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Krasinkiewicz Maps from Compacta to Polyhedra

by

Eiichi MATSUHASHI

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Summary. We prove that the set of all Krasinkiewicz maps from a compact metric space to a polyhedron (or a 1-dimensional locally connected continuum, or an n-dimensional Menger manifold, $n \geq 1$) is a dense G_δ-subset of the space of all maps. We also investigate the existence of surjective Krasinkiewicz maps from continua to polyhedra.

1. Introduction. In this paper all spaces are separable and metrizable, and all maps are continuous. We denote the interval [0, 1] by I. A compact metric space is called a *compactum*, and a *continuum* means a connected compactum. Let X and Y be compacta. Then $C(X, Y)$ denotes the set of all continuous maps from X to Y endowed with the sup metric.

If X, Y are compacta, a map $f : X \to Y$ is called a *Krasinkiewicz map* if any continuum in X either contains a component of a fiber of f or is contained in a fiber of f (cf. [5] and [8]). In [5] J. Krasinkiewicz showed that the set of all Krasinkiewicz maps from a compactum to a 1-dimensional manifold is a dense subset of the space of all maps. In fact, he proved this result as follows. First, he defined a map $f : X \to Y$ to be *singular* if there exists a Bing map (see [3], [4], [6] and [10]) $F : X \times I \to Y$ such that $F_0 = f$. Next he proved that the set of all singular maps from a compactum to an n-dimensional manifold ($n \geq 1$) is a dense subset of the space of all maps, and also that a singular map from a compactum to a 1-dimensional manifold is a Krasinkiewicz map. On the other hand, in [8] M. Levin and W. Lewis proved that the set of Krasinkiewicz maps from a compactum to I is a dense subset of the space of all maps by their own method.

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In this paper, we prove the following theorem.

THEOREM 1.1. *Let X be a compactum and P a polyhedron. Then the set of all Krasinkiewicz maps from X to P is a dense G_δ-subset of $C(X,P)$.*

In Section 3, as an application of Theorem 1.1, we prove that the set of all Krasinkiewicz maps from a compact metric space to a 1-dimensional locally connected continuum (or an n-dimensional Menger manifold, $n \geq 1$) is a dense G_δ-subset of the space of all maps. Also, we investigate the existence of surjective Krasinkiewicz maps from continua to polyhedra.

2. Main theorem. In this section we prove Theorem 1.1. First we indroduce our notation and terminology. By $\mathring{I}$ we denote the manifold interior of I, $\mathring{I} = (0,1)$, and by ∂I its manifold boundary, $\partial I = \{0,1\}$. Analogous symbols are used for the unit cube I^n. Let $f : X \to Y$ be a mapping. For a point $x \in X$ we denote by $C(x,f)$ the component of the fiber $f^{-1}(f(x))$ containing x, and call it the *component of f at x*. Any component of f at a point is said to be a *component of f*. A mapping $g : X \to Y$ is said to be an *alteration of f on U over V*, where $U \subset X$ and $V \subset Y$ are arbitrary sets, if $g(x) = f(x)$ for each $x \notin U \cap f^{-1}(V)$, and $g(U \cap f^{-1}(V)) \subset V$.

The easy proof of the next lemma is left to the reader.

LEMMA 2.1. *Let $q : X \to X'$ be a mapping between compacta, and let $u : X \to I$ and $u' : X' \to I$ be mappings such that $u = u' \circ q$.*

(i) *If $v' : X' \to I$ is an alteration of u' on G' over (a,b), then $v = v' \circ q$ is an alteration of u on $G = q^{-1}(G')$ over (a,b). Moreover, $d(u,v) \leq d(u',v')$.*
(ii) *If q is a monotone surjection and C' is a component of u' then $C = q^{-1}(C')$ is a component of u.*

The next lemma is a strengthening of a result proved by M. Levin [7].

LEMMA 2.2. *Let $u : X \to I$ be a mapping of a compactum X and let $(a,b) \subset I$. Suppose Z is a nonvoid closed subset of X whose components lie in fibers of u, and $u(Z) \subset (a,b)$. Then, for any open neighborhood G of Z in X, u can be approximated by mappings $v : X \to I$ such that:*

(i) *v is an alteration of u on $G \setminus Z$ over (a,b),*
(ii) *each component of Z is a component of v.*

Proof. Let X' denote the quotient space obtained from X by shrinking the components of Z to points, and let $q : X \to X'$ denote the quotient mapping. Then $Z' = q(Z)$ is a closed 0-dimensional subset of X' and $G' = q(G)$ is an open neighborhood of Z' in X'. Since components of Z lie in fibers of u, there is a map $u' : X' \to I$ such that $u = u' \circ q$. Fix $\varepsilon > 0$. In view of Lemma 2.1, if $v' : X' \to I$ is an alteration of u' on $G' \setminus Z'$ over (a,b)

with $d(v', u') < \varepsilon$, then $v = v' \circ q$ satisfies the conclusion of our lemma, and $d(v, u) < \varepsilon$. Therefore, without loss of generality, we may assume that

(1) $\dim Z = 0$.

Let $U = G \cap u^{-1}((a, b))$. Since $U \subset X$ is open and $Z \subset U$, by (1) there exist open sets $U_1, \ldots, U_{n(\emptyset)}$ such that for each $i, i' = 1, \ldots, n(\emptyset)$ we have:

(2_1) $Z \subset U_1 \cup \cdots \cup U_{n(\emptyset)}$, $Z \cap U_i \neq \emptyset$, $\operatorname{cl} U_i \subset U$,
(3_1) $\operatorname{cl} U_i \cap \operatorname{cl} U_{i'} = \emptyset$ for $i \neq i'$,
(4_1) $\operatorname{diam} U_i < 1/2^1$, $\operatorname{diam} u(\operatorname{cl} U_i) < \varepsilon/2^1$.

Hence $Z_i = Z \cap U_i$ is a nonvoid closed subset of the open set U_i. Therefore, we can repeat the procedure for U_i, Z_i in place of U, Z; and so on. Thus, for each $k \geq 1$, we construct open sets U_α, where $\alpha = (i_1, \ldots, i_k)$, satisfying conditions (2_k)–(4_k). Then we pick open sets V_α, W_α and intervals $[a_\alpha, b_\alpha] \subset [a, b]$ such that

(5_k) $\operatorname{cl} U_\alpha \subset V_\alpha \subset \operatorname{cl} V_\alpha \subset W_\alpha \subset \operatorname{cl} W_\alpha \subset U_\beta$, where $\beta = (i_1, \ldots, i_{k-1})$,
(6_k) $\operatorname{cl} W_\alpha \cap \operatorname{cl} W_{\alpha'} = \emptyset$, where $\alpha' = (i_1, \ldots, i_{k-1}, i'_k)$, $i_k \neq i'_k$,
(7_k) $\operatorname{diam} W_\alpha < 1/2^k$,
(8_k) $u(\operatorname{cl} W_\alpha) \subset [a_\alpha, b_\alpha]$ and $\operatorname{diam}[a_\alpha, b_\alpha] < \varepsilon/2^k$.

Moreover, we can choose the intervals so that

(9) all the initial points a_α are different.

Then take any mappings $v_\alpha : \operatorname{cl} W_\alpha \setminus U_\alpha \to [a_\alpha, b_\alpha]$ which map ∂V_α to a_α, and which coincide with u on $\partial W_\alpha \cup \partial U_\alpha$. Finally, define $v : X \to I$ to be the union of the maps v_α on $\operatorname{cl} W_\alpha \setminus U_\alpha$, and u on the complement of these sets.

It follows from (8) that $d(v, u) < \varepsilon$. Suppose there is a nondegenerate component D of v which meets Z. It follows from (5) and (7) that D meets two different boundaries ∂V_α. Hence $v(D)$ contains at least two different points a_α, a contradiction. ∎

LEMMA 2.3. *Let $f : X \to I$ be a mapping of a compactum X and let G be an open subset of X. Then f can be approximated by mappings $g : X \to I$ such that*

(i) *g is an alteration of f on G over $\mathring{I}$,*
(ii) *if $L \subset G$ is a continuum lying in no fiber of g, then L contains a component of g.*

Proof. Fix $\varepsilon > 0$. We need a map $g : X \to I$ which is ε-close to f and satisfies the conclusion of our lemma. It will be defined as the limit of a sequence of maps $u_0, u_1, \ldots$ from X to I. To define the latter, first choose a sequence of closed intervals $[a_1, b_1], [a_2, b_2], \ldots$ in $\mathring{I}$, and a sequence of Cantor sets $C_1, C_2, \ldots$ satisfying the following conditions (for $i, j = 1, 2, \ldots$):

(1) diam $[a_i, b_i] < \varepsilon/2^i$,
(2) $C_i \subset (a_i, b_i)$,
(3) $([a_i, b_i] \cap [a_j, b_j] \neq \emptyset$ and $i < j) \Rightarrow ([a_j, b_j] \subset (a_i, b_i) \setminus C_i)$,
(4) each nonvoid open subset of I contains some $[a_i, b_i]$.

Next, take an increasing sequence $G_0, G_1, \ldots$ of open subsets of X such that $G = G_0 \cup G_1 \cup \cdots$ and $\operatorname{cl} G_{i-1} \subset G_i$ for each i. Then define the mappings u_{i-1} by induction. Put

(5) $u_0 = f$.

If u_{i-1} has been defined, define u_i as follows. Take a continuous surjection $\varphi_i : C_i \to 2^X$, where 2^X denotes the hyperspace of closed subsets of X, and set

$$Z_i = \bigcup \{\varphi_i(c) \cap u_{i-1}^{-1}(c) \cap \operatorname{cl} G_{i-1} \mid c \in C_i\}.$$

Then Z_i is a closed subset of X, its components lie in fibers of u_{i-1}, and $u_{i-1}(Z_i) \subset C_i$. Therefore, by Lemma 2.2, there is a mapping $u_i : X \to I$ such that

(6) u_i is an alteration of u_{i-1} on $G_i \setminus Z_i$ over (a_i, b_i),
(7) each component of Z_i is a component of $u_i^{-1}(c)$ for some $c \in C_i$.

Let us observe that

(8) if $L \subset G_{i-1}$ is a continuum and $C_i \subset u_{i-1}(L)$, then L contains a component of $u_i^{-1}(c)$ for some $c \in C_i$.

In fact, there is $c_0 \in C_i$ such that $\varphi_i(c_0) = L$. Since $c_0 \in u_{i-1}(L)$, we have $L \cap u_{i-1}^{-1}(c_0) \neq \emptyset$. On the other hand,

$$L \cap u_{i-1}^{-1}(c_0) = \varphi_i(c_0) \cap u_{i-1}^{-1}(c_0) \cap \operatorname{cl} G_{i-1}$$

is a subset of Z_i. Let D be a component of $L \cap u_{i-1}^{-1}(c_0)$. Then D is a component of Z_i. Since $u_i(D) = u_{i-1}(D) = \{c_0\}$, by (7) we infer that D is a component of $u_i^{-1}(c_0)$. Thus, L contains a component of $u_i^{-1}(c_0)$, which proves (8).

From (3) and (6) we infer that

(9) if $u_i(x) \in [a_j, b_j]$ for some $j > i$, and $j_0 = \min\{j > i \mid u_i(x) \in [a_j, b_j]\}$, then $u_k(x) \in [a_{j_0}, b_{j_0}]$ for each $k > i$.

Now we are ready to complete the proof. We are going to show that $g = \lim u_i$ satisfies the conclusion of our lemma. The limit is well defined in view of (6) and (1). Then, by (1), (5) and (6), g is ε-close to f. Condition (i) follows from (6). It remains to prove (ii). To this end, consider a continuum $L \subset G$ such that $g(L)$ is not a singleton. Then there is $i \geq 1$ such that $L \subset G_{i-1}$ and $[a_i, b_i] \subset u_{i-1}(L)$. By (2) and (8), L contains a component of $u_i^{-1}(c)$ for some $c \in C_i$. Therefore, it is enough to show that

(10) $g^{-1}(c) = u_i^{-1}(c)$.

By (3) and (6) one easily sees that $u_i^{-1}(c) \subset g^{-1}(c)$. To prove the converse, suppose on the contrary that there is a point $x \in g^{-1}(c) \setminus u_i^{-1}(c)$. Then $g(x) = c \neq u_i(x)$. It follows that $u_i(x) \in [a_j, b_j]$ for some $j > i$ (otherwise $g(x) = u_i(x)$). By (9), $u_k(x) \in [a_{j_0}, b_{j_0}]$ for each $k \geq i$, where $j_0 > i$. It follows that $g(x) \in [a_{j_0}, b_{j_0}]$. On the other hand, $[a_{j_0}, b_{j_0}] \subset (a_i, b_i) \setminus C_i$, by (3). Hence $c = g(x) \notin C_i$, a contradiction. This proves (10), and ends the entire proof. ∎

LEMMA 2.4. *Let X be a compactum and let $f = (f_1, \ldots, f_n) : X \to I^n$ be a mapping such that $f|f^{-1}(\partial I^n) : f^{-1}(\partial I^n) \to \partial I^n$ is a Krasinkiewicz map. Then f can be approximated by Krasinkiewicz maps $g : X \to I^n$ which are alterations of f on $f^{-1}(\mathring{I}^n)$ over $\mathring{I}^n$. In other words, each g extends $f|f^{-1}(\partial I^n)$ and transforms $f^{-1}(\mathring{I}^n)$ into $\mathring{I}^n$.*

Proof. Fix $\varepsilon > 0$ and let $G = f^{-1}(\mathring{I}^n)$. By Lemma 2.3, for each $i = 1, \ldots, n$, there is a mapping $g_i : X \to I$ which is $(\varepsilon/\sqrt{n})$-close to f_i and such that

(1) g_i is an alteration of f_i on G over $\mathring{I}$,
(2) if $L \subset G$ is a continuum lying in no fiber of g_i, then L contains a component of g_i.

We shall show that $g = (g_1, \ldots, g_n) : X \to I^n$ satisfies the conclusion. First note that g is ε-close to f, and g is an alteration of f on $f^{-1}(\mathring{I}^n)$ over $\mathring{I}^n$. It remains to show that g is a Krasinkiewicz map. To this end, consider a continuum $L \subset X$ such that $g(L)$ is not a singleton. We must show that L contains a component of g. If $L \cap G = \emptyset$, then $L \subset f^{-1}(\partial I^n)$ hence, by our hypothesis L contains a component of $f|f^{-1}(\partial I^n)$, which is also a component of f. Next, assume $L \cap G \neq \emptyset$. Then there is a continuum $L' \subset L \cap G$ such that $g(L')$ is not a singleton. Hence $g_i(L')$ is not a singleton for some i. By Lemma 2.3, L' contains a component $C(x, g_i)$. Since $C(x, g) \subset C(x, g_i) \subset L' \subset L$, this ends the proof. ∎

Before the proof of Theorem 2.5 we give some notations. If X and Y are compacta, then $K(X, Y)$ denotes the set of all Krasinkiewicz maps from X to Y. If $\mathcal{K}$ is a simplicial complex, we write $\mathcal{K}^n = \{\sigma \in \mathcal{K} \mid \dim \sigma \leq n\}$ and denote the polyhedron of $\mathcal{K}$ by $|\mathcal{K}|$. Let X be a compactum, $\mathcal{K}$ a simplicial complex and $f : X \to |\mathcal{K}|$ a mapping. A mapping $g : X \to |\mathcal{K}|$ is said to be a *$\mathcal{K}$-modification* of f if $f(x) \in \mathring{\sigma}$ implies $g(x) \in \mathring{\sigma}$ for every $x \in X$ and $\sigma \in \mathcal{K}$. One easily sees that

(∗) *if g is a $\mathcal{K}$-modification of f and $\mathcal{L}$ is a subcomplex of $\mathcal{K}$ then $g^{-1}(|\mathcal{L}|) = f^{-1}(|\mathcal{L}|)$ and the mapping $g^{-1}(|\mathcal{L}|) \to \mathcal{L}$ determined by g is an $\mathcal{L}$-modification of the mapping $f^{-1}(|\mathcal{L}|) \to \mathcal{L}$ determined by f.*

THEOREM 2.5. *Let X be a compactum and P a polyhedron. Then $K(X,P)$ is a dense subset of $C(X,P)$.*

Proof. For any $\varepsilon > 0$ there is a simplicial complex $\mathcal{K}$ such that $P = |\mathcal{K}|$ and $\operatorname{mesh}\mathcal{K} < \varepsilon$. Therefore our theorem readily follows from the following assertion:

(∗∗) *for any mapping $f : X \to |\mathcal{K}|$, where X is a compactum and $\mathcal{K}$ is a simplicial complex, there is a Krasinkiewicz map $g : X \to |\mathcal{K}|$ which is a $\mathcal{K}$-modification of f.*

We prove (∗∗) by induction on $\dim\mathcal{K}$. For $\dim\mathcal{K} = 0$ it is obvious because every map from a compactum into a discrete space is a Krasinkiewicz map. Then consider any mapping $f : X \to |\mathcal{K}|$, where $n = \dim\mathcal{K} > 0$, and assume (∗∗) holds for any $(n-1)$-dimensional complex. It suffices to find a Krasinkiewicz map $g : X \to |\mathcal{K}|$ which is a $\mathcal{K}$-modification of f. To this end consider the mapping $f_0 : f^{-1}(|\mathcal{K}^{n-1}|) \to |\mathcal{K}^{n-1}|$ determined by f (i.e. $f_0(x) = f(x)$ for every x). By the inductive assumption there is a Krasinkiewicz map $g_0 : f^{-1}(|\mathcal{K}^{n-1}|) \to |\mathcal{K}^{n-1}|$ which is a $\mathcal{K}^{n-1}$-modification of f_0. For any n-simplex $\sigma \in \mathcal{K} \setminus \mathcal{K}^{n-1}$, by (∗), $g_0^{-1}(\partial\sigma) = f^{-1}(\partial\sigma)$ and the mapping $g_{\partial\sigma} : g_0^{-1}(\partial\sigma) \to \partial\sigma$ determined by g_0 is a $\partial\sigma$-modification of the mapping $f^{-1}(\partial\sigma) \to \partial\sigma$ determined by f. Since σ is homeomorphic to I^n and the mapping $g_{\partial\sigma}$ is a Krasinkiewicz map, by Lemma 2.4, there is a Krasinkiewicz map $g_\sigma : f^{-1}(\sigma) \to \sigma$ such that $g_\sigma(x) = g_0(x)$ for all $x \in f^{-1}(\partial\sigma)$, and $g_\sigma(x) \in \mathring{\sigma}$ if $f(x) \in \mathring{\sigma}$. Now we define the mapping $g : X \to |\mathcal{K}|$ to be g_0 on the inverse $f^{-1}(|\mathcal{K}^{n-1}|)$, and g_σ on $f^{-1}(\sigma)$ for each $\sigma \in \mathcal{K} \setminus \mathcal{K}^{n-1}$. One easily verifies that g is well defined and has the desired properties, which ends the proof. ∎

A set $A \subset X$ is said to be *residual in* X if A contains a dense G_δ-subset in X. In [8] M. Levin and W. Lewis claimed that if X is a compactum, then $K(X,I)$ is a residual set of $C(X,I)$. In fact, they claimed that the set of maps f in $C(X,I)$ satisfying the following condition is a dense G_δ-subset of $C(X,I)$:

(#) for every continuum $F \subset X$ such that $f(F)$ is not a singleton there exists a subset $D \subset f(F)$ dense in $f(F)$ such that for every $d \in D$, $f^{-1}(d) \cap F$ is the union of some components of $f^{-1}(d)$.

Contrary to this assertion, one can show that there exist continua X such that the set of maps in $C(X,I)$ satisfying (#) is not dense in $C(X,I)$. For instance, take any mapping $f : I \times I \to I$ close enough to the first projection $\mathrm{pr}_1 : I \times I \to I$. Then f does not satisfy (#). In fact, let $F = I \times \{1/2\}$. Then $f(F)$ is not a singleton, but if $d \in f(F)$ and $1/2 \in f(F)$ are suffciently near, then there exists a component C of $f^{-1}(d)$ such that $C \cap (I \times \{0\}) \neq \emptyset \neq C \cap (I \times \{1\})$. So there does not exist a subset $D \subset f(F)$ as in (#).

So $K(X,I)$ need not be a residual subset of $C(X,I)$. However, we prove that if X and Y are compacta, then $K(X,Y)$ is a G_δ-subset of $C(X,Y)$.

If $\delta > 0$ and $A \subset X$, we denote the set $\{x \in X \mid d(x,A) < \delta\}$ by $B(A,\delta)$.

THEOREM 2.6. *Let X and Y be compacta. Then $K(X,Y)$ is a G_δ-subset of $C(X,Y)$.*

Proof. For each $m, n \in \mathbb{N}$, let $H_{m,n}$ be the set of all maps $f \in C(X,Y)$ satisfying the condition:

($\star$) if $L \subset X$ is a subcontinuum with $\operatorname{diam} f(L) \geq 1/n$, then there exists $x \in L$ such that $C(x,f) \subset B(L,1/m)$.

We claim that

(A) $H_{m,n}$ is an open subset of $C(X,Y)$,
(B) $K(X,Y) = \bigcap_{m,n\in\mathbb{N}} H_{m,n}$.

To prove (A), we show that $C(X,Y) \setminus H_{m,n}$ is a closed subset of $C(X,Y)$. Note that $C(X,Y) \setminus H_{m,n}$ is the set of all maps $f \in C(X,Y)$ satisfying:

($\star\star$) there exists a subcontinuum $K \subset X$ such that $\operatorname{diam} f(K) \geq 1/n$ and $C(x,f) \not\subset B(K,1/m)$ for each $x \in K$.

Let $f \in \operatorname{cl}(C(X,Y) \setminus H_{m,n})$. Then there exists a sequence of maps $\{f_i\}_{i=1}^\infty \subset C(X,Y) \setminus H_{m,n}$ such that $\lim f_i = f$. For each $i = 1,2,\ldots$, there exists a subcontinuum $K_i \subset X$ such that $\operatorname{diam} f_i(K_i) \geq 1/n$ and $C(x,f_i) \not\subset B(K_i,1/m)$ for each $x \in K_i$. We may assume that K_i converges to a subcontinuum $K \subset X$. Then it is easy to see that $\operatorname{diam} f(K) \geq 1/n$. Let $x \in K$. Then there exists a sequence $\{x_i\}_{i=1}^\infty \subset X$ such that $x_i \in K_i$ for each $i = 1,2,\ldots$, and $\lim x_i = x$. Hence $C(x_i,f_i) \not\subset B(K_i,1/m)$ for each $i = 1,2,\ldots$. We may assume that $C(x_i,f_i)$ converges to a subcontinuum $C \subset X$. Then it is easy to see that $x \in C \subset C(x,f)$ and $C \not\subset B(K,1/m)$. So $f \in C(X,Y) \setminus H_{m,n}$. This completes the proof of (A).

Next we prove (B). It is easy to see that $K(X,Y) \subset \bigcap_{m,n\in\mathbb{N}} H_{m,n}$. So we only prove the reverse inclusion. Let $f \in \bigcap_{m,n\in\mathbb{N}} H_{m,n}$ and let $L \subset X$ be a subcontinuum such that $\operatorname{diam} f(L) > 0$. Let $L_1 = L$. Now we prove that there exists a subcontinuum $L_2 \subset L_1$ such that $\operatorname{diam} f(L_2) > 0$ and $C(x,f) \subset B(L_1,1/2)$ for each $x \in L_2$. Since $\operatorname{diam} f(L_1) > 0$, there exists $n_1 \in \mathbb{N}$ such that $\operatorname{diam} f(L_1) \geq 1/n_1$. Since $f \in H_{2,n_1}$, there exists $x_0 \in L_1$ such that $C(x_0,f) \subset B(L_1,1/2)$. Since $B(L_1,1/2)$ is open and $C(x_0,f)$ is a component of the compact set $f^{-1}(f(x_0))$, there exist two disjoint open sets U, U' such that

(1) $f^{-1}(f(x_0)) \subset U \cup U'$,
(2) $C(x_0,f) \subset U \subset B(L_1,1/2)$.

Since f is a closed mapping, by (1) there is an open neighborhood V of $f(x_0)$ in Y such that

(3) $f^{-1}(\mathrm{cl}\, V) \subset U \cup U'$.

Since $\mathrm{diam}\, f(L_1) > 0$, we can also assume that

(4) $f(L_1) \setminus V \neq \emptyset$.

Consider the open set $W = f^{-1}(V) \cap U$. Note that

(5) $C(x_0, f) \subset W$.

Let C_0 denote the component of $W \cap L_1$ which contains x_0. We are going to show that $L_2 = \mathrm{cl}\, C_0$ is a continuum with the desired properties. First we show that

(6) $\mathrm{diam}\, f(L_2) > 0$.

Since L_1 is a continuum and $L_1 \not\subset W$ by (4), we infer that $\mathrm{cl}\, C_0 \cap \partial_L(W \cap L_1) \neq \emptyset$. Note that $\partial_L(W \cap L_1) \subset \partial W$. The set $f^{-1}(\mathrm{cl}\, V) \cap U$ is closed, because it is a closed subset of the closed set $f^{-1}(\mathrm{cl}\, V)$, by (3). It follows that $\mathrm{cl}\, W \subset f^{-1}(\mathrm{cl}\, V) \cap U$, hence $\partial W = \mathrm{cl}\, W \setminus W \subset (f^{-1}(\mathrm{cl}\, V) \cap U) \setminus (f^{-1}(V) \cap U) \subset f^{-1}(\partial V)$. Consequently, $\partial_L(W \cap L_1) \subset f^{-1}(\partial V)$, hence $\mathrm{cl}\, C_0 \cap f^{-1}(\partial V) \neq \emptyset$. Therefore, $f(\mathrm{cl}\, C_0)$ contains $f(x_0)$ and meets ∂V, which proves (6).

Now consider any $x \in \mathrm{cl}\, C_0$. In order to end the proof it is enough to show that

(7) $C(x, f) \subset B(L_1, 1/2)$.

Since $f^{-1}(\mathrm{cl}\, V) \cap U$ is closed, $\mathrm{cl}\, C_0 \subset f^{-1}(\mathrm{cl}\, V) \cap U$, so $x \in f^{-1}(\mathrm{cl}\, V) \cap U$. Therefore, $x \in f^{-1}(f(x)) \cap U$. The set $f^{-1}(f(x)) \cap U$ is closed and open in $f^{-1}(f(x))$, as $f^{-1}(\mathrm{cl}\, V) \cap U$ is closed and open in $f^{-1}(\mathrm{cl}\, V)$, by (3), and $f^{-1}(f(x)) \subset f^{-1}(\mathrm{cl}\, V)$. It follows that $C(x, f) \subset f^{-1}(f(x)) \cap U \subset U$. Thus, by (2), we get (7).

By induction, we can find a decreasing sequence $\{L_k\}_{k=1}^{\infty}$ of subcontinua of L such that if $k \in \mathbb{N}$, then

- $\mathrm{diam}\, f(L_k) > 0$,
- $C(x, f) \subset B(L_k, 1/(k+1))$ for each $x \in L_{k+1}$.

Then it is easy to see that $C(x, f) \subset L$ for each $x \in \bigcap_{k=1}^{\infty} L_k$ $(\subset L)$. This implies $f \in K(X, Y)$, and completes the proof. ■

By Theorems 2.5 and 2.6, we get Theorem 1.1.

3. Applications. In this section we give some applications of Theorem 1.1. First we prove the following result.

PROPOSITION 3.1. *Let X, Y and Z be compacta. If $f : X \to Y$ is a Krasinkiewicz map and $g : Y \to Z$ is a 0-dimensional map, then $g \circ f : X \to Z$ is a Krasinkiewicz map.*

Proof. Let $h = g \circ f$ and $L \subset X$ a continuum such that $h(L)$ is not a singleton. Since $f(L)$ is not a singleton and f is a Krasinkiewicz map, there exists $y \in Y$ such that L contains a component C of $f^{-1}(y)$. Since g is a 0-dimensional map, $\{y\}$ is a component of $g^{-1}(g(y))$. Then C is a component of $h^{-1}(g(y))$. This completes the proof. ■

By Theorem 1.1, Proposition 3.1 and the proofs of Corollaries 4.3 and 4.4 in [10] we obtain the following results.

THEOREM 3.2. *Let X be a compactum and Y a 1-dimensional locally connected continuum. Then $K(X,Y)$ is a dense G_δ-subset of $C(X,Y)$.*

THEOREM 3.3. *Let X be a compactum and M an n-dimensional Menger manifold ($n \geq 1$). Then $K(X,M)$ is a dense G_δ-subset of $C(X,M)$.*

(See [1] for properties of Menger manifolds.)

We denote the space of all surjective maps from X to Y by $C_{\mathrm{s}}(X,Y)$. Also, we denote the set of all surjective Krasinkiewicz maps from X to Y by $K_{\mathrm{s}}(X,Y)$. By Theorem 1.1, Proposition 3.1 and the argument in [3] we get the following result.

THEOREM 3.4. *Let X be a continuum and P a connected polyhedron. Then $K_{\mathrm{s}}(X,P)$ is a dense G_δ-subset of $C_s(X,P)$.*

By Proposition 3.1, Theorem 3.4 and the proofs of Theorem 6 and Corollary 7 in [3] we get the following results.

THEOREM 3.5. *Let X be a continuum and Y a 1-dimensional locally connected continuum. Then $K_{\mathrm{s}}(X,Y)$ is a dense G_δ-subset of $C_{\mathrm{s}}(X,Y)$.*

THEOREM 3.6. *Let X be a continuum and M an n-dimensional Menger manifold ($n \geq 1$). Then $K_{\mathrm{s}}(X,M)$ is a dense G_δ-subset of $C_{\mathrm{s}}(X,M)$.*

Also we give an application of Theorem 3.4. We need the following well known theorem.

THEOREM 3.7 (M. Brown [2]). *Let $\{X_i, f_i\}$ be an inverse sequence such that X_i is compact for each $i = 1, 2, \ldots$. Then there exist $\varepsilon_1 > \varepsilon_2 > \cdots > 0$ such that if $\{g_i\}_{i=1}^{\infty}$ ($g_i : X_{i+1} \to X_i$ for each $i = 1, 2, \ldots$) satisfies $d(f_i, g_i) < \varepsilon_i$ for each $i = 1, 2, \ldots$, then $\varprojlim\{X_i, g_i\}$ is homeomorphic to $\varprojlim\{X_i, f_i\}$.*

By Theorems 3.4 and 3.7, we obtain the following result.

COROLLARY 3.8. *For each continuum X, there exists an inverse sequence $\{P_i, g_i\}$ such that P_i is a compact connected polyhedron, $g_i : P_{i+1} \to P_i$ is a surjective Krasinkiewicz map for each $i = 1, 2, \ldots$, and $X = \varprojlim\{P_i, g_i\}$.*

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Eiichi Matsuhashi
Institute of Mathematics
University of Tsukuba
Ibaraki, 305-8571 Japan
E-mail: aichi@math.tsukuba.ac.jp

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