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PROBABILITY THEORY AND STOCHASTIC PROCESSES

Sharp Norm Inequalities for Martingales and their Differential Subordinates

by

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Summary. Suppose $f = (f_n)$, $g = (g_n)$ are martingales with respect to the same filtration, satisfying

$$|f_n - f_{n-1}| \leq |g_n - g_{n-1}|, \quad n = 1, 2, \ldots,$$

with probability 1. Under some assumptions on f_0, g_0 and an additional condition that one of the processes is nonnegative, some sharp inequalities between the pth norms of f and g, $0 < p < \infty$, are established. As an application, related sharp inequalities for stochastic integrals and harmonic functions are obtained.

1. Introduction. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a discrete filtration $(\mathcal{F}_n)_{n\geq 0}$. Let $f = (f_n)$, $g = (g_n)$ be two adapted martingales taking values in a separable Hilbert space $\mathcal{H}$, with

$$f_n = \sum_{k=0}^{n} df_k, \quad g_n = \sum_{k=0}^{n} dg_k.$$

According to Burkholder, we say that f is *differentially subordinate* to g if

$$|df_n| \leq |dg_n|$$

almost surely for any nonnegative n.

As proved by Burkholder in [1], we have the following sharp estimate:

$$\|f_n\|_p \leq \alpha_p \|g_n\|_p, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where $\alpha_p = \max\{p, p/(p-1)\} - 1$ for $1 < p < \infty$. If $0 < p \leq 1$, the inequality fails to hold for any finite α_p.

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The above inequalities were the subject of many papers in which weaker or stronger assumptions on f, g were imposed and it was investigated how it affects the constant α_p (e.g. see [5] and the references therein). In particular, Burkholder [4] proved that if we assume additionally that $g_n \geq 0$ almost surely for all n, then (1.1) holds for $1 < p < \infty$ and the optimal constant equals

$$\alpha'_p = \begin{cases} 1/(p-1) & \text{if } p \in (1,2], \\ p^{1/p}[(p-1)/2]^{(p-1)/p} & \text{if } p \in (2,\infty). \end{cases}$$

We see that $\alpha_p = \alpha'_p$ for $1 < p \leq 2$ and $\alpha_p > \alpha'_p$ for $p > 2$.

We continue this line of research in two directions. The inequality (1.1) still fails to hold if $p \in (0,1)$ and $g \geq 0$, but it turns out that if the differential subordination is replaced by a slightly different condition, then we have the following fact.

THEOREM 1.1. *Suppose f is a martingale taking values in $\mathcal{H}$ and g is a nonnegative martingale. Assume that for some deterministic $\beta > 0$ we have*

$$\beta|f_0| \geq g_0 \quad \textit{and} \quad |df_n| \leq |dg_n|, \qquad n = 1,2,\ldots,$$

with probability 1. Then for $p \in (0,1)$,

$$\|f_n\|_p \geq C_{p,\beta}\|g_n\|_p, \qquad n = 0,1,2,\ldots, \tag{1.2}$$

where $C_{p,\beta} = 0$ if $\beta \geq 1$ and

$$C_{p,\beta} = \left[\left(\frac{p(1-\beta)}{2(1+\beta-p)}\right)^{1-p} \frac{2(1+\beta)(1-p)+p^2}{p(1+\beta-p)}\right]^{1/p}$$

if $\beta < 1$. The inequality is sharp if $2\beta > p$.

By sharpness we mean that for any $C > C_{p,\beta}$, there exists a pair (f,g) satisfying the assumptions of the theorem and an integer n for which we have $\|f_n\|_p < C\|g_n\|_p$.

The second result we obtain is the following.

THEOREM 1.2. *Suppose g is an $\mathcal{H}$-valued martingale and f is nonnegative and differentially subordinate to g. Then for $0 < p < \infty$,*

$$\|f_n\|_p \leq C_p\|g_n\|_p, \qquad n = 0,1,2,\ldots, \tag{1.3}$$

where

$$C_p = \begin{cases} \infty & \textit{if } p \in (0,1), \\ 1 & \textit{if } p = 1, \\ p^{-1/p}[2/(p-1)]^{(p-1)/p} & \textit{if } p \in (1,2), \\ p-1 & \textit{if } p \in [2,\infty). \end{cases}$$

The inequality is sharp.

Therefore, compared to the general case, the constant decreases for $p \in [1,2)$.

Let us comment upon the method of proof. In [1] (see also [2]) Burkholder proves the inequality (1.1) for general f, g constructing a quite complicated special function U_p satisfying some convex-type properties. It turns out that a certain integration trick is available, which enables one to build U_p from much simpler functions and to reduce the complexity of the proof significantly (cf. [5]). In [4], the proof of (1.1) for nonnegative g follows the same pattern and the special function U_p' is even more complicated than U_p. In this paper we discover an integral identity which expresses U_p' in terms of much simpler objects. Related identities yield special functions leading to the inequalities (1.2) and (1.3).

The paper is organized as follows. In the next section we introduce the simple special functions, study their properties and present the crucial integral identities. Section 3 contains the proof of Theorems 1.1 and 1.2. The last two sections are devoted to applications of these theorems to stochastic integrals and harmonic functions on Euclidean domains.

2. The special functions. For a fixed number $s>1$, set

$$D=\left\{(x,y)\in\mathbb{R}_+^2 : y\leq \min\left(x+1,\frac{s+1}{s-1}-x\right)\right\}.$$

Define $u_{1,s}:\mathcal{H}\times\mathbb{R}_+\to\mathbb{R}$, $u_{2,s}:\mathbb{R}_+\times\mathcal{H}\to\mathbb{R}$, $u_{\infty,s}:\mathcal{H}\times\mathbb{R}_+\to\mathbb{R}$ by

$$u_{1,s}(x,y)=\begin{cases}\frac{s-1}{s+1}\left(|x|^2-y^2\right)-\frac{2}{s+1}|x|+\frac{2s}{s+1}y & \text{if } (|x|,y)\in D,\\ 1 & \text{if } (|x|,y)\notin D,\end{cases}$$

$$u_{2,s}(x,y)=\begin{cases}\frac{s-1}{s+1}\left(x^2-|y|^2\right) & \text{if } (x,|y|)\in D,\\ \frac{2}{s+1}x-\frac{2s}{s+1}|y|+1 & \text{if } (x,|y|)\notin D,\end{cases}$$

$$u_{\infty,s}(x,y)=\begin{cases}0 & \text{if } (y,|x|)\in D,\\ \frac{s-1}{s+1}\left(|x|^2-y^2\right)+\frac{2}{s+1}y-\frac{2s}{s+1}|x|+1 & \text{if } (y,|x|)\notin D.\end{cases}$$

It is easy to check that these functions are continuous. Furthermore, let $\phi_{1,s}$, $\psi_{1,s}$, $\phi_{2,s}$, $\psi_{2,s}$, $\phi_{\infty,s}$, $\psi_{\infty,s}$ be defined by

$$\begin{aligned}&(\phi_{1,s}(x,y),\psi_{1,s}(x,y))\\&\quad=\begin{cases}\left(\frac{2(s-1)}{s+1}x-\frac{2}{s+1}x',-\frac{2(s-1)}{s+1}y+\frac{2s}{s+1}\right) & \text{if } (|x|,y)\in D,\\ (0,0) & \text{if } (|x|,y)\in D,\end{cases}\end{aligned}$$

$$(\phi_{2,s}(x,y),\psi_{2,s}(x,y)) = \begin{cases} \left(\frac{2(s-1)}{s+1}\,x, -\frac{2(s-1)}{s+1}\,y\right) & \text{if } (x,|y|)\in D,\\ \left(\frac{2}{s+1}, -\frac{2s}{s+1}y'\right) & \text{if } (x,|y|)\in D,\end{cases}$$

$$(\phi_{\infty,s}(x,y),\psi_{\infty,s}(x,y)) = \begin{cases} (0,0) & \text{if } (y,|x|)\in D,\\ \left(\frac{2(s-1)}{s+1}\,x-\frac{2s}{s+1}\,x', -\frac{2(s-1)}{s+1}\,y+\frac{2}{s+1}\right) & \\ & \text{if } (y,|x|)\in D,\end{cases}$$

where $x' = x/|x|$ for $x \neq 0$ and $x' = 0$ if $x = 0$.

The key properties of the above functions are described in the following lemma.

LEMMA 2.1. *Let $s > 1$ be a fixed number.*

(i) *We have*

$$u_{1,s}(x,y) \le 1, \tag{2.1}$$

$$u_{2,s}(x,y) \le \frac{2}{s+1}\,x - \frac{2s}{s+1}\,|y| + 1, \tag{2.2}$$

$$u_{\infty,s}(x,y) \le \frac{s-1}{s+1}\,(|x|^2-y^2) + \frac{2}{s+1}\,y - \frac{2s}{s+1}\,|x| + 1. \tag{2.3}$$

(ii) *Suppose $x,h\in\mathcal{H}$, $y, y+k \ge 0$ and $|h|\le|k|$. Then*

$$u_{1,s}(x+h,y+k) \le u_{1,s}(x,y) + \phi_{1,s}(x,y)\cdot h + \psi_{1,s}(x,y)k, \tag{2.4}$$

$$u_{\infty,s}(x+h,y+k) \le u_{\infty,s}(x,y) + \phi_{\infty,s}(x,y)\cdot h + \psi_{\infty,s}(x,y)k. \tag{2.5}$$

Suppose $x, x+h \ge 0$, $y,k\in\mathcal{H}$ and $|h|\le|k|$. Then

$$u_{2,s}(x+h,y+k) \le u_{2,s}(x,y) + \phi_{2,s}(x,y)h + \psi_{2,s}(x,y)\cdot k. \tag{2.6}$$

Proof. (i) It is easy to see that the inequalities (2.1)–(2.3) are equivalent and therefore it suffices to prove the first one. To this end, note that for $(|x|,y)\in D$ the partial derivative of $u_{1,s}$ with respect to y equals

$$\frac{2(s-1)}{s+1}\left(\frac{s}{s-1} - y\right) \ge 0$$

and the inequality follows by the continuity of $u_{1,s}$.

(ii) This is done by a well-known procedure (cf. [2]–[4]). Consider a function

$$G_{1,s}(t) = u_{1,s}(x+th, y+tk),$$

defined on $\{t : y+tk \ge 0\}$. The inequality (2.4) is equivalent to

$$G_{1,s}(1) \le G_{1,s}(0) + G'_{1,s}(0)$$

(with $(G_{1,s})'_-(0)$, $(G_{1,s})'_+(0)$ or 0 instead of $G'_{1,s}(0)$ if the latter does not exist) and will follow once we have established the concavity of $G_{1,s}$. Consider the sets

$$(2.7)\qquad E_{1,s}=\{t:(|x+th|,y+tk)\notin D\},\qquad F_{1,s}=\{t:(|x+th|,y+tk)\in D\}.$$

On $E_{1,s}$ we have $G_{1,s}\equiv 1$, which is clearly concave, while on $F_{1,s}$, $G_{1,s}(t)$ equals

$$\frac{s-1}{s+1}(|h|^2-k^2)t^2+\frac{s-1}{s+1}[|x|^2+2tx\cdot h-y^2-2tyk] - \frac{2}{s+1}|x+th|+\frac{2s}{s+1}(y+tk)$$

and the concavity follows from $|h|^2\le k^2$ and the concavity of the function $t\mapsto -|x+th|$. It remains to note that $E_{1,s}$, $F_{1,s}$ are intervals and, by (2.1), $G(t)\le 1$ on $F_{1,s}$.

For the functions $u_{2,s}$, $u_{\infty,s}$ the argument is essentially the same; we introduce the functions $G_{2,s}$ and $G_{\infty,s}$ in a similar manner and reduce the proof of (2.5), (2.6) to the concavity of these functions. The concavity is clear on the sets $E_{2,s}$, $F_{2,s}$ and $E_{\infty,s}$, $F_{\infty,s}$, defined as in (2.7), and the inequality for one-sided derivatives follows from (2.2), (2.3). The sets $E_{2,s}$, $E_{\infty,s}$ may happen to be unions of two intervals, but this does not change the argument. ■

Now let us introduce the special functions corresponding to the moment inequalites. For $p\in(0,1)$, $x\in\mathcal{H}$, $y\ge 0$, let

$$(2.8)\qquad U_{p,s}(x,y)=\frac{p(1-p)(2-p)(s+1)}{2}\int_0^\infty t^{p-1}u_{1,s}(x/t,y/t)\,dt,$$

while for $p\in(1,2)$, $x\ge 0$, $y\in\mathcal{H}$,

$$(2.9)\qquad U_{p,s}(x,y)=\frac{p(p-1)(2-p)(s+1)}{2}\int_0^\infty t^{p-1}u_{2,s}(x/t,y/t)\,dt.$$

Finally, for $p\in(2,\infty)$, $x\in\mathcal{H}$, $y\ge 0$, set

$$(2.10)\qquad U_{p,s}(x,y)=\frac{p(p-1)(p-2)(s+1)}{2}\int_0^\infty t^{p-1}u_{\infty,s}(x/t,y/t)\,dt.$$

The formulas for $U_{p,s}$ are as follows. Suppose $p\in(0,1)$. If $y\le s|x|$, then

$$U_{p,s}(x,y)=\left(\frac{s-1}{s+1}\right)^{p-1}(|x|+y)^{p-1}[y(s-1+p)+|x|(s-sp-1)],$$

while for $y\ge s|x|$,

$$U_{p,s}(x,y)=(y-|x|)^{p-1}[y(s+1-p)+|x|(sp-s-1)].$$

In case $p \in (1, 2)$, if $|y| \leq sx$, then

$$U_{p,s}(x, y) = \left(\frac{s-1}{s+1}\right)^{p-1} (x + |y|)^{p-1}[|y|(-s - p + 1) + x(sp - s + 1)],$$

while for $|y| \geq sx$,

$$U_{p,s}(x, y) = (|y| - x)^{p-1}[|y|(p - s - 1) + x(s - sp + 1)].$$

Finally, let $p \in (2, \infty)$. Then, if $sy \leq |x|$,

$$U_{p,s}(x, y) = (|x| - y)^{p-1}[y(sp - s - 1) + |x|(s - p + 1)]$$

and for $sy \geq |x|$,

$$U_{p,s}(x, y) = \left(\frac{s-1}{s+1}\right)^{p-1} (|x| + y)^{p-1}[y(s - ps - 1) + |x|(s + p - 1)].$$

The following functions will also play a role. If $p \in (0, 1)$ and $s > 1$, let $V_{p,s} : \mathcal{H} \times \mathbb{R}_+ \to \mathbb{R}$ be given by

$$V_{p,s}(x, y) = (s + 1 - p)[y^p - K_{p,s}|x|^p]$$

and for $p \in (1, 2)$, $s > 1$, define $V_{p,s} : \mathbb{R}_+ \times \mathcal{H} \to \mathbb{R}$ by

$$V_{p,s}(x, y) = (s + 1 - p)[-|y|^p + K_{p,s}x^p].$$

Here

$$K_{p,s} = \left(\frac{s-1}{2}\right)^{p-1} \cdot \frac{p}{s + 1 - p}.$$

We will need the following fact about the functions defined above.

LEMMA 2.2. *Suppose $p \in (0, 2)$, $p \neq 1$ and $s > 1$. Then*

$$U_{p,s} \geq V_{p,s}. \tag{2.11}$$

Proof. It suffices to prove the inequality in the special case $\mathcal{H} = \mathbb{R}$. Consider the functions $F, G : (0, 1) \to \mathbb{R}$ given by

$$F(t) = V_{p,s}(t, 1 - t), \qquad G(t) = U_{p,s}(t, 1 - t).$$

The function F is convex on $(0, t_0)$ and concave on $(t_0, 1)$ for some $t_0 \in (0, 1)$, while G is concave on $(0, (s + 1)^{-1})$ and linear on $((s + 1)^{-1}, 1)$. Moreover,

$$F(0) = G(0), \qquad F'(0) < G'(0),$$
$$F\left(\frac{2}{s+1}\right) = G\left(\frac{2}{s+1}\right), \qquad F'\left(\frac{2}{s+1}\right) = G'\left(\frac{2}{s+1}\right).$$

Thus $F \leq G$, which yields (2.11) by homogeneity. ■

REMARK 2.1. If $x = 0$ or $2|y| = (s - 1)|x|$, then $U_{p,s}(x, y) = V_{p,s}(x, y)$. This is a consequence of $F(0) = G(0)$ and $F(2/(s + 1)) = G(2/(s + 1))$.

3. The proofs of the theorems. The inequalities (2.4)–(2.6) yield the following estimates.

LEMMA 3.1. *Let $s > 1$ and suppose f, g are martingales satisfying*

$$|df_n| \le |dg_n|, \qquad n = 1, 2, \ldots,$$

with probability 1.

(i) *Suppose f is $\mathcal{H}$-valued and g is nonnegative. Then*

$$\mathbb{E}u_{1,s}(f_n, g_n) \le \mathbb{E}u_{1,s}(f_0, g_0), \qquad n = 0, 1, 2, \ldots. \tag{3.1}$$

(ii) *Suppose f is $\mathcal{H}$-valued and g is nonnegative. Furthermore, assume that both f and g are square integrable. Then*

$$\mathbb{E}u_{\infty,s}(f_n, g_n) \le \mathbb{E}u_{\infty,s}(f_0, g_0), \qquad n = 0, 1, 2, \ldots. \tag{3.2}$$

(iii) *Suppose f is nonnegative and g is $\mathcal{H}$-valued. Then*

$$\mathbb{E}u_{2,s}(f_n, g_n) \le \mathbb{E}u_{2,s}(f_0, g_0), \qquad n = 0, 1, 2, \ldots. \tag{3.3}$$

Proof. We will only prove (i); the remaining statements can be established in the same manner. It suffices to show that for any $1 \le k \le n$,

$$\mathbb{E}u_{1,s}(f_k, g_k) \le \mathbb{E}u_{1,s}(f_{k-1}, g_{k-1}). \tag{3.4}$$

Since $|df_k| \le |dg_k|$ almost surely, the inequality (2.4) gives

$$u_{1,s}(f_k, g_k) \le u_{1,s}(f_{k-1}, g_{k-1}) + \phi_{1,s}(f_{k-1}, g_{k-1}) \cdot df_k + \psi_{1,s}(f_{k-1}, g_{k-1})dg_k.$$

Both sides of the inequality above are integrable; taking the conditional expectation with respect to $\mathcal{F}_{k-1}$ gives

$$\mathbb{E}[u_{1,s}(f_k, g_k) \,|\, \mathcal{F}_{k-1}] \le u_{1,s}(f_{k-1}, g_{k-1}).$$

This implies (3.4) and completes the proof. ∎

Proof of (1.2). If $\beta \ge 1$, then $C_{p,\beta} = 0$ and the inequality is trivial. Assume that $\beta < 1$. The identity (2.8) together with Lemmas 2.2 and 3.1 yields

$$\begin{aligned}(s+1-p)\mathbb{E}[g_n^p - K_{p,s}|f_n|^p] &= \mathbb{E}V_{p,s}(f_n, g_n) \le \mathbb{E}U_{p,s}(f_n, g_n) \\ &\le \mathbb{E}U_{p,s}(f_0, g_0)\end{aligned} \tag{3.5}$$

for any n. Now set

$$s = \frac{1+\beta-\beta p}{1+\beta-p} > 1.$$

Then $\mathbb{E}U_{p,s}(f_0, g_0) \le 0$, which follows from the fact that for $x \in \mathcal{H}$ and $y \in \mathbb{R}_+$ satisfying $\beta|x| \ge y$ we have

$$U_{p,s}(x, y) \le U_{p,s}(x, \beta|x|) = c[\beta(s-1+p) + s - sp - 1] = 0$$

for some nonnegative c. To complete the proof, note that $K_{p,s} = C_{p,\beta}^{-p}$. ∎

Proof of (1.3). It suffices to prove the inequality for $p \in (1,2)$, as for $p \leq 1$ it is trivial and for $p \geq 2$ it holds for general f, g. We proceed as previously. The identity (2.9) and Lemmas 2.2 and 3.1 give

$$(3.6)\qquad (s+1-p)\mathbb{E}[-|g_n|^p + K_{p,s}f_n^p] = \mathbb{E}V_{p,s}(f_n,g_n) \leq \mathbb{E}U_{p,s}(f_n,g_n) \leq \mathbb{E}U_{p,s}(f_0,g_0)$$

for any n. Now the choice $s=p$ implies $\mathbb{E}U_{p,s}(f_0,g_0) \leq 0$, since $U_{p,p}(x,y) \leq 0$ if $x \leq |y|$. All that is left is to observe that $C_p^{-p} = K_{p,p}$. ∎

Remark 3.1. For $p>2$, the function $U_{p,p}$ can be used to establish the inequality (1.1) for $\mathcal{H}$-valued f differentially subordinate to $g \geq 0$ (with the optimal constant α_p'). In [4], Burkholder uses a slightly different function

$$U_p'(x,y) = \begin{cases} U_{p,p}(x,y) & \text{if } (p-1)y \leq 2|x|, \\ p\Big(\dfrac{p-1}{2}\Big)^{p-1}|y|^p - |x|^p & \text{if } (p-1)y \geq 2|x|, \end{cases}$$

and proves $\mathbb{E}U_p'(f_n,g_n) \leq \mathbb{E}U_p'(f_0,g_0) \leq 0$ by showing an inequality analogous to (2.4)–(2.6). Our approach (through identity (2.10)) enables us to avoid technical computations.

Remark 3.2. The inequalities (3.5), (3.6) can be used to obtain variations of (1.2), (1.3), involving the initial variables f_0, g_0. For example, assume that f is $\mathcal{H}$-valued and differentially subordinate to a nonnegative g with $|f_0| = g_0$. If $0<p<1$, then (3.5) yields

$$\mathbb{E}g_n^p \leq \frac{(s-1)^{p-1}}{s+1-p}\left[\frac{p}{2^{p-1}}\,\mathbb{E}|f_n|^p + \frac{2^{p-1}(s-1)(2-p)}{(s+1)^{p-1}}\,\mathbb{E}|f_0|^p\right]$$

for any $s>1$. Take $s \to \infty$ to obtain

$$\|g_n\|_p \leq 2\Big(1-\frac{p}{2}\Big)^{1/p}\|f_0\|_p.$$

Sharpness. This will be shown in a few steps. Assume $\mathcal{H} = \mathbb{R}$ and $p \in (0,2)$, $p \neq 1$.

Step 1. Let us consider the following process, a modification of the one used by Burkholder in [4]. Let $s>1$, $\delta \in (0,1)$ be fixed and set

$$x_n = \Big(1+\frac{2\delta}{s-1}\Big)^n, \qquad p_n = \left[\frac{(1-\delta)(s-1)}{(1+\delta)(s-1+2\delta)}\right]^n$$

for $n = 0,1,2,\ldots$. Consider a Markov chain $H = H(s,\delta)$ with values in $\mathbb{R}_+^2$, starting from $(1,s)$, such that for $n=0,1,\ldots,$

$$\mathbb{P}(H_{2n+1} = (x_n(1-\delta), x_n(s+\delta)) \,|\, H_{2n} = (x_n, sx_n)) = \frac{1}{1+\delta},$$

$$\mathbb{P}(H_{2n+1} = (2x_n, (s-1)x_n) \,|\, H_{2n} = (x_n, sx_n)) = \frac{\delta}{1+\delta},$$

$$\mathbb{P}(H_{2n+2} = (0, x_n(s-1+2\delta)) \,|\, H_{2n+1} = (x_n(1-\delta), x_n(s+\delta))) = \frac{\delta(s+1)}{s-1+2\delta},$$

$$\mathbb{P}(H_{2n+2} = (x_{n+1}, sx_{n+1}) \,|\, H_{2n+1} = (x_n(1-\delta), x_n(s+\delta))) = \frac{(1-\delta)(s-1)}{s-1+2\delta}$$

with the further condition that all the states lying on the lines $2y = (s-1)x$ and $x = 0$ are absorbing. Then the processes $F = F(s,\delta)$, $G = G(s,\delta)$, defined by $H_n = (F_n, G_n)$, are martingales such that for $n \geq 1$, $dF_n = \pm dG_n$.

STEP 2. Now we will show that the sequence $(\mathbb{E}U_{p,s}(H_n))_{n\geq 0}$ is almost constant. For any nonnegative integer n, let $A_n = \{H_{n+1} \neq H_n\}$. Note that

$$A_{2n} = \{H_{2n} = (x_n, sx_n)\}, \qquad A_{2n+1} = \{H_{2n+1} = (x_n(1-\delta), x_n(s+\delta))\}.$$

LEMMA 3.2. *Let n be a nonnegative integer.*

(i) $\mathbb{P}(A_{2n}) = p_n$.

(ii) *The following equalities hold true*:

$$\mathbb{E}U_{p,s}(H_{2n+2}) = \mathbb{E}U_{p,s}(H_{2n+1}), \tag{3.7}$$

$$\mathbb{E}U_{p,s}(H_{2n+1}) = \mathbb{E}U_{p,s}(H_{2n}) - x_n^p R(\delta)\mathbb{P}(A_{2n}), \tag{3.8}$$

for some function $R = R_{p,s} : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $R_{p,s}(\delta)/\delta \to 0$ as $\delta \to 0$.

Proof. (i) We have $P(A_0) = 1 = p_0$ and $\mathbb{P}(A_{2k} \,|\, A_{2k-2}) = p_1$ for any $k \geq 1$.

(ii) On A_{2n+1}, the variable H_{2n+2} takes values $(0, x_n(s-1+2\delta))$ and

$$(x_{n+1}, sx_{n+1}) = \left(x_n\left(1 + \frac{2\delta}{s-1}\right), x_n(s-1+2\delta) + x_n\left(1 + \frac{2\delta}{s-1}\right)\right).$$

But the function $t \mapsto U_{p,s}(t, x_n(s-1+2\delta)+t)$ is linear on the interval $[0, x_n(1+2\delta/(s-1))]$; this proves the first estimate. For the second one, the argument is similar: on A_{2n},

$$H_{2n+1} \in \{(2x_n, (s-1)x_n),\ (x_n(1-\delta), x_n(s+\delta))\}, \qquad H_{2n} = (x_n, sx_n)$$

and the function $t \mapsto U_{p,s}(x_n+t, sx_n-t)$ has a continuous derivative on $(-\delta, x_n)$ and is linear on $[0, x_n]$. It remains to use the fact that $U_{p,s}$ is homogeneous of order p to get the special form of the remainder. ■

STEP 3. Let us study the following estimate:

$$\mathbb{E}V_{p,s}(H_{2n}) + \varepsilon \mathbb{E}F_{2n}^p \geq \mathbb{E}U_{p,s}(H_0). \tag{3.9}$$

LEMMA 3.3. *Let $\varepsilon > 0$ be fixed.*

(i) *Suppose $p \in (0,1)$ and $s > 1$. Then there exists $\delta > 0$ such that the inequality* (3.9) *holds for large n.*
(ii) *Suppose $p \in (1,2)$. Then there exist $s < p$ and $\delta > 0$ such that the inequality* (3.9) *holds for large n.*

Proof. Outside A_{2n}, the variable H_{2n} takes values on one of the lines $2y = (s-1)x$, $x = 0$. Since $U_{p,s}$, $V_{p,s}$ coincide on these lines, we have, by Lemma 3.2,

$$\begin{aligned}\mathbb{E}V_{p,s}(H_{2n}) &= \mathbb{E}U_{p,s}(H_{2n}) + \mathbb{P}(A_{2n})[V_{p,s}(x_n, sx_n) - U_{p,s}(x_n, sx_n)] \\ &= \mathbb{E}U_{p,s}(H_0) - R(\delta)\sum_{k=0}^{n-1} x_k^p p_k - cx_n^p p_n,\end{aligned} \tag{3.10}$$

where $c = -V_{p,s}(1,s) + U_{p,s}(1,s) \geq 0$.

On the other hand,

$$\mathbb{E}F_{2n}^p \geq \sum_{k=0}^{n-1}(2x_n)^p \frac{p_n\delta}{1+\delta} \geq 2^{-1}\delta \sum_{k=0}^{n-1} x_n^p p_n = 2^{-1}\delta \sum_{k=0}^{n-1} r^k, \tag{3.11}$$

where

$$r = r(\delta) = x_1^p p_1 = \left(1 + \frac{2\delta}{s-1}\right)^{p-1} \frac{1-\delta}{1+\delta}. \tag{3.12}$$

(i) Fix $\varepsilon > 0$, $p \in (0,1)$ and $s > 1$. By (3.11), there exists δ such that

$$R(\delta)\sum_{k=0}^{n-1} x_k^p p_k \leq \frac{2R(\delta)}{\delta}\,\mathbb{E}F_{2n}^p \leq \frac{\varepsilon}{2}\,\mathbb{E}F_{2n}^p \tag{3.13}$$

for any n. Furthermore, since $p < 1$, we have $r(\delta) < 1$; hence $cx_n^p p_n = cr^n \leq \varepsilon\delta/4 < 2^{-1}\varepsilon\mathbb{E}F_{2n}^p$ for large n. Combining this estimate with (3.10) and (3.13) yields (3.9).

(ii) Fix $\varepsilon > 0$ and $p \in (1,2)$. We have $r'(0) = 2(p-s)/(s-1)$, so there exist $s \in (1,p)$ and $\delta(\varepsilon)$ such that if $\delta \in (0, \delta(\varepsilon))$, then $1 < r(\delta) < 1 + \varepsilon\delta/8c$. Then, by (3.11),

$$cx_n^p p_n = cr^n \leq c\left[\frac{2(r-1)}{\delta}\,\mathbb{E}F_{2n}^p + 1\right] \leq \frac{\varepsilon}{4}\,\mathbb{E}F_{2n}^p + 1 < \frac{\varepsilon}{2}\,\mathbb{E}F_{2n}^p$$

if n is large enough; the last inequality follows from $\mathbb{E}F_{2n}^p \to \infty$ as $n \to \infty$. We conclude the proof by observing that (3.13) holds for sufficiently small δ, and applying (3.10). ∎

STEP 4: *sharpness of (1.2).* Let $\beta \in (p/2, 1)$, $\delta > 0$, $\varepsilon > 0$ and set

$$s = \frac{1+\beta-\beta p}{1+\beta-p} > 1, \qquad a = \frac{2\beta - s + 1}{1+\beta} < 1.$$

The inequality $p < 2\beta$ implies $a > 0$. Consider martingales $F = (F_n)_{n\geq -1}$, $G = (G_n)_{n\geq -1}$ satisfying

(I) $F_{-1} = 2 - a$, $G_{-1} = a + s - 1$ almost surely,
(II) $\mathbb{P}((F_0, G_0) = (1, s)) = a = 1 - \mathbb{P}((F_0, G_0) = (2, s-1))$,
(III) on $\{F_0 = 2\}$, the process (F_n, G_n) is constant,
(IV) on $\{F_0 = 1\}$, the conditional distribution of the process (F_n, G_n) is the distribution of $H(s, \delta)$ constructed in Step 1.

By the choice of a, we have $\beta F_{-1} = G_{-1}$ and $\mathbb{E}U_{p,s}(F_0, G_0) = 0$. Clearly,

$$\mathbb{E}V_{p,s}(F_{2n}, G_{2n}) = \mathbb{E}V_{p,s}(F_{2n}, G_{2n})\chi_{\{F_0=1\}} + \mathbb{E}V_{p,s}(F_{2n}, G_{2n})\chi_{\{F_0=2\}}.$$

On the set $\{F_0 = 1\}$ we can use Lemma 3.3: a proper choice of δ and n implies

$$\mathbb{E}V_{p,s}(F_{2n}, G_{2n})\chi_{\{F_0=1\}} + \varepsilon\mathbb{E}F_{2n}^p\chi_{\{F_0=1\}} \geq \mathbb{E}U_{p,s}(F_0, G_0)\chi_{\{F_0=1\}}.$$

On the set $\{F_0 = 2\}$ the pair $(F_{2n}, G_{2n}) = (F_0, G_0)$ lies on the line $2y = (s-1)x$, which implies $V_{p,s}(F_{2n}, G_{2n}) = U_{p,s}(F_0, G_0)$. Combining these two facts we get

$$\mathbb{E}V_{p,s}(F_{2n}, G_{2n}) + \varepsilon\mathbb{E}F_{2n}^p \geq \mathbb{E}U_{p,s}(F_0, G_0), \tag{3.14}$$

so

$$\mathbb{E}G_{2n}^p \geq \left(C_{p,\beta}^{-p} - \frac{\varepsilon}{s+1-p}\right)\mathbb{E}F_{2n}^p > (C_{p,\beta}^{-p} - \varepsilon)\mathbb{E}F_{2n}^p.$$

This proves that (1.2) is sharp. For the case $\beta \geq 1$, observe that $C_{p,\beta}$ is nonincreasing as a function of β and $C_{p,\beta} \to 0$ as $\beta \uparrow 1$.

STEP 5: *sharpness of (1.3).* The cases $p \leq 1$, $p = 2$ are trivial; for $p \geq 2$, we use the example on page 669 of [1]. The only case left is $p \in (1, 2)$.

For $\varepsilon > 0$, let $s \in (1, p)$ and $\delta > 0$ be the numbers guaranteed by Lemma 3.3. Consider martingales F, G satisfying (I)–(IV) with $a = (3-s)/2$. By arguments similar to those above, (3.9) leads to the inequality (3.14), valid for large n. Since $\mathbb{E}F_{2n}^p \to \infty$, we have $\mathbb{E}U_{p,s}(F_0, G_0) \geq -\varepsilon\mathbb{E}F_{2n}^p$ for large n, which combined with (3.14) implies

$$\mathbb{E}G_{2n}^p \leq \left(K_{p,s} + \frac{2\varepsilon}{s+p-1}\right)\mathbb{E}F_{2n}^p < (C_p^{-p} + 2\varepsilon)\mathbb{E}F_{2n}^p.$$

Therefore C_p is the best possible in (1.3).

REMARK 3.3. It is clear from the examples above that the inequalities (1.2) and (1.3) are sharp even in the case of ± 1 transforms, i.e. if we assume that $df_n = \pm dg_n$ for $n = 1, 2, \ldots$.

4. Sharp inequalities for stochastic integrals. Suppose $X = (X_t)_{t\geq 0}$ is a càdlàg martingale on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is filtered by a nondecreasing right-continuous family $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-fields of $\mathcal{F}$. In addition, assume that $\mathcal{F}_0$ contains all the events of probability 0. Let Y be the Itô integral of H with respect to X, where H is a predictable process:

$$Y_t = H_0X_0 + \int_{(0,t]} H_s\,dX_s.$$

The continuous-time versions of Theorems 1.1 and 1.2 are stated below.

THEOREM 4.1. *Suppose $p \in (0,1)$, X is nonnegative and for any $t > 0$, the variable H_t takes values in a closed unit ball of $\mathcal{H}$. If $\beta > 0$ satisfies $\mathbb{P}(\beta|H_0| \geq 1) = 1$, then for any $t > 0$,*

$$\|Y_t\|_p \geq C_{p,\beta}\|X_t\|_p, \tag{4.1}$$

and the inequality is sharp if $p < 2\beta$.

THEOREM 4.2. *Suppose $p \in (0,\infty)$, X is nonnegative and H takes values outside the open unit ball of $\mathcal{H}$. Then for any $t > 0$,*

$$\|X_t\|_p \leq C_p\|Y_t\|_p, \tag{4.2}$$

and the inequality is sharp.

The inequalities (4.1), (4.2) follow from (1.2), (1.3) by discretizing the argument; see [3], where an analogous submartingale inequality follows from the corresponding discrete-time version. The sharpness follows from the fact that the constants $C_{p,\beta}$, C_p are the best possible in (1.2), (1.3) in the case when f is a ± 1 transform of g (see Remark 3.3).

5. Inequalities for harmonic functions. In this section we study harmonic extensions of inequalities (1.2), (1.3). Let N be a fixed positive integer and D be an open connected subset of $\mathbb{R}^N$. Fix $\xi \in D$ and consider two harmonic functions u, v on D, taking values in Hilbert spaces $\mathcal{H}$, $\mathcal{K}$. Suppose u is differentially subordinate to v, that is,

$$|\nabla u| \leq |\nabla v| \quad \text{ on } D.$$

Let D_0 be a bounded subdomain of D with $\xi \in D_0 \subset D_0 \cup \partial D_0 \subset D$. Let $\mu^{\xi}_{D_0}$ stand for the harmonic measure on ∂D_0 with respect to ξ, and

$$\|u\|_{D_0,p} = \Big[\int_{\partial D_0} |u(z)|^p\,\mu^{\xi}_{D_0}(dz)\Big]^{1/p}, \quad 0 < p < \infty.$$

We now give some norm inequalities for smooth functions.

THEOREM 5.1. *Let u, v, D_0 be as above.*

(i) *Assume that $p \in (0,1)$ and v is nonnegative. Then*

$$\|u\|_{D_0,p} \geq C_{p,\beta}\|v\|_{D_0,p}, \tag{5.1}$$

where $\beta = v(\xi)/|u(\xi)|$.

(ii) *Assume that* $p \in (0,\infty)$, u *is nonnegative and* $u(\xi) \le |v(\xi)|$. *Then*

$$\|u\|_{D_0,p} \le C_p \|v\|_{D_0,p}. \tag{5.2}$$

Proof. We will prove only the first part, the second one can be established similarly. As $C_{p,\beta} = 0$ for $\beta \ge 1$, we may assume that $\beta < 1$. Let

$$s = \frac{1+\beta-\beta p}{1+\beta-p} > 1.$$

It is easy to check that the function $u_{1,s}(u,v)$ is superharmonic. Therefore

$$\int_{D_0} u_{1,s}(u(z),v(z))\,\mu^{\xi}_{D_0}(dz) \le u_{1,s}(u(\xi),v(\xi)).$$

Applying the identity (2.8) we obtain

$$\int_{D_0} U_{p,s}(u(z),v(z))\,\mu^{\xi}_{D_0}(dz) \le U_{p,s}(u(\xi),v(\xi)) = 0,$$

since $\beta|u(\xi)| = v(\xi)$. It suffices to use the inequality (2.11) to get (5.1). ∎

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