function $\chi(i) = \delta_i(0)$ is the characteristic function for a separating set for the ranges of f and g. ∎

Theorem 7 (RCA_0). *The following are equivalent*:

(1) WKL_0.
(2) *If* $\langle \delta_i \rangle_{i\in\mathbb{N}}$ *is a sequence of decimal expansions then there is a sequence* $\langle \lambda_i \rangle_{i\in\mathbb{N}}$ *of Dedekind cuts such that for each* $i \in \mathbb{N}$, $\delta_i = \lambda_i$.

Proof. To prove that (1) implies (2), assume WKL_0 and let $\langle \delta_i \rangle_{i\in\mathbb{N}}$ be a sequence of decimal expansions. Fix an enumeration of $\mathbb{Q}$. Note that the sign of a decimal expansion δ_i can be determined from $\delta_i(0)$. For each δ_k define a pair of functions f_k and g_k as follows. If δ_k is greater than 0 or equal to 0, let $f_k(m) = q$, where q is the first element of $\mathbb{Q}$ that is not in $[f_k(m)] \cup [g_k(m)]$ (the ranges of f_k and g_k on values less than m) that satisfies $q < \delta_k(m)$. Let $g_k(m) = q$ where q is the first element of $\mathbb{Q}$ that is not in $[f_k(m+1)] \cup [g_k(m)]$ that satisfies $q > \delta_k(m) + 10^{-m}$. If δ_k is less than 0 or equal to -0, let $f_k(m) = q$ where q is the first element of $\mathbb{Q}$ that is not in $[f_k(m)] \cup [g_k(m)]$ that satisfies $q < \delta_k(m) - 10^{-m}$. Let $g_k(m) = q$ where q is the first element of $\mathbb{Q}$ that is not in $[f_k(m+1)] \cup [g_k(m)]$ that satisfies $q > \delta_k(m)$. RCA_0 suffices to prove the existence of the sequences $\langle f_k \rangle_{k\in\mathbb{N}}$ and $\langle g_k \rangle_{k\in\mathbb{N}}$. By Lemma 5, WKL_0 proves the existence of a sequence $\langle \lambda_i \rangle_{i\in\mathbb{N}}$ such that for each k, λ_k contains the range of f_k and is disjoint from the range of g_k.

We will show that λ_k is a Dedekind cut and $\delta_k = \lambda_k$. Suppose that $\delta_k(0)$ is greater than 0 or equal to 0. If $q \in \mathbb{Q}$ and $q < \delta_k$, then for some m, $q < \delta_k(m)$. Since δ_k is an increasing function, for some $n > m$, $f_k(n) = q$, so $q \in \lambda_k$. If $q \in \mathbb{Q}$ and $q > \delta_k$, then for some m, $q > \delta_k(m) + 10^{-m}$. Since $\delta_k(j) + 10^{-j}$ is a decreasing function in j, for some $n > m$, $g_k(n) = q$ and so $q \notin \lambda_k$. Thus λ_k is a Dedekind cut equal to δ_k. Since λ_k is a separating set, if δ_k is a rational then δ_k may or may not be an element of λ_k. Thus, we have not shown that λ_k is an open cut. The proof that λ_k is the desired Dedekind cut when $\delta_k(0)$ is negative or -0 is similar.

It remains to show that (2) implies WKL_0. As in the preceding theorem, we will use (2) to separate the ranges of disjoint functions. Let f and g be injections such that for all m and n, $f(m) \neq g(n)$. For the following, let $[d]^n$

denote a string of n copies of the digit d. Define a sequence $\langle\delta_i\rangle_{i\in\mathbb{N}}$ of decimal expansions by setting

$$\delta_k(n) = \begin{cases} .[1]^t[2]^{n-t} & \text{if } t < n \wedge g(t) = k, \\ .[1]^t[0]^{n-t} & \text{if } t < n \wedge f(t) = k, \\ .[1]^n & \text{otherwise.} \end{cases}$$

Let $\langle\lambda_i\rangle_{i\in\mathbb{N}}$ be a sequence of Dedekind cuts such that for each $i \in \mathbb{N}$, $\delta_i = \lambda_i$. Then the set $S = \{i \mid \frac{1}{9} \in \lambda_i\}$ contains every element of the range of f and no elements of the range of g. ■

THEOREM 8 (RCA_0). *The following are equivalent*:

(1) ACA_0.
(2) *If $\langle\lambda_i\rangle_{i\in\mathbb{N}}$ is a sequence of Dedekind cuts, then there is a sequence $\langle\sigma_i\rangle_{i\in\mathbb{N}}$ of open cuts such that for each $i \in \mathbb{N}$, $\lambda_i = \sigma_i$.*

Proof. First, assume (1) and let $\langle\lambda_i\rangle_{i\in\mathbb{N}}$ be a sequence of Dedekind cuts. For each $i \in \mathbb{N}$, if $\exists q \in \lambda_i\ \forall q' \in \lambda_i\ (q' \le q)$, then let $\sigma_i = \lambda_i - \{q\}$. Otherwise, let $\sigma_i = \lambda_i$. ACA_0 proves that the sequence $\langle\sigma_i\rangle_{i\in\mathbb{N}}$ exists, and the omission of maxima guarantees that each σ_i is an open cut.

To prove the converse, we will use (2) to find the range of an injection [9, Lemma III.1.3]. Let $f : \mathbb{N}^+ \to \mathbb{N}$ be an injection. Define the sequence $\langle\lambda_i\rangle_{i\in\mathbb{N}}$ of Dedekind cuts by putting $q \in \mathbb{Q}$ in λ_i if and only if $q \le 0$ or

$$q > 0 \wedge (\exists t < 1/q)\big(f(t) = i\big).$$

RCA_0 suffices to prove that the sequence $\langle\lambda\rangle_{i\in\mathbb{N}}$ exists and that each λ_i is a Dedekind cut. (Indeed, each λ_i is a closed lower Dedekind cut for some rational.) By (2), there is a sequence $\langle\sigma_i\rangle_{i\in\mathbb{N}}$ of open cuts satisfying $\sigma_i = \lambda_i$ for each $i \in \mathbb{N}$. Since $\exists t\ (f(t) = k)$ if and only if $0 \in \sigma_k$, recursive comprehension proves that the range of f exists. ■

The remaining analysis of the conversions of the representations of sequences of reals consists of two easy corollaries to the preceding theorems.

COROLLARY 9 (RCA_0). *The following are equivalent*:

(1) WKL_0.
(2) *If $\langle\varrho_i\rangle_{i\in\mathbb{N}}$ is a sequence of rapidly converging Cauchy sequences, then there is a sequence $\langle\lambda_i\rangle_{i\in\mathbb{N}}$ of Dedekind cuts such that for all $i \in \mathbb{N}$, $\varrho_i = \lambda_i$.*

Proof. To prove that (1) implies (2), concatenate Theorems 6 and 7. Since every decimal expansion is a rapidly converging Cauchy sequence, (2) above implies (2) of Theorem 7, so WKL_0 follows by Theorem 7. ■

COROLLARY 10 (RCA_0). *The following are equivalent*:

(1) ACA_0.

(2) *If $\langle \delta_i \rangle_{i\in\mathbb{N}}$ is a sequence of decimal expansions, then there is a sequence $\langle \sigma_i \rangle_{i\in\mathbb{N}}$ of open cuts such that for all $i \in \mathbb{N}$, $\delta_i = \sigma_i$.*

(3) *If $\langle \varrho_i \rangle_{i\in\mathbb{N}}$ is a sequence of rapidly converging Cauchy sequences, then there is a sequence $\langle \sigma_i \rangle_{i\in\mathbb{N}}$ of open cuts such that for all $i \in \mathbb{N}$, $\varrho_i = \sigma_i$.*

Proof. Since ACA_0 implies WKL_0, the proof of (3) from (1) follows from a concatenation of Theorems 6, 7, and 8. Since every decimal expansion is a rapidly converging Cauchy sequence, (2) is a special case of (3). It remains to show that (2) implies (1). By Theorem 1, RCA_0 proves that every sequence of Dedekind cuts can be converted to a sequence of decimal expansions, so (2) above implies (2) of Theorem 8, and ACA_0 follows by Theorem 8. (Theorem 6 of [4] includes a computable analysis counterexample corresponding to a direct proof of (1) from (2).) ■

We summarize the results of the preceding two sections in the following table. Each table entry corresponds to a conversion from a sequence of the row type to a sequence of the column type. Row and column labels are:

- ϱ: rapidly converging Cauchy sequence,
- δ: decimal expansion,
- λ: Dedekind cut,
- σ: open cut.

The conversion results are either provable in RCA_0 (as shown in §2), or equivalent to the designated subsystem (as shown in this section).

from \ to	ϱ	δ	λ	σ
ϱ	RCA_0	WKL_0	WKL_0	ACA_0
δ	RCA_0	RCA_0	WKL_0	ACA_0
λ	RCA_0	RCA_0	RCA_0	ACA_0
σ	RCA_0	RCA_0	RCA_0	RCA_0

4. Related results. As noted in the reversal of Theorem 8, conversions from Dedekind cuts to open cuts require ACA_0, even for sequences consisting only of rationals. On the other hand, conversions of purely irrational sequences can be carried out in RCA_0, as shown by the following theorem and corollary.

THEOREM 11 (RCA_0). *If $\langle \varrho_i \rangle_{i\in\mathbb{N}}$ is a sequence of rapidly converging Cauchy sequences each of which converges to an irrational number, then there is a sequence $\langle \sigma_i \rangle_{i\in\mathbb{N}}$ of open cuts such that for all $i \in \mathbb{N}$, $\sigma_i = \varrho_i$.*

Proof. Given the sequence $\langle \varrho_i \rangle_{i\in\mathbb{N}}$, determine if $q \in \mathbb{Q}$ is in σ_k as follows. Since ϱ_k is irrational, $\varrho_k \neq q$. Find n so large that $\varrho_k(n) - 2^{-n} > q$ or $\varrho_k(n) + 2^{-n} < q$. If the first inequality holds, include q in σ_k. If the second holds then exclude q from σ_k. RCA_0 suffices to prove that $\langle \sigma_i \rangle_{i \in \mathbb{N}}$ exists, each σ_i is an open cut, and for each i, $\sigma_i = \varrho_i$. ■

Corollary 12 (RCA_0). *Any sequence of irrationals in any of the four representations can be converted to a sequence in any other representation.*

Proof. Immediate from Corollary 2 and Theorem 11. ■

In general, separating rationals and irrationals requires ACA_0 as shown by the following theorem and corollary.

Theorem 13 (RCA_0). *The following are equivalent*:

(1) ACA_0.
(2) *If* $\langle \sigma_i \rangle_{i \in \mathbb{N}}$ *is a sequence of open cuts then the set* $\{i \in \mathbb{N} \mid \sigma_i \in \mathbb{Q}\}$ *exists.*

Proof. First, assume (1) and suppose $\langle \sigma_i \rangle_{i \in \mathbb{N}}$ is a sequence of open cuts. Note that $\sigma_i \in \mathbb{Q}$ if and only if

$$\exists q \in \mathbb{Q}\ \forall q' \in \mathbb{Q}\ (q' \notin \sigma_i \to q \le q').$$

Since each rational can be encoded by a natural number, this formula is arithmetical. Thus, the desired set exists by arithmetical comprehension.

To prove that (2) implies (1), assume RCA_0 and let f be an injection. Include q in σ_i if and only if

- $\exists k\ (q < -2^{-k} \wedge \forall t \le k\ (f(t) \ne i))$, or
- $\exists t\ (f(t) = i \wedge q < -2^{-t}/\pi)$.

RCA_0 suffices to prove that $\langle \sigma_i \rangle_{i \in \mathbb{N}}$ exists, that each σ_i is an open cut, and that $\sigma_i = 0$ if $i \notin \mathrm{Range}(f)$ and σ_i is irrational otherwise. The complement of $\{i \in \mathbb{N} \mid \sigma_i \in \mathbb{Q}\}$ is the range of f, so an application of [9, Lemma III.1.3] yields ACA_0. ■

Corollary 14 (RCA_0). *For any of the four representations of reals, the following are equivalent.*

(1) ACA_0.
(2) *If* $\langle \tau_i \rangle_{i \in \mathbb{N}}$ *is a sequence of reals in the specified representation, then the set* $\{i \in \mathbb{N} \mid \tau_i \in \mathbb{Q}\}$ *exists.*

Proof. To prove that (1) implies (2), assume ACA_0 and let $\langle \tau_i \rangle_{i \in \mathbb{N}}$ be a sequence of reals. Apply results from Section 3 to convert $\langle \tau_i \rangle_{i \in \mathbb{N}}$ to open cuts. An application of Theorem 13 yields the desired set.

To prove the converse, assume RCA_0 and suppose (2) holds. By Corollary 2, RCA_0 proves that (2) above implies (2) of Theorem 13. ACA_0 follows from Theorem 13. ■

In Theorems 3 and 5 of [4], Mostowski analyzed change of basis for sequences of decimal expansions in a computable analysis setting. Theorems 16 and 17 give the reverse mathematical analogs of his results. The following terminology is useful in the proofs. A *base b expansion* is defined in the same manner as a decimal expansion, using b in place of 10 and integers less than b as digits. A base b expansion is *terminating* if there is some point after which every digit is zero or every digit is $b-1$. By the definition of equality between rapidly converging Cauchy sequences, this means that a terminating base b expansion is always equal to (but not necessarily the same as) an expansion ending in zeros. The next lemma shows that termination may or may not be conserved under change of basis. For natural numbers a and b, we will use the notation $a \mid b$ to denote "a divides b" and $a \nmid b$ to denote "a does not divide b."

Lemma 15 (RCA_0). *For all b and c, there is an n such that $c \mid b^n$ if and only if every real with a terminating base c expansion has a terminating base b expansion. In particular, if for all n we have $c \nmid b^n$, then the base b expansion of $1/c$ is nonterminating.*

Proof. Suppose that for some t and n, $tc = b^n$. Let σ be a terminating base c expansion. We may assume that σ terminates in zeros, so for some j,

$$\sigma = \sigma(0) + \operatorname{sign}(\sigma(0)) \sum_{i=1}^{j} \frac{\sigma_i}{c^i}$$

where $0 \leq \sigma_i \leq c-1$ for each $i \leq j$. Since

$$\frac{\sigma_i}{c^i} = \frac{t^i \sigma_i}{t^i c^i} = \frac{t^i \sigma_i}{b^{ni}},$$

we have

$$\sigma = \sigma(0) + \operatorname{sign}(\sigma(0)) \sum_{i=1}^{j} \frac{t^i \sigma_i}{b^{ni}},$$

so σ can be expressed as a terminating base b expansion.

To prove the converse, suppose that for every value of n, $c \nmid b^n$. Suppose by way of contradiction that $1/c$ has a terminating base b expansion. Then we may write

$$\frac{1}{c} = \sum_{i=1}^{j} \frac{\beta_i}{b^i} = \frac{t}{b^j}$$

for some $t \in \mathbb{N}$. Thus $ct = b^j$, contradicting our divisibility assumption. Thus, $1/c$ has no terminating base b expansion. ∎

Theorem 16 (RCA_0). *If $c \mid b^n$ for some n, then for every sequence $\langle \beta_i \rangle_{i \in \mathbb{N}}$ of base b expansions there is a sequence $\langle \gamma_i \rangle_{i \in \mathbb{N}}$ of base c expansions such that for all $i \in \mathbb{N}$, $\beta_i = \gamma_i$.*

Proof. This argument is essentially a formalization of the proof of Theorem 3 of [4]. Suppose $c \mid b^n$. By Lemma 15, whenever $\gamma = \beta$ where γ is a base c expansion and β is a base b expansion, if γ terminates then so does β.

Consider a single base b expansion; call it β. As usual, let $\beta(k)$ denote the result of truncating β after the first k digits to the right of the decimal point. Let $(\beta(k))_c$ denote the base c expansion of $\beta(k)$. Suppose by way of contradiction that there is a j such that for all k, $(\beta(k))_c$ and $(\beta(k) + b^{-k})_c$ disagree somewhere in the first j digits. In this case there are two base c expansions γ_0 and γ_1 such that $\beta = \gamma_0 = \gamma_1$ and γ_0 and γ_1 disagree somewhere in the first j digits. This implies that γ_0 and γ_1 must be terminating. Let γ denote the element of $\{\gamma_0, \gamma_1\}$ that terminates in zeros. Since $\beta = \gamma$ and γ terminates, β must terminate also, and we may assume that β ends in zeros. Choose m so large that $m > j$ and for all $k > m$, $\beta(k) = \beta(m)$ and $\gamma(k) = \gamma(m)$. Choose $p > m$ such that for all $k > p$, $b^{-k} < c^{-m-1}$. Thus when $k > p$, $(\beta(k))_c = \gamma(m)$, $(\beta(k) + b^{-k})_c < \gamma(m) + c^{-m-1}$, and $(\beta(k))_c$ must agree with $(\beta(k) + b^{-k})_c$ on the first j digits, contradicting our assumption. Thus for every j there is a k such that $(\beta(k))_c$ and $(\beta(k) + b^{-k})_c$ agree on the first j digits. Furthermore, for any m greater than such k, $(\beta(m))_c$ and $(\beta(k))_c$ agree on the first j digits.

Now we can present the algorithm for converting $\langle \beta_i \rangle_{i \in \mathbb{N}}$ to $\langle \gamma_i \rangle_{i \in \mathbb{N}}$. For any i and j, find a k so large that $(\beta_i(k))_c$ and $(\beta_i(k) + b^{-k})_c$ agree on the first j digits. Let $\gamma_i(j)$ consist of those j digits. RCA_0 suffices to prove that $\langle \gamma_i \rangle_{i \in \mathbb{N}}$ exists and is a sequence of base c expansions, and that $\beta_i = \gamma_i$ for all $i \in \mathbb{N}$. ∎

THEOREM 17 (RCA_0). *If for all n we have $c \nmid b^n$, then the following are equivalent*:

(1) WKL_0.
(2) *For every sequence $\langle \beta_i \rangle_{i \in \mathbb{N}}$ of base b expansions there is a sequence $\langle \gamma_i \rangle_{i \in \mathbb{N}}$ of base c expansions such that for all $i \in \mathbb{N}$, $\beta_i = \gamma_i$.*

Proof. Suppose that for all n, c does not divide b^n. Since base b expansions are rapidly converging Cauchy sequences, and 10 can be replaced by c in the proof of Theorem 6, the statement that (1) implies (2) can be proved by adapting the proof of Theorem 6.

The proof of the converse is essentially a formalization of the construction in Theorem 5 of [4]. By Lemma 15, let β be the nonterminating base b expansion of $1/c$. Since β does not terminate, after any given point in the expansion a digit greater than 0 must occur and a digit less than $b - 1$ must occur. For any k, let $n_k > k$ be the first location to the right of the kth decimal place in β that has a value less than $b-1$ and define

$$\beta_k^{\uparrow}(j) = \begin{cases} \beta(j) & \text{for } j < n_k, \\ b-1 & \text{for } j = n_k, \\ 0 & \text{for } j > n_k. \end{cases}$$

Note that $\beta_k^{\uparrow} > 1/c$. Similarly, when $n_k > k$ is the first location to the right of the kth decimal place in β that has a value greater than 0, define

$$\beta_k^{\downarrow}(j) = \begin{cases} \beta(j) & \text{for } j < n_k, \\ 0 & \text{for } j \geq n_k. \end{cases}$$

Note that $\beta_k^{\downarrow} < 1/c$. Let f and g be functions with disjoint ranges, and define $\langle \beta_i \rangle_{i\in\mathbb{N}}$ by

$$\beta_i(j) = \begin{cases} \beta_t^{\uparrow}(j) & \text{if } t \leq j \text{ and } f(t) = i, \\ \beta_t^{\downarrow}(j) & \text{if } t \leq j \text{ and } g(t) = i, \\ \beta(j) & \text{otherwise.} \end{cases}$$

Apply (2) to find a sequence $\langle \gamma_i \rangle_{i\in\mathbb{N}}$ of base c expansions such that $\beta_i = \gamma_i$ for all $i \in \mathbb{N}$. The set $S = \{i \mid \gamma_i(1) \geq 1/c\}$ is a separating set for the ranges of f and g. ∎

We close by observing that many of the reversals of the results on sequences can be converted to arguments in constructive analysis for negative statements about single reals. As an example, consider the proof of (2)⇒(1) in Theorem 17. To increase the concreteness of the discussion, suppose $b = 2$ and $c = 10$. Thus β is the base 2 expansion of $1/10$, that is, $\beta = .000\overline{1100}$ in standard base 2 notation. Let P denote a formal theory that is assumed to be consistent and that has proofs that can be Gödel numbered. (A reasonable choice would be Peano arithmetic.) Let S denote a statement whose status is completely open. That is, S might or might not be provable in P and $\neg S$ might or might not be provable in P. (At the moment, S could be the Goldbach conjecture.) Define β_0 by setting

$$\beta_0(j) = \begin{cases} \beta_t^{\uparrow}(j) & \text{if some } t \leq j \text{ encodes a proof of } S \text{ in } P, \\ \beta_t^{\downarrow}(j) & \text{if some } t \leq j \text{ encodes a proof of } \neg S \text{ in } P, \\ \beta(j) & \text{otherwise.} \end{cases}$$

β_0 is a constructive base 2 expansion. Note that if $\gamma = \beta_0$ and γ is a base 10 expansion, then $\gamma(1) \geq 1/10$ implies there is no proof of the negation of S in P and $\gamma(1) < 1/10$ implies there is no proof of S in P. Since we lack sufficient information about the provability of S to determine the value of $\gamma(1)$, there is no constructive base 10 expansion that is equal to β_0. A constructivist might summarize by saying that some base 2 expansions cannot be converted to base 10 expansions. For more on constructive representations of reals, see [10].

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