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There are no Phantom Pairs of Mappings to 1-Dimensional CW-Complexes

by

Sibe MARDEŠIĆ

Presented by Czesław BESSAGA

Summary. Two mappings from a CW-complex to a 1-dimensional CW-complex are homotopic if and only if their restrictions to finite subcomplexes are homotopic.

1. Introduction. Let $f, g: P \to P'$ be two mappings between CW-complexes. Clearly, if f and g are homotopic, $f \simeq g$, then for every finite subcomplex $Q \subseteq P$, the restrictions $f|Q, g|Q$ are also homotopic. In this paper we will prove that the converse implication holds provided P' has dimension ≤ 1, i.e., we will prove the following theorem.

THEOREM 1. *Let P, P' be CW-complexes and let $f, g: P \to P'$ be mappings such that, for every finite subcomplex $Q \subseteq P$, the restrictions $f|Q, g|Q$ are homotopic. If $\dim P' \leq 1$, then $f \simeq g$.*

In homotopy theory a mapping $f: P \to Y$ from a CW-complex P to a topological space Y is called an *essential phantom mapping of the second kind* provided f is essential, i.e., it is not homotopic to a constant mapping, but its restriction to any finite subcomplex Q of P is homotopic to a constant mapping [5]. A generalization of this notion is the notion of an *essential phantom pair of mappings of the second kind*. This is a pair of nonhomotopic mappings $f, g: P \to Y$ whose restrictions $f|Q, g|Q$ to every finite subcomplex Q of P are homotopic. Consequently, Theorem 1 can be restated as follows.

THEOREM 1$'$. *There are no essential phantom pairs of mappings of the second kind from a CW-complex P to a CW-complex P' with $\dim P' \leq 1$.*

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REMARK 1. In Theorem 1 the assumption $f|Q \simeq g|Q$ for finite subcomplexes $Q \subseteq P$ cannot be replaced by the assumption that the restrictions of f and g to 1-cells of P be homotopic (see Example 2 in Section 4).

2. Equivalent forms of Theorem 1. Theorem 1 is equivalent to the following theorem (the author owes this remark to J. Dydak).

THEOREM 2. *Let P, P' be connected CW-complexes and let $f, g\colon P \to P'$ be mappings such that, for every finite subcomplex $Q \subseteq P$, the restrictions $f|Q, g|Q$ are homotopic. If $\dim P = \dim P' = 1$, then $f \simeq g$.*

Obviously, Theorem 1 implies Theorem 2. To prove the converse, consider $f, g\colon P \to P'$ such that $f|Q \simeq g|Q$ for all finite subcomplexes Q of P. Since every point $u \in P$ belongs to a finite subcomplex Q of P, the points $f(u), g(u)$ belong to the same component of P'. Hence, f and g map a component of P to the same component of P'. Therefore, it suffices to prove Theorem 1 under the additional assumption that P and P' are connected.

If $\dim P' = 0$, then P' is a point $*'$, and thus $f = g$. Therefore, it suffices to consider the case when $\dim P' = 1$. If $\dim P = 0$, then P is a point $*$, and thus $f = f|* \simeq g|* = g$. Therefore, we can assume that $\dim P \geq 1$. Consider the 1-skeleton P^1 of P. Clearly, the restrictions $f|P^1, g|P^1\colon P^1 \to P'$ satisfy the assumptions of Theorem 2, and therefore there exists a homotopy $h^1\colon P^1 \times I \to P'$ which connects $f|P^1$ and $g|P^1$.

It is well known that every connected CW-complex of dimension 1 is an Eilenberg–Mac Lane complex of type $K(G, 1)$ (see, e.g., [1, Example 1B.1]). Therefore, the homotopy groups $\pi_n(P')$ are zero for $n \geq 2$. Consequently, there are no obstructions to extending a homotopy $h^{n-1}\colon P^{n-1} \times I \to P'$ of the $(n-1)$-skeleton P^{n-1} of P to a homotopy $h^n\colon P^n \times I \to P'$ of its n-skeleton P^n (use, e.g., Lemma 4.7 of [1]). Proceeding in this way one obtains a homotopy connecting f to g.

We will now prove that Theorem 2 is equivalent to the following theorem.

THEOREM 3. *Let P, P' be connected 1-dimensional CW-complexes having only one 0-cell $*$ and $*'$, respectively and let $f, g\colon P \to P'$. If for every finite subcomplex $Q \subseteq P$ the restrictions $f|Q, g|Q$ are homotopic, then $f \simeq g$.*

Obviously, Theorem 2 implies Theorem 3. To prove the converse, consider $f, g\colon P \to P'$ such that $f|Q \simeq g|Q$ for all finite subcomplexes Q of P. It is well known that every connected 1-dimensional CW-complex contains a maximal tree (see, e.g., [1, Proposition 1A.1]). Let T and T' be maximal trees in P and P', respectively. Since the pair (P, T) has the homotopy extension property and T is contractible, the quotient mapping $q\colon P \to P/T$ is a homotopy equivalence, and thus admits a homotopy inverse $r\colon P/T \to P$ (see, e.g., [1, Proposition 0.17]). Analogously, the quotient mapping $q'\colon P' \to P'/T'$ admits a homotopy inverse $r'\colon P'/T' \to P'$. If $P' = T'$, then P' is contractible, and thus $f \simeq g$. Otherwise, $R' = P'/T'$ is a connected 1-dimensional CW-complex having the point $*' = T'$ as its only 0-cell. Analogously, if $P = T$, then P contracts to a point $u_0 \in P$, which is a 0-cell of P. Consequently, f is homotopic to the constant $f(u_0)$ and g is homotopic to the constant $g(u_0)$. But $f|u_0 \simeq g|u_0$, and thus $f \simeq g$. Otherwise, $R = P/T$ is a connected 1-dimensional CW-complex having the point $* = T$ as its only 0-cell. To complete the proof it suffices to consider the case when $P = T$ and $P' = T'$. Let $f', g' \colon R \to R'$ be defined by $f' = q'fr$ and $g' = q'gr$. If $S \subseteq R$ is a finite subcomplex of R, then S and $r(S)$ are compact. Therefore, $r(S)$ is contained in a finite subcomplex Q of P. By assumption, $f|Q \simeq g|Q$ and thus also $fr|S \simeq gr|S$. It follows that $f'|S = q'fr|S \simeq q'gr|S = g'|S$, i.e., the restrictions of f' and g' to all finite subcomplexes of R are homotopic. Now Theorem 3 shows that $f' \simeq g'$, i.e., $q'fr \simeq q'gr$. Since q' and r are homotopy equivalences, it follows that also $f \simeq g$.

3. A theorem on free groups. To prove Theorem 3 we need the following theorem on free groups.

THEOREM 4. *Let F be a free group and let $(\alpha_i), (\beta_i)$, $i \in M$, be two collections of elements from F. If for every finite subset $L \subseteq M$, there exists an element $\gamma_L \in F$ such that $\alpha_i = \gamma_L \beta_i \gamma_L^{-1}$ for every $i \in L$, then there exists an element $\gamma \in F$ such that $\alpha_i = \gamma \beta_i \gamma^{-1}$ for every $i \in M$.*

In the proof of Theorem 4 we will use some well-known facts concerning free groups. They are stated in the following proposition.

PROPOSITION 1. *In a free group F the following statements hold*:

(i) *If $a \in F$, $a \neq 1$, $n \in \mathbb{Z}$ and $a^n = 1$, then $n = 0$.*
(ii) *If $a, b \in F$ and $m, n \in \mathbb{Z}$ are integers different from 0 such that a^m and b^n commute, then there exist an element $c \in F$ and integers $r, s \in \mathbb{Z}$ such that $a = c^r$, $b = c^s$.*
(iii) *If $a, b \in F$, $a \neq 1$, $b \neq 1$ and $n \in \mathbb{Z}$, $n \neq 0$, then $a^n = b^n$ implies $a = b$.*
(iv) *If $a, b, c \in F$ are different from 1, if a and b commute and if b and c commute, then also a and c commute.*

For a proof of (i) see [3, Corollary 1.2.2] or [2, Proposition 2.16]. For (ii) see [3, 1.4, Problems 4 and 6] or [2, Proposition 2.17]. For (iii) note that a^n and b^n commute, and therefore, by (ii), there exist $c \in F$ and $r, s \in \mathbb{Z}$ such that $a = c^r$ and $b = c^s$. Consequently, $c^{nr} = c^{ns}$, and thus $c^{nr-ns} = 1$. Since $c \neq 1$, (i) implies that $nr - ns = 0$, and thus $r = s$, which yields the desired conclusion that $a = b$. For (iv) see [2, Proposition 2.18].

Proof of Theorem 4. For an arbitrary $i \in M$, consider the singleton $\{i\}$ and put $\gamma_i = \gamma_{\{i\}}$. Note that $\alpha_i = \gamma_i \beta_i \gamma_i^{-1}$. If for a given $i \in M$, $\beta_i = 1$, then

$\alpha_i = 1$, and thus $\alpha_i = \gamma\beta_i\gamma^{-1}$ for any $\gamma \in F$. Therefore, there is no loss of generality in assuming that $\beta_i \neq 1$ for all $i \in M$. Denote by B the subgroup of F generated by all β_i, $i \in M$. Being a subgroup of a free group, B is also a free group (see [3, Corollary 2.9] or [2, Proposition 3.3]). We distinguish two cases: I, when B is commutative, and II, when B is not commutative.

In case I, we fix an arbitrary $k \in M$. We will show that $\gamma = \gamma_k$ is as required, i.e., $\alpha_i = \gamma_k\beta_i\gamma_k^{-1}$ for all $i \in M$. Indeed, since the only commutative group which is free is the free cyclic group (see [3, 2.4, Problem 2] or [2, Proposition 3.1]), B is cyclic. Let $\{\beta\}$ be a basis of B. Then every element β_i, $i \in M$, is of the form $\beta_i = \beta^{r_i}$, where $r_i \in \mathbb{Z}$. By the assumptions of Theorem 4, for an arbitrary $i \in M$ and the finite set $L = \{i, k\} \subseteq M$, there exists $\gamma_{ik} = \gamma_{\{i,k\}} \in F$ such that $\alpha_i = \gamma_{ik}\beta_i\gamma_{ik}^{-1}$ and $\alpha_k = \gamma_{ik}\beta_k\gamma_{ik}^{-1}$. Since $\beta_i = \beta^{r_i}$, we see that

$$\alpha_i = \gamma_i\beta^{r_i}\gamma_i^{-1} = (\gamma_i\beta\gamma_i^{-1})^{r_i} \quad \text{and} \quad \alpha_i = \gamma_{ik}\beta^{r_i}\gamma_{ik}^{-1} = (\gamma_{ik}\beta\gamma_{ik}^{-1})^{r_i}.$$

Therefore, $(\gamma_i\beta\gamma_i^{-1})^{r_i} = (\gamma_{ik}\beta\gamma_{ik}^{-1})^{r_i}$. Note that $r_i \neq 0$, because $r_i = 0$ would imply $\beta_i = 1$. Moreover, $\gamma_i\beta\gamma_i^{-1} \neq 1$, because $\gamma_i\beta\gamma_i^{-1} = 1$ would imply $\beta = 1$ hence, also $\beta_i = 1$. Similarly, $\gamma_{ik}\beta\gamma_{ik}^{-1} \neq 1$. By Proposition 1(iii), one concludes that $\gamma_i\beta\gamma_i^{-1} = \gamma_{ik}\beta\gamma_{ik}^{-1}$. An analogous argument shows that $\gamma_k\beta\gamma_k^{-1} = \gamma_{ik}\beta\gamma_{ik}^{-1}$. Consequently, $\gamma_i\beta\gamma_i^{-1} = \gamma_k\beta\gamma_k^{-1}$, and thus $(\gamma_i\beta\gamma_i^{-1})^{r_i} = (\gamma_k\beta\gamma_k^{-1})^{r_i}$. Since $\alpha_i = (\gamma_i\beta\gamma_i^{-1})^{r_i}$ and $\gamma_k\beta_i\gamma_k^{-1} = (\gamma_k\beta\gamma_k^{-1})^{r_i}$, we obtain the desired relation $\alpha_i = \gamma_k\beta_i\gamma_k^{-1}$ for all $i \in M$.

In case II, B is not commutative, so there exist $k, l \in M$ such that β_k and β_l do not commute. Consider the fixed subset $\{k, l\} \subseteq M$ and put $\gamma_{kl} = \gamma_{\{k,l\}}$. For an arbitrary $i \in M$, consider the subset $\{i, k, l\} \subseteq M$ and put $\gamma_{ikl} = \gamma_{\{i,k,l\}}$. Let us show that

$$\gamma_{ikl} = \gamma_{kl} \quad \text{for all } i \in M.$$

Indeed, since $k \in \{k, l\} \cap \{i, k, l\}$, we see that $\alpha_k = \gamma_{kl}\beta_k\gamma_{kl}^{-1}$ and $\alpha_k = \gamma_{ikl}\beta_k\gamma_{ikl}^{-1}$. Consequently,

$$\gamma_{kl}\beta_k\gamma_{kl}^{-1} = \gamma_{ikl}\beta_k\gamma_{ikl}^{-1}.$$

This shows that $\gamma_{kl}^{-1}\gamma_{ikl}$ commutes with β_k. Analogously, $\gamma_{kl}^{-1}\gamma_{ikl}$ commutes with β_l. Recall that $\beta_k \neq 1$ and $\beta_l \neq 1$. Therefore, if one would also have $\gamma_{kl}^{-1}\gamma_{ikl} \neq 1$, Proposition 1(iv) would imply that β_k commutes with β_l, which is not the case. We have thus proved that $\gamma_{kl}^{-1}\gamma_{ikl} = 1$, i.e., $\gamma_{kl} = \gamma_{ikl}$, as desired.

Since $\alpha_i = \gamma_{ikl}\beta_i\gamma_{ikl}^{-1}$ for all $i \in M$, the equality $\gamma_{ikl} = \gamma_{kl}$ implies that $\alpha_i = \gamma_{kl}\beta_i\gamma_{kl}^{-1}$ for all $i \in M$. Consequently, $\gamma = \gamma_{kl}$ is as required. ∎

EXAMPLE 1. *Let F be the free group of rank 2 with basis $\{\beta_1, \beta_2\}$. Let $\alpha_1 = \beta_2^{-1}\beta_1\beta_2$ and $\alpha_2 = \beta_1^{-1}\beta_2\beta_1$. Then there is no $\gamma \in F$ such that $\alpha_i = \gamma\beta_i\gamma^{-1}$ for $i = 1, 2$.*

To verify the assertion, assume that $\gamma \in F$ is such that $\alpha_i = \gamma\beta_i\gamma^{-1}$ for $i = 1, 2$, i.e.,

$$\beta_2^{-1}\beta_1\beta_2 = \gamma\beta_1\gamma^{-1} \quad \text{and} \quad \beta_1^{-1}\beta_2\beta_1 = \gamma\beta_2\gamma^{-1}.$$

The first of these relations shows that $\beta_2\gamma$ commutes with β_1. By Proposition 1(ii), there exist $\xi \in F$ and $r, s \in \mathbb{Z}$ such that $\beta_1 = \xi^r$ and $\beta_2\gamma = \xi^s$. There is no loss of generality in assuming that $r \geq 0$ (if not, replace ξ by ξ^{-1}). Since β_1 belongs to a basis of F, one cannot have $r \geq 2$, and thus $r = 1$. Consequently, $\beta_2\gamma = (\beta_1)^s$. Analogously, there exists an integer s' such that $\beta_1\gamma = (\beta_2)^{s'}$. It follows that

$$\beta_2^{-1}(\beta_1)^s = \gamma = \beta_1^{-1}(\beta_2)^{s'}.$$

However, this is impossible because $\beta_2^{-1}(\beta_1)^s$ and $\beta_1^{-1}(\beta_2)^{s'}$ are reduced words, beginning with β_2^{-1} and β_1^{-1}, respectively. Therefore, they cannot represent the same element of F.

4. Proof of Theorem 3. To prove Theorem 3, we will use Theorem 4 and some elementary facts concerning the homotopy of loops in a pointed space $(Y, *')$. In particular, denote by $(S^1, *)$ the standard 1-sphere $\{z \in \mathbb{C} : |z| = 1\}$ with the basepoint $* = 1$ and let $e\colon I \to S^1$ be the exponential mapping, $e(t) = e^{2\pi i t}$, $t \in I = [0, 1]$. By a loop a in Y, based at $*'$, we mean a mapping $a\colon S^1 \to Y$ such that $a(*) = *'$. Note that a determines the path $\widetilde{a} = ae\colon I \to Y$, which has the property that $\widetilde{a}(0) = \widetilde{a}(1) = *'$. Conversely, every path $\widetilde{a}\colon I \to S^1$ having the latter property determines a unique loop a such that $\widetilde{a} = ae$. The composition of two loops $a_1, a_2\colon S^1 \to Y$, based at $*'$, is the only loop a_1a_2 such that $(a_1a_2)e = \widetilde{a}_1\widetilde{a}_2$. We will say that the loops a, b, based at $*'$, are (*freely*) *homotopic* provided there exists a homotopy $H\colon S^1 \times I \to Y$ such that $H(u, 0) = a(u)$ and $H(u, 1) = b(u)$ for $u \in S^1$. Note that the formula $\widetilde{c}(t) = H(*, t)$ determines a path $\widetilde{c}\colon I \to Y$ such that $\widetilde{c}(0) = H(*, 0) = a(*) = *'$ and $\widetilde{c}(1) = H(*, 1) = b(*) = *'$. Therefore, $\widetilde{c}$ determines a loop $c\colon S^1 \to Y$ based at $*'$. We will say that H is a *c-homotopy* and the loops a and b are *c-homotopic*. Let $\alpha, \beta, \gamma \in \pi_1(Y, *')$ be the homotopy classes of the loops a, b, c. Then the following elementary lemma holds (see [4, Theorem II.8.2]).

LEMMA 1. *Let a, b, c be loops in a pointed space $(Y, *')$ and let α, β, γ be the corresponding classes in $\pi_1(Y, *')$. Then the following two conditions are equivalent*:

(i) *the loops a and b are c-homotopic*;
(ii) $\alpha = \gamma\beta\gamma^{-1}$.

Proof. If (i) holds, then there is a c-homotopy $H\colon S^1 \times I \to Y$ which connects a and b and $H(*, t) = \widetilde{c}(t)$ for $t \in I$. Therefore, $\widetilde{H}\colon I \times I \to Y$ given by $\widetilde{H} = H(e \times 1)$ is a homotopy which connects $\widetilde{a}$ to $\widetilde{b}$. Moreover, $\widetilde{H}(0, t) =$

$H(e(0),t) = H(*,t) = \widetilde{c}(t)$ and $\widetilde{H}(1,t) = H(e(1),t) = H(*,t) = \widetilde{c}(t)$. Clearly, $\widetilde{H}$ gives rise to a homotopy $\widetilde{G}\colon I \times I \to Y$ which connects the loops $\widetilde{a}$ and $\widetilde{c}\,\widetilde{b}\,\widetilde{c}^{-1}$ and is fixed at the two end-points 0, 1, i.e., it is a homotopy rel ∂I. Now $\widetilde{G}$ determines a homotopy $G\colon S^1 \times I \to Y$ such that $\widetilde{G} = G(e \times 1)$. Note that G connects the loops a and cbc^{-1} and is fixed at the basepoint $*$, i.e., it is a homotopy rel $*$. Indeed, if $s \in I$ and $u = e(s)$, then

$$G(u,0) = G(e(s),0) = \widetilde{G}(s,0) = \widetilde{a}(s) = ae(s) = a(u).$$

Similarly,

$$G(u,1) = G(e(s),1) = \widetilde{G}(s,1) = \widetilde{c}\,\widetilde{b}\,\widetilde{c}^{-1}(s) = cbc^{-1}(e(s)) = cbc^{-1}(u).$$

Moreover, $G(*,t) = G(e(0),t) = \widetilde{G}(0,t) = *'$. It follows that $\alpha = \gamma\beta\gamma^{-1}$, as required by (ii).

To prove (ii)⇒(i), it suffices to follow the steps of the above proof in the opposite order. ∎

Proof of Theorem 3. First note that every mapping $f\colon P \to P'$ is homotopic to a mapping $f'\colon P \to P'$ such that $f'(*) = *'$. Indeed, since P' is pathwise connected, there is a path $\omega\colon I \to P'$ such that $\omega(0) = f(*)$ and $\omega(1) = *'$. By the homotopy extension property for the pair $(P,*)$, there is a homotopy $H\colon P \times I \to P'$ such that $H(u,0) = f(u)$ for $u \in P$ and $H(*,s) = \omega(s)$ for $s \in I$. Define $f'\colon P \to P'$ by putting $f'(u) = H(u,1)$. Clearly, $f \simeq f'$ implies $f'|Q \simeq f|Q$ for every finite subcomplex $Q \subseteq P$. Moreover, $f'(*) = H(*,1) = \omega(1) = *'$. Repeating the argument for g, we see that there is no loss of generality in assuming that f and g preserve the basepoints, i.e., $f(*) = g(*) = *'$.

Being a connected 1-dimensional CW-complex with a single 0-cell $*$, P is the wedge $\bigvee_{i\in M} P_i$ of a collection of copies $(P_i,*_i)$ of $(S^1,*)$, $i \in M$. It is obtained from the coproduct $\coprod_{i\in M} P_i$ by identifying all the basepoints $*_i \in P_i$ to a single basepoint $*$ of P. Let $e_i\colon I \to P_i$, $i \in M$, denote the exponential mappings. It is well known that $\pi_1(P,*)$ is a free group, having as a basis the collection $[e_i]$, $i \in M$, of homotopy classes (rel ∂I) of the loops e_i (see, e.g., [1, Proposition 1A.2]). Analogous assertions hold for $P' = \bigvee_{i\in M'} P'_i$.

For every $i \in M$, consider the loops $a_i = f|P_i$ and $b_i = g|P_i$ in P', based at $*'$. By the assumptions of Theorem 3, $f|Q \simeq g|Q$ for every finite subcomplex Q of P. In particular, this holds for $Q = P_L = \bigvee_{i\in L} P_i$ for any finite subset $L \subseteq M$. Therefore, there exists a homotopy $H_L\colon P_L \times I \to P'$ which connects $f|Q$ and $g|Q$. Let $\widetilde{c}_L\colon I \to P'$ be given by $\widetilde{c}_L(t) = H(*,t)$. Clearly, $H_L|P_i \times I$ is a c_L-homotopy connecting a_i to b_i. Therefore, the implication (i)⇒(ii) in Lemma 1 shows that the homotopy classes $\alpha_i = [a_i], \beta_i = [b_i], \gamma_L = [c_L] \in \pi_1(P',*')$ satisfy $\alpha_i = \gamma_L\beta_i\gamma_L^{-1}$ for all $i \in L$. Now

Theorem 4 shows that there exists $\gamma \in \pi_1(P', *')$ such that $\alpha_i = \gamma\beta_i\gamma^{-1}$ for all $i \in M$. Let c be a representative of the class γ. Using the implication (ii)⇒(i) of Lemma 1, we conclude that, for every $i \in M$, there is a c-homotopy $H_i\colon P_i \times I \to P'$, which connects the loops a_i and b_i. Since $H_i(*, t) = \widetilde{c}(t)$ does not depend on i, the homotopies H_i, $i \in M$, extend to a well-defined homotopy $H\colon P \times I \to P'$ which connects f and g, because $H|P_i \times 0 = H_i|P_i \times 0 = a_i = f|P_i$ and $H|P_i \times 1 = H_i|P_i \times 1 = b_i = g|P_i$. ■

Example 2. *Let $P = P_1 \vee P_2$ be the wedge of two copies of S^1. Let $\alpha_i, \beta_i \in \pi_1(P, *)$, $i = 1, 2$, be as in Example* 1. *Let a_i, b_i be loops in P, based at $*$, such that $\alpha_i = [a_i]$, $\beta_i = [b_i]$, $i = 1, 2$, and let $f, g\colon P \to P$ be defined by $f|P_i = a_i$, $g|P_i = b_i$ for $i = 1, 2$. Then $f|P_i \simeq g|P_i$ for $i \in M$, but $f \not\simeq g$.*

Consider the loops $c_1 = b_2^{-1}$ and $c_2 = b_1^{-1}$ and the corresponding classes $\gamma_1 = [c_1]$ and $\gamma_2 = [c_2]$. Since $\alpha_1 = \gamma_1\beta_1\gamma_1^{-1}$ and $\alpha_2 = \gamma_2\beta_2\gamma_2^{-1}$, Lemma 1 shows that $f|P_i \simeq g|P_i$ for $i \in M$. Now assume that $f \simeq g$. More precisely, let g be c-homotopic to f, where $c\colon I \to P$ is a loop based at $*$. If γ denotes the class of c, Lemma 1 shows that $\alpha_i = \gamma\beta_i\gamma^{-1}$ for $i = 1, 2$, which contradicts the assertions of Example 1.

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Sibe Mardešić
Department of Mathematics
University of Zagreb
Bijenička cesta 30
10 002 Zagreb, P.O. Box 335, Croatia
E-mail: smardes@math.hr

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