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FUNCTIONAL ANALYSIS

Schroeder–Bernstein Quintuples for Banach Spaces

by

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Summary. Let X and Y be two Banach spaces, each isomorphic to a complemented subspace of the other. In 1996, W. T. Gowers solved the Schroeder–Bernstein Problem for Banach spaces by showing that X is not necessarily isomorphic to Y. In this paper, we obtain necessary and sufficient conditions on the quintuples (p, q, r, s, t) in $\mathbb{N}$ for X to be isomorphic to Y whenever

$$\begin{cases} X \sim X^p \oplus Y^q, \\ Y^t \sim X^r \oplus Y^s. \end{cases}$$

Such quintuples are called Schroeder–Bernstein quintuples for Banach spaces and they yield a unification of the known decomposition methods in Banach spaces involving finite sums of X and Y, similar to Pełczyński's decomposition method. Inspired by this result, we also introduce the notion of Schroeder–Bernstein sextuples for Banach spaces and pose a conjecture which would complete their characterization.

1. Introduction. Let X and Y be Banach spaces. We write $X \overset{c}{\hookrightarrow} Y$ if X is isomorphic to a complemented subspace of Y, and $X \sim Y$ if X is isomorphic to Y. If $n \in \mathbb{N}^* = \{1, 2, \ldots\}$, then X^n denotes the sum of n copies of X. It is useful to define $X^0 = \{0\}$.

Suppose that X and Y are Banach spaces satisfying

$$X \overset{c}{\hookrightarrow} Y \quad \text{and} \quad Y \overset{c}{\hookrightarrow} X. \tag{1.1}$$

In 1996, W. T. Gowers [4] solved the so-called Schroeder–Bernstein Problem for Banach spaces by showing that X is not necessarily isomorphic to Y. On the other hand, in [2] a quadruple (p, q, r, s) in $\mathbb{N}$, with $p + q \geq 2$ and $r + s \geq 2$, was said to be a *Schroeder–Bernstein quadruple for Banach spaces*

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(for short, SBQ) if $X \sim Y$ whenever these spaces satisfy (1.1) and

$$\begin{cases} X \sim X^p \oplus Y^q, \\ Y \sim X^r \oplus Y^s. \end{cases} \tag{1.2}$$

In Remark 2.4 we recall the characterization of the SBQ obtained in [2]. Notice that the SBQ $(2, 0, 0, 2)$ corresponds to the classical decomposition method in Banach spaces called Pełczyński's decomposition method (see [1, p. 64]).

The starting point of this work is the fact that there are some decomposition methods in Banach spaces involving only finite sums of X and Y which are not of the form (1.2). For instance, it is not difficult to check (see Theorem 1.2) that $X \sim Y$ whenever these Banach spaces satisfy (1.1) and

$$X \sim X^2 \quad \text{and} \quad Y^2 \sim Y^3. \tag{1.3}$$

Let us point out that it is an open problem whether for every Banach space Y satisfying $Y^2 \sim Y^3$ we have $Y \sim Y^2$ (the *Square-cube Problem for Banach spaces*, see [6, p. 367]).

It is then natural, in the spirit of [2], to ask whether it is possible to determine all conditions similar to (1.3) which added to (1.1) also yield $X \sim Y$. To be more precise, we define:

Definition 1.1. A quintuple (p, q, r, s, t) in $\mathbb{N}$ with $p+q \geq 2$, $r+s+t \geq 3$, $(r, s) \neq (0, 0)$ and $t \geq 1$ is a *Schroeder–Bernstein quintuple for Banach spaces* (for short, SBq) if $X \sim Y$ whenever the Banach spaces X and Y satisfy (1.1) and

$$\begin{cases} X \sim X^p \oplus Y^q, \\ Y^t \sim X^r \oplus Y^s. \end{cases} \tag{1.4}$$

We also say that the number $\nabla = (p-1)(s-t) - rq$ is the *discriminant* of the quintuple (p, q, r, s, t).

The conditions $p+q \geq 2$, $r+s+t \geq 3$ and $(r, s) \neq (0, 0)$ are posed to avoid trivial cases. The main aim of Section 3 is to present the following characterization of SBq's in terms of their discriminants ∇:

Theorem 1.2. *Let (p, q, r, s, t) be a quintuple in $\mathbb{N}$ with $p+q \geq 2$, $r+s+t \geq 3$, $(r, s) \neq (0, 0)$ and $t \geq 1$. Then (p, q, r, s, t) is a SBq if and only if*

(a) $\nabla \neq 0$;
(b) ∇ *divides* $p+q-1$ *and* $r+s-t$.

This is an extension of the main result of [2], where $t = 1$ (see Remark 2.4). To prove our result, the Banach spaces constructed by W. T. Gowers and B. Maurey [5] will be fundamental (see Remark 2.1 and the proofs of Lemmas 3.1 and 3.2). However, we do not know enough Banach spaces satisfying (1.1) to complete the characterization of the *Schroeder–Bernstein sextuples for Banach spaces* (see Definition 4.1, Corollary 4.2 and Conjecture 4.3). Finally, as a particular case of the conjecture, we stand the *Square-cube Schroeder–Bernstein Problem for Banach spaces* (see Problem 4.4).

2. Preliminaries. To determine the quintuples (p,q,r,s,t) in $\mathbb{N}$ which are SBq, we need to recall some recent results on Banach spaces which are isomorphic to complemented subspaces of each other. If m,n are integer numbers, then $m \mid n$ means that m divides n.

REMARK 2.1. In [5, p. 563] Gowers and Maurey constructed Banach spaces X_u, for every $u \in \mathbb{N}$, $u \geq 2$, having the following property: $X_u^m \sim X_u^n$ with $m,n \in \mathbb{N}^*$ if and only if $m \equiv n \bmod u$.

REMARK 2.2. In [3, Lemma 2.1] it was proved that if X and Y are Banach spaces satisfying (1.1) and

$$X^i \oplus Y^j \sim X^k \oplus Y^l$$

for some $i,j,k,l \in \mathbb{N}$, then:

(a) If $d \in \mathbb{N}, d \leq j$ and $d \leq l$, then $X^{i+d} \oplus Y^{j-d} \sim X^{k+d} \oplus Y^{l-d}$.
(b) If $d \in \mathbb{N}, d \leq i$ and $d \leq k$, then $X^{i-d} \oplus Y^{j+d} \sim X^{k-d} \oplus Y^{l+d}$.

REMARK 2.3 ([3, Remark 3.5]). Suppose that X and Y are Banach spaces satisfying (1.1) and $X^p \sim X^{p-1} \oplus Y$ for some $p \in \mathbb{N}$, $p \geq 2$. If $X \sim X^r \oplus Y^s$ for some $r,s \in \mathbb{N}$, $r+s \geq 2$, then $X \sim Y$.

The discriminant of a quadruple (p,q,r,s) in $\mathbb{N}$ was defined in [2] to be the number $\Delta = (p-1)(s-1) - rq$ and in this same paper the following characterization of SBQ was proved:

REMARK 2.4 ([2, Theorem 2.1]). A quadruple (p,q,r,s) in $\mathbb{N}$ with $p+q \geq 2$ and $r+s \geq 2$ is a SBQ if and only if $\Delta \neq 0$, $\Delta \mid (p+q-1)$ and $\Delta \mid (r+s-1)$.

3. Characterization of the Schroeder–Bernstein quintuples for Banach spaces. The main goal of this section is to prove Theorem 1.2. It is an immediate consequence of Propositions 3.4 and 3.5 below. In order to prove Proposition 3.4 we state three lemmas.

LEMMA 3.1. *Let $q,r,s,t \in \mathbb{N}$ with $q \geq 2$, $r \geq 1$ and $s \geq t \geq 1$. Then there exist non-isomorphic Banach spaces X and Y satisfying* (1.1) *with*

$$\begin{cases} X \sim Y^q, \\ Y^t \sim X^r \oplus Y^s. \end{cases} \tag{3.1}$$

Proof. Take $u = s-t+rq$, $i = s-t$, $j = 2(s-t)+r(q+1)$ and $n \in \mathbb{N}$ such that $(n-2)(s-t)+(n-1)rq > r$. So $u \geq 2$ and $i,j,j-i,u-i,nu-j \in \mathbb{N}$.

Since $(j-i)+(nu-j)-(u-i)=(n-1)u$ and $(u-i)+j-(j-i)=u$, by the property of X_u mentioned in Remark 2.1 we have

$$\tag{3.2} \begin{cases} X_u^{u-i} \sim X_u^{j-i} \oplus X_u^{nu-j}, \\ X_u^{j-i} \sim X_u^{u-i} \oplus X_u^{j}. \end{cases}$$

Furthermore, observe that

$$\begin{aligned} q(j-i)-(u-i) &= q(s-t+r(q+1))-rq = q(s-t+rq) = qu, \\ r(u-i)+s(j-i)-t(j-i) &= r(rq)+(s-t)(s-t+r(q+1)) \\ &= r(rq)+(s-t)(s-t+rq)+(s-t)r \\ &= r(rq+s-t)+(s-t)(s-t+rq) \\ &= (r+s-t)(s-t+rq) = (r+s-t)u. \end{aligned}$$

Thus again by Remark 2.1, we obtain

$$\tag{3.3} \begin{cases} X_u^{u-i} \sim X_u^{q(j-i)}, \\ X_u^{t(j-i)} \sim X_u^{r(u-i)} \oplus X_u^{s(j-i)}. \end{cases}$$

Put $X = X_u^{u-i}$ and $Y = X_u^{j-i}$. So by (3.2), X and Y satisfy (1.1) and according to (3.3), they also satisfy (3.1). Moreover, since $q \geq 2$ and $r \geq 1$, X is not isomorphic to Y, because $u = s-t+rq$ does not divide

$$\begin{aligned} (u-i)-(j-i) = u-j &= (s-t+rq)-2(s-t)-r(q+1) \\ &= -(s-t+r). \ \blacksquare \end{aligned}$$

LEMMA 3.2. *Let $q,s,t \in \mathbb{N}$ with $q \geq 2$, $s > t \geq 1$ and suppose that $s-t$ does not divide $q-1$. Then there exist non-isomorphic Banach spaces X and Y satisfying* (1.1) *with*

$$\tag{3.4} \begin{cases} X \sim Y^q, \\ Y^t \sim Y^s. \end{cases}$$

Proof. Since $(s-t)\nmid(q-1)$, it follows that there exist $m \in \mathbb{N}^*$ such that also $(s-t)\nmid(m(q-1))$ and

$$\frac{1}{m} < \frac{q}{s-t}.$$

Now fix $n \in \mathbb{N}^*$ satisfying

$$\tag{3.5} \frac{n-1}{m} < \frac{q}{s-t} \leq \frac{n}{m}.$$

Take $u = s-t$, $i = n(s-t)-mq$ and $j = m(1-q)+n(s-t)$. So $u \geq 2$ and by (3.5), $i, j, j-i, u-i \in \mathbb{N}$. Moreover $nu-j = m(q-1) \in \mathbb{N}$. Hence as in the proof of Lemma 3.1, we see that (3.2) holds. Furthermore, note that

$$q(j-i)-(u-i) = qm-((1-n)(s-t)+qm) = (n+1)u,$$
$$s(j-i)-t(j-i) = (s-t)(j-i) = u(j-i).$$

Hence once more Remark 2.1 implies that

$$\begin{cases} X_u^{u-i} \sim X_u^{q(j-i)}, \\ X_u^{t(j-i)} \sim X_u^{s(j-i)}. \end{cases} \tag{3.6}$$

Put $X = X_u^{u-i}$ and $Y = X_u^{j-i}$. By (3.2), X and Y satisfy (1.1) and according to (3.6), they also satisfy (3.4). Now notice that $u = s-t$ does not divide $(u-i)-(j-i) = u-j$, because by the choice of m, $s-t$ does not divide j. Consequently, X is not isomorphic to Y. ∎

LEMMA 3.3. *If a quintuple* (p,q,r,s,t) *in* $\mathbb{N}$ *with* $p+q \geq 2$, $t \geq 1$, $r+s \geq t+1$ *and* $s \geq t-1$ *is a SBq, then* $\nabla \neq 0$, $\nabla \mid (p+q-1)$ *and* $\nabla \mid (r+s-t)$.

Proof. Let X and Y be Banach spaces satisfying (1.1) and

$$\begin{cases} X \sim X^p \oplus Y^q, \\ Y \sim X^r \oplus Y^{s-(t-1)}. \end{cases} \tag{3.7}$$

Adding Y^{t-1} to both sides of the second condition of (3.7) we see that (1.4) holds. Since (p,q,r,s,t) is a SBq, it follows from (1.4) that $X \sim Y$. Moreover, by hypothesis $r+(s-(t-1)) \geq 2$. So the quadruple $(p,q,r,s-(t-1))$ is a SBQ. Note also that its discriminant is equal to $(p-1)(s-t)-rq = \nabla$. Hence according to Remark 2.4, $\nabla \neq 0$, $\nabla \mid (p+q-1)$ and $\nabla \mid (r+(s-(t-1))-1) = r+s-t$. ∎

PROPOSITION 3.4. *If a quintuple* (p,q,r,s,t) *in* $\mathbb{N}$ *with* $p+q \geq 2$, $r+s+t \geq 3$, $(r,s) \neq (0,0)$ *and* $t \geq 1$ *is a SBq, then* $\nabla \neq 0$, $\nabla \mid (p+q-1)$ *and* $\nabla \mid (r+s-t)$.

Proof. It will be convenient to distinguish three cases: $p \geq 1$ and $q \geq 1$; $p \geq 2$ and $q = 0$; $p = 0$ and $q \geq 2$.

CASE 1: $p \geq 1$ and $q \geq 1$. There are two subcases: $r \geq 1$ and $r = 0$.

SUBCASE 1.1: $r \geq 1$. Let $n \in \mathbb{N}^*$ be such that $nqr + s \geq t+1$ and suppose that (p,q,r,s,t) is a SBq. Let X and Y be Banach spaces satisfying (1.1) and

$$\begin{cases} X \sim X^p \oplus Y^q, \\ Y^t \sim X^{r(p+(n-1)(p-1))} \oplus Y^{nqr+s}. \end{cases} \tag{3.8}$$

Adding $X^{p-1} \oplus Y^q$ to both sides of the first condition of (1.4) we obtain

$X \sim X^p \oplus Y^q \sim X^{p+(p-1)} \oplus Y^{2q}$. Therefore, by induction, we have

$$X \sim X^{p+(m-1)(p-1)} \oplus Y^{mq}, \quad \forall m \in \mathbb{N}^*. \tag{3.9}$$

By using (3.9) with $m = n$ in the second condition of (3.8) we deduce that (1.4) holds. Since (p, q, r, s, t) is a SBq, it follows from (1.4) that $X \sim Y$. By the choice of n, we infer that $r(p+(n-1)(p-1))+(nqr+s)+t \geq 2t+1 \geq 3$. Therefore $(p, q, r(p + (n-1)(p-1)), nqr + s, t)$ is also a SBq. Moreover, $r(p+(n-1)(p-1))+(nqr+s) \geq t+1$, $nqr+s \geq t-1$ and the discriminant of the last quintuple is equal to

$$(p-1)(nqr + s - t) - (r(p + (n-1)(p-1))q = (p-1)(s-t) - rq = \nabla.$$

Hence by Lemma 3.3, $\nabla \neq 0$, $\nabla \mid (p + q + 1)$ and

$$\nabla \mid (r(p + (n-1)(p-1)) + (nqr + s) - t) = nr(p + q + 1) + (r + s - t).$$

So we also have $\nabla \mid (r + s - t)$.

Subcase 1.2: $r = 0$. Then by hypothesis $s \neq 0$. Let $n \in \mathbb{N}^*$ be such that $s(q + 1 + (n-1)q) \geq t + 1$. Assume that $(p, q, 0, s, t)$ is a SBq and let X and Y be Banach spaces satisfying (1.1) and

$$\begin{cases} X \sim X^p \oplus Y^q, \\ Y^t \sim X^{sn(p-1)} \oplus Y^{s(q+1+(n-1)q)}. \end{cases} \tag{3.10}$$

Applying Remark 2.2(b) with $d = 1$ in the first condition of (3.10) we deduce that $Y \sim X^{p-1} \oplus Y^{q+1}$. Next adding $X^{p-1} \oplus Y^q$ to both sides of the last isomorphism we obtain $Y \sim X^{p-1} \oplus Y^q \sim X^{2(p-1)} \oplus Y^{q+1+q}$. Therefore, by induction, we have

$$Y \sim X^{n(p-1)} \oplus Y^{q+1+(n-1)q}. \tag{3.11}$$

By using (3.11) in the second condition of (3.10) we have

$$\begin{cases} X \sim X^p \oplus Y^q, \\ Y^t \sim Y^s. \end{cases} \tag{3.12}$$

Since $(p, q, 0, s, t)$ is a SBq, it follows from (3.12) that $X \sim Y$. By the choice of n, we infer that $sn(p-1) + s(q + 1 + (n-1))q + t \geq 2t + 1 \geq 3$. Hence $(p, q, sn(p-1), s(q+1+(n-1)q)$ is also a SBq. Moreover, $sn(p-1)+s(q+1+(n-1)q) \geq t+1$, $s(q+1+(n-1)q) \geq t-1$ and the discriminant of the last quintuple is equal to

$$(p-1)(s(q + 1 + (n-1)q) - t) - qsn(p-1) = (p-1)(s-t) - rq = \nabla.$$

Therefore according to Lemma 3.3, $\nabla \neq 0$, $\nabla \mid (p + q + 1)$ and

$$\nabla \mid (sn(p-1) + s(q + 1 + (n-1)q) - t) = sn(p + q - 1) + s - t.$$

So we also conclude that $\nabla \mid (s - t)$.

Case 2: $p \geq 2$ and $q = 0$. There are two subcases: $s \geq t$ and $s < t$.

Subcase 2.1: $s \geq t$. Suppose that $(p, 0, r, s, t)$ is a SBq and let X and Y be Banach spaces satisfying (1.1) and

$$\begin{cases} Y \sim Y \oplus X^{p-1}, \\ X^t \sim Y^{s-t} \oplus X^{r+t}. \end{cases} \tag{3.13}$$

By applying Remark 2.2(a) with $d = 1$ in the first condition of (3.13) and Remark 2.2(b) with $d = t$ in the second, we obtain

$$\begin{cases} X \sim X^p, \\ Y^t \sim X^r \oplus Y^s. \end{cases} \tag{3.14}$$

Hence $X \sim Y$. Notice that $(s - t) + (r + t) + t \geq 3$. Consequently, $(1, p-1, s-t, r+t, t)$ is also a SBq. Since the discriminant of this quintuple is equal to $-(p-1)(s-t) = -\nabla$, by Case 1, we see that $\nabla \neq 0$, $\nabla \mid (p-1)$ and $\nabla \mid ((s-t) + (r+t) - t) = r + s - t$.

Subcase 2.2: $s < t$. Assume that $(p, 0, r, s, t)$ is a SBq and let X and Y be Banach spaces satisfying (1.1) and

$$\begin{cases} Y \sim Y \oplus X^{p-1}, \\ X^{r+s} \sim Y^{t-s} \oplus X^s. \end{cases} \tag{3.15}$$

By applying Remark 2.2(a) with $d = 1$ in the first condition of (3.15) and Remark 2.2(b) with $d = s$ in the second, we conclude that (3.14) holds. Thus $X \sim Y$. Observe that $(t - s) + s + (r + s) = r + s + t \geq 3$. So $(1, p-1, t-s, s, r+s)$ is also a SBq. Since the discriminant of this quintuple is equal to $(p-1)(t-s) = -\nabla$, by Case 1, we see that $\nabla \neq 0$, $\nabla \mid (p-1)$ and $\nabla \mid ((t-s) + s - (r+s)) = -(r+s-t)$.

Case 3: $p = 0$ and $q \geq 2$. So $\nabla = -(s-t) - rq$. There are two subcases: $s \geq t$ and $s < t$.

Subcase 3.1: $s \geq t$. First of all note that it is enough to show:

(a) If $\nabla = 0$ then $(0, q, r, s, t)$ is not a SBq;
(b) If $\nabla \neq 0$ and $\nabla \nmid (q-1)$, then $(0, q, r, s, t)$ is not a SBq;
(c) If $\nabla \neq 0$ and $\nabla \nmid (r+s-t)$, then $(0, q, r, s, t)$ is not a SBq.

Next we turn to proving (a), (b) and (c).

(a) Since $\nabla = 0$, it follows that $s = t$ and $r = 0$. Hence $(0, q, 0, t, t)$ is not a SBq, because taking $X = X_{q+1}$ and $Y = X_{q+1}^q$, we see by Remark 2.1 that X and Y are non-isomorphic Banach spaces satisfying (1.1) and $X \sim Y^q$.

(b) Suppose that $\nabla \neq 0$ and note that $\nabla \mid (q-1)$ if and only if $r = 0$ and $(s-t) \mid (q-1)$. Thus it suffices to consider the subcases: $r \geq 1$ and $r = 0$ with $s - t$ not dividing $q - 1$.

Subcase 3.1.1: $r \geq 1$. Then, by Lemma 3.1, $(0, q, r, s, t)$ is not a SBq.

Subcase 3.1.2: $r=0$ and $(s-t)\nmid(q-1)$. Then according to Lemma 3.2, $(0,q,0,s,t)$ is not a SBq.

(c) Assume that $\nabla\neq 0$ and notice that $\nabla\,|\,(r+s-t)$ if and only if $r=0$. Therefore we must consider the case where $r\geq 1$. But in this case, Lemma 3.1 implies that $(0,q,0,s,t)$ is not a SBq.

Subcase 3.2: $s<t$. There are two subcases: $t\leq rq+s$ and $t>rq+s$.

Subcase 3.2.1: $t\leq rq+s$. Suppose that $(0,q,r,s,t)$ is a SBq and let X and Y be Banach spaces satisfying (1.1) and

$$\begin{cases} X\sim Y^q, \\ Y^t\sim Y^{rq+s}. \end{cases} \tag{3.16}$$

By using the first condition of (3.16) in the second, we obtain

$$\begin{cases} X\sim Y^q, \\ Y^t\sim X^r\oplus Y^s. \end{cases} \tag{3.17}$$

Therefore $X\sim Y$. So $(0,q,0,rq+s,t)$ is also a SBq. Thus by Subcase 3.1, it follows that $-(rq+s-t)=\nabla\neq 0$, $\nabla\,|\,(q-1)$ and $\nabla\,|\,((rq+s)-t)$.

Subcase 3.2.2: $t>rq+s$. Assume that $(0,q,r,s,t)$ is a SBq and let X and Y be Banach spaces satisfying (1.1) and

$$\begin{cases} X\sim Y^q, \\ Y^{rq+s}\sim Y^t. \end{cases} \tag{3.18}$$

By using the first condition of (3.18) in the second, we again deduce (3.17). Hence $X\sim Y$. In particular, $(0,q,0,t,rq+s)$ is also a SBq. Consequently, according to Subcase 3.1, $-(t-(rq+s))=-\nabla\neq 0$, $\nabla\,|\,(q-1)$ and $\nabla\,|\,(t-(rq+s))$. ∎

Proposition 3.5. *Suppose that (p,q,r,s,t) is a quintuple in $\mathbb{N}$ with $p+q\geq 2$, $r+s\geq 2$, $t\geq 1$, $\nabla\neq 0$, $\nabla\,|\,(p+q-1)$ and $\nabla\,|\,(r+s-t)$. Then (p,q,r,s,t) is a SBq.*

Proof. Let X and Y be Banach spaces satisfying (1.1) and (1.4). We will prove that $X\sim Y$. If $t=1$ then the result follows from Remark 2.4. Thus we can suppose that $t\geq 2$. We will distinguish five cases: $s\geq t$, $p\geq 1$ and $q\geq 1$; $s\geq t$, $p\geq 2$ and $q=0$; $s\geq t$, $p=0$ and $q\geq 2$; $s<t$, $p\geq 1$; $s<t$, $p=0$ and $q\geq 2$.

Case 1: $s\geq t$, $p\geq 1$ and $q\geq 1$. Let B be a Banach space such that $Y\sim X\oplus B$. Thus, according to (3.9) with $m=t$ and the second condition of (1.4), we get

$$\begin{aligned} Y &\sim X^{p+(t-1)(p-1)} \oplus Y^{tq} \oplus B \sim X^{p+(t-1)(p-1)} \oplus Y^{t} \oplus Y^{t(q-1)} \oplus B \\ &\sim X^{p+(t-1)(p-1)} \oplus (X^r \oplus Y^s) \oplus Y^{t(q-1)} \oplus B \\ &\sim (X^{p+(t-1)(p-1)} \oplus Y^{tq} \oplus B) \oplus X^r \oplus Y^{s-t} \\ &\sim Y \oplus X^r \oplus Y^{s-t} \sim X^r \oplus Y^{s-(t-1)}. \end{aligned}$$

Therefore (3.7) holds. Since the discriminant of the quadruple $(p, q, r, s-t+1)$ is equal to $(p-1)(s-t) - rq = \nabla \neq 0$, it follows that $s \neq t$ or $r \neq 0$. So $r + (s - (t-1)) \geq 2$. Therefore our hypothesis on ∇ and Remark 2.4 imply that this quadruple is a SBQ. Consequently, by (3.7), $X \sim Y$.

Case 2: $s \geq t, p \geq 2$ and $q = 0$. Then $\nabla = (p-1)(s-t)$ and $\nabla \mid (p-1)$. So $s-t = 1$. Applying Remark 2.2(b) with $d = 1$ in the first condition of (1.4), we deduce that $Y \sim X^{p-1} \oplus Y$. Now, if we apply Remark 2.2(a) with $d = t$ in the second condition of (1.4) we conclude that $X^t \sim X^{r+t} \oplus Y^{s-t} \sim X^{r+t} \oplus Y$. Hence

$$\text{(3.19)} \qquad \begin{cases} Y \sim Y \oplus X^{p-1}, \\ X^t \sim Y \oplus X^{r+t}. \end{cases}$$

Next note that the discriminant of the quintuple $(1, p-1, 1, r+t, t)$ is equal to $1-p = -\nabla$. By our hypothesis on ∇, $\nabla \mid (1+(r+t)-t) = r+1$. Therefore by (3.19) and Case 1, we conclude that $Y \sim X$.

Case 3: $s \geq t$, $p = 0$ and $q \geq 2$. Then $\nabla = -(s-t) - rq$. By hypothesis $(rq + s - t) \mid (r + s - t)$. Since $q \geq 2$, we see that $r = 0$. According to (1.4) we know that

$$\text{(3.20)} \qquad \begin{cases} X \sim Y^q, \\ Y^t \sim Y^s. \end{cases}$$

Again by hypothesis $(s-t) \mid (q-1)$. Let $n \in \mathbb{N}^*$ be such that $q-1 = n(s-t)$. Adding Y^{s-t} to both sides of the second condition of (3.20) we have $Y^s \sim Y^{t+(s-t)} \sim Y^{s+(s-t)}$. Hence, by induction, we obtain

$$Y^s \sim Y^{s+n(s-t)}.$$

Therefore by the first condition of (3.20) we get

$$\text{(3.21)} \qquad Y^s \sim Y^s \oplus Y^{n(s-t)} \sim Y^s \oplus Y^{q-1} \sim Y^{s-1} \oplus Y^q \sim Y^{s-1} \oplus X.$$

Next applying Remark 2.2(a) with $d = s-1$ in (3.21) we see that

$$\text{(3.22)} \qquad X^s \sim X^{s-1} \oplus Y.$$

Thus by (3.22), the first condition of (3.20) and Remark 2.3, we deduce $X \sim Y$.

Case 4: $s < t$, $p \geq 1$. There are two subcases: $r = 0$ and $r \geq 1$.

Subcase 4.1: $r = 0$. We rewrite (1.4) as follows:

$$\begin{cases} X \sim X^p \oplus Y^q, \\ Y^s \sim Y^t. \end{cases} \tag{3.23}$$

Notice that the discriminant of $(p, q, 0, t, s)$ is equal to $(p-1)(t-s) = -\nabla$. So by (3.23), our hypothesis, and Case 1 when $q \geq 1$, and Case 2 when $q = 0$, $X \sim Y$.

Subcase 4.2: $r \geq 1$. There are two subcases: $q \geq 1$ and $q = 0$.

Subcase 4.2.1: $q \geq 1$. Let $n \in \mathbb{N}^*$ be such that $nqr + s > t$. From (3.9) with $m = n$ we infer that

$$X^r \sim X^{r(p+(n-1)(p-1))} \oplus Y^{nqr}.$$

Then, by (1.4), we know that

$$\begin{cases} X \sim X^p \oplus Y^q, \\ Y^t \sim X^{r(p+(n-1)(p-1))} \oplus Y^{nqr+s}. \end{cases} \tag{3.24}$$

Now observe that the discriminant of the quintuple $(p, q, r(p+(n-1)(p-1)), nqr + s, t)$ is equal to

$$(p-1)(nqr + s - t) - rq(p + (n-1)(p-1)) = (p-1)(s-t) - rq = \nabla.$$

Moreover, it follows from our hypothesis on ∇ that

$$\nabla \mid (r(p + (n-1)(p-1)) + nqr + s - t) = nr(p + q - 1) + (s + r - t).$$

Therefore according to (3.24) and Case 3, $X \sim Y$.

Subcase 4.2.2: $q = 0$. Then $p \geq 2$ and (3.24) holds. By applying Remark 2.2(b) with $d = 1$ in the first condition of (3.14), and Remark 2.2(a) with $d = s$ in the second, we see that (3.15) holds. Next note that the discriminant of $(1, p-1, t-s, s, r+s)$ is equal to $(p-1)(t-s) = -\nabla$. Furthermore, our hypothesis on ∇ implies that

$$\nabla \mid ((t-s) + s - (r+s)) = -(r + s - t).$$

Hence by (3.15) and Subcase 4.2.1, $X \sim Y$.

Case 5: $s < t$, $p = 0$ and $q \geq 2$. In this case, $\nabla = t - s - rq$ and (1.4) implies that $X \sim Y^q$ and $Y^t \sim X^r \oplus Y^s$. Hence $Y^t \sim Y^{rq} \oplus Y^s \sim Y^{rq+s}$. There are two subcases: $t \leq rq + s$ and $t > rq + s$.

Subcase 5.1: $t \leq rq + s$. Then (3.16) holds. Since the discriminant of $(0, q, 0, rq+s, t)$ is equal to $-(rq+s-t) = \nabla$, by (3.16), our hypothesis and Case 3, $X \sim Y$.

Subcase 5.2: $t > rq + s$. Then (3.18) holds. Now notice that the discriminant of the quintuple $(0, q, 0, t, rq+s)$ is equal to $-(t-(rq+s)) = -\nabla$. According to (3.18), our hypothesis and Case 3, $X \sim Y$. ∎

4. Schroeder–Bernstein sextuples for Banach spaces. In this short section, inspired by Theorem 1.2, we introduce the notion of Schroeder–Bernstein sextuples for Banach spaces and pose an intriguing problem on decomposition methods in Banach spaces which will be called the *Square-cube Schroeder–Bernstein Problem for Banach spaces* (see Problem 4.4).

Definition 4.1. A sextuple (p, q, r, s, u, v) in $\mathbb{N}$ with $p + q + u \geq 3$, $(p, q) \neq (0, 0)$, $r + s + t \geq 3$, $(r, s) \neq (0, 0)$, $u \geq 1$ and $v \geq 1$ is a *Schroeder–Bernstein sextuple for Banach spaces* (for short, SBs) if $X \sim Y$ whenever the Banach spaces X and Y satisfy (1.1) and

$$\begin{cases} X^u \sim X^p \oplus Y^q, \\ Y^v \sim X^r \oplus Y^s. \end{cases}$$

We also say that the number $\diamondsuit = (p-u)(s-v) - rq$ is the *discriminant* of the sextuple (p, q, r, s, u, v).

As a direct consequence of Theorem 1.2 we deduce:

Corollary 4.2. *Let (p, q, r, s, u, v) be a sextuple in $\mathbb{N}$ with $p+q+u \geq 3$, $(p, q) \neq (0, 0)$ $r + s + v \geq 3$, $(r, s) \neq (0, 0)$, $u = 1$ or $v = 1$ or $(p, q) = (1, 0)$ or $(r, s) = (0, 1)$. Then (p, q, r, s, u, v) is a SBs if and only if*

(a) $\diamondsuit \neq 0$;
(b) $\diamondsuit \mid (p + q - u)$ *and* $\diamondsuit \mid (r + s - v)$.

The following conjecture seems to be natural:

Conjecture 4.3. Let (p, q, r, s, u, v) be a sextuple in $\mathbb{N}$ with $p + q + u \geq 3$, $(p, q) \neq (0, 0)$, $r + s + v \geq 3$, $(r, s) \neq (0, 0)$, $u \geq 1$ and $v \geq 1$. If (p, q, r, s, u, v) is a SBs then $u = 1$ or $v = 1$ or $(p, q) = (1, 0)$ or $(r, s) = (0, 1)$.

Note that, in view of Corollary 4.2, a positive answer to this conjecture would characterize the SBs. But we do not even know how to prove the following simple case of it:

Problem 4.4. Give non-isomorphic Banach spaces X and Y satisfying (1.1) with

$$X^2 \sim X^3 \quad \text{and} \quad Y^2 \sim Y^3.$$

Finally, observe that, by Pełczyński's decomposition method, a positive solution to this Square-cube Schroeder–Bernstein Problem would imply a negative solution to the Square-cube Problem mentioned in the introduction.

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