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On Functions with the Cauchy Difference Bounded by a Functional

by

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Summary. K. Baron and Z. Kominek [2] have studied the functional inequality

$$f(x+y)-f(x)-f(y)\geq\phi(x,y),\qquad x,y\in X,$$

under the assumptions that X is a real linear space, ϕ is homogeneous with respect to the second variable and f satisfies certain regularity conditions. In particular, they have shown that ϕ is bilinear and symmetric and f has a representation of the form $f(x)=\frac{1}{2}\phi(x,x)+L(x)$ for $x\in X$, where L is a linear function.

The purpose of the present paper is to consider this functional inequality under different assumptions upon X, f and ϕ. In particular we will give conditions which force biadditivity and symmetry of ϕ and the representation $f(x)=\frac{1}{2}\phi(x,x)-A(x)$ for $x\in X$, where A is a subadditive function.

Let $(X,+)$ be an abelian group. We consider the functional inequality

$$f(x+y)-f(x)-f(y)\geq\phi(x,y),\qquad x,y\in X, \tag{1}$$

where ϕ: $X\times X\to\mathbb{R}$ and f: $X\to\mathbb{R}$ are unknown mappings.

It is easy to check that if ϕ: $X\times X\to\mathbb{R}$ is biadditive and symmetric, A: $X\to\mathbb{R}$ is subadditive and f: $X\to\mathbb{R}$ is defined by the formula $f(x):=\frac{1}{2}\phi(x,x)-A(x)$ for $x\in X$, then (1) holds. We are going to provide conditions under which the converse implication is valid.

PROPOSITION. *If* f: $X\to\mathbb{R}$ *and* ϕ: $X\times X\to\mathbb{R}$ *satisfy* (1) *and*

$$\phi(x,-x)\geq-\phi(x,x),\quad x\in X, \tag{2}$$

then: (a) $f(0)\leq 0$; (b) $f(x)+f(-x)\leq\phi(x,x)$ for $x\in X$; (c) $f(2x)\geq 3f(x)+f(-x)$ for $x\in X$.

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Proof. The assumption (2) implies that $\phi(0,0) \geq 0$; thus applying (1) with $x = 0$ and $y = 0$ we get $f(0) \leq 0$. Using this and substituting $y := -x$ in (1) we derive (b). Substituting $y := x$ in (1) and using (b) proves (c). This completes the proof.

In what follows we make use of a result of Karol Baron (see S. Rolewicz [4, Lemma 5.7]). A careful inspection of the original proof allows us to weaken certain assumptions of this lemma. The original result reads as follows.

Lemma (K. Baron). *Assume that $f\colon X \to \mathbb{R}$ and $\phi\colon X \times X \to \mathbb{R}$ satisfy* (1)*. If f is even, $f(2x) = 4f(x)$ and $\phi(x, \cdot)$ is odd for every $x \in X$, then there exists a biadditive and symmetric functional $B\colon X \times X \to \mathbb{R}$ such that $\phi = 2B$ and $f(x) = B(x,x)$ for every $x \in X$.*

We have the following modification of this lemma.

Lemma 1. *Assume that $f\colon X \to \mathbb{R}$ and $\phi\colon X \times X \to \mathbb{R}$ satisfy* (1)*. If*

$$\phi(x,-y) \geq -\phi(x,y), \qquad x, y \in X, \tag{3}$$

and

$$f(2x) \leq 4f(x), \qquad x \in X, \tag{4}$$

then

$$f(x) = \frac{1}{2}\,\phi(x,x), \qquad x \in X.$$

Moreover, ϕ is biadditive and symmetric.

Proof. Using the inequality (c) of the Proposition and (4) we see that $f(x) \geq f(-x)$ for $x \in X$, which proves that f is even. Setting $-y$ instead of y in (1) we obtain

$$f(x-y) - f(x) - f(-y) \geq \phi(x,-y) \geq -\phi(x,y), \qquad x, y \in X.$$

Adding this to (1) and using the evenness of f leads to

$$f(x+y) + f(x-y) \geq 2f(x) + 2f(y), \qquad x, y \in X.$$

Fix $u, v \in X$. Applying the above inequality with $x = u + v$ and $y = u - v$ we infer that

$$4f(u) + 4f(v) \geq f(2u) + f(2v) \geq 2f(u+v) + 2f(u-v), \qquad u, v \in X.$$

Therefore f is a quadratic function, i.e.

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \qquad x, y \in X.$$

So, there exists a biadditive and symmetric functional $B\colon X \times X \to \mathbb{R}$ such that $f(x) = B(x,x)$ for $x \in X$ (see e.g. J. Aczél & J. Dhombres [1, Chapter 11, Proposition 1]). It is easy to check that

$$B(x,y) = \frac{1}{2}\,[f(x+y) - f(x) - f(y)], \qquad x, y \in X.$$

Now, assumption (3) and the biadditivity of B imply that $2B = \phi$. This completes the proof.

THEOREM 1. *Assume that* $f\colon X \to \mathbb{R}$ *and* $\phi\colon X \times X \to \mathbb{R}$ *satisfy* (1), (3) *and*

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{4^n}\, \phi(2^n x, 2^n x) < \infty, && x \in X, \\ &\liminf_{n\to\infty} \frac{1}{4^n}\, \phi(2^n x, 2^n y) \geq \phi(x, y), && x, y \in X. \end{aligned} \tag{5}$$

If f is even, then there exists a subadditive function $A\colon X \to \mathbb{R}$ *such that*

$$f(x) = \frac{1}{2}\, \phi(x, x) - A(x), \qquad x \in X.$$

Moreover, ϕ is biadditive and symmetric.

Proof. Fix an $x \in X$ and a positive integer n. By the evenness of f and the Proposition we get

$$\frac{1}{4^{n-1}}\, f(2^{n-1}x) \leq \frac{1}{4^n}\, f(2^n x) \leq \frac{1}{4^n} \cdot \frac{1}{2}\, \phi(2^n x, 2^n x).$$

The first part of the assumption (5) implies that the right-hand side of this inequality is bounded by a real constant which does not depend on n. Therefore the formula

$$Q(x) := \lim_{n\to+\infty} \frac{1}{4^n}\, f(2^n x), \qquad x \in X,$$

correctly defines a map $Q\colon X \to \mathbb{R}$. Moreover, $Q(2x) = 4Q(x)$ for $x \in X$ and the following inequality is satisfied:

$$\begin{aligned} Q(x+y) - Q(x) - Q(y) &= \lim_{n\to\infty} \left[\frac{1}{4^n} f(2^n x + 2^n y) - \frac{1}{4^n} f(2^n x) - \frac{1}{4^n} f(2^n y) \right] \\ &\geq \liminf_{n\to\infty} \frac{1}{4^n}\, \phi(2^n x, 2^n y) \geq \phi(x, y), \qquad x, y \in X. \end{aligned}$$

Lemma 1 states that ϕ is biadditive and symmetric and $Q(x) = \frac{1}{2}\phi(x, x)$ for $x \in X$. In particular $Q(x + y) - Q(x) - Q(y) = \phi(x, y)$ for $x, y \in X$. From this and (1), it is easy to check that $A := Q - f$ is subadditive. This completes the proof.

COROLLARY 1. *Assume that* $f\colon X \to \mathbb{R}$ *and* $\phi\colon X \times X \to \mathbb{R}$ *satisfy* (1), (3), (5) *and*

$$\phi(-x, -y) = \phi(x, y), \qquad x, y \in X. \tag{6}$$

Then there exists a subadditive function $A\colon X \to \mathbb{R}$ *such that*

$$f(x) = \frac{1}{2}\, \phi(x, x) - A(x), \qquad x \in X.$$

Moreover, ϕ is biadditive and symmetric.

Proof. Define $h\colon X \to \mathbb{R}$ by $h(x) := \frac{1}{2}(f(x) + f(-x))$ for $x \in X$. Assumption (6) implies that

$$h(x+y) - h(x) - h(y) \geq \phi(x,y), \qquad x, y \in X.$$

Using Theorem 1 with f replaced by h we get the biadditivity and symmetry of ϕ. Now, one may easily check that the map $A\colon X \to \mathbb{R}$ given by $A(x) := \frac{1}{2}\phi(x,x) - f(x)$ for $x \in X$ is subadditive. This completes the proof.

Now, we are going to provide conditions which, in particular, allow us to omit the assumption (6) and to weaken (5). We start with a lemma.

LEMMA 2. *If $f\colon X \to \mathbb{R}$ and $\phi\colon X \times X \to \mathbb{R}$ satisfy* (1), (2) *and f is odd then $f(2x) = 2f(x)$ and $\phi(x,x) = 0$ for $x \in X$. Moreover, if ϕ satisfies* (3), *then f is additive and $\phi = 0$.*

Proof. Fix an $x \in X$. Since f is odd, we get

$$\begin{aligned} f(2x) - 2f(x) = -[f(-2x) - 2f(-x)] \leq -\phi(-x,-x) \leq \phi(-x,x) \\ \leq f(-x+x) - f(-x) - f(x) = 0, \end{aligned}$$

whence, again by the oddness of f, we obtain $f(2x) = 2f(x)$ for $x \in X$ and, in consequence, $\phi(x,x) = 0$ for $x \in X$.

Now, assume (3) and let $x, y \in X$. Using the assumption (3) and (1) twice we obtain

$$f(x-y) - f(x) - f(-y) \geq \phi(x,-y) \geq -\phi(x,y) \geq -f(x+y) + f(x) + f(y),$$

which means that

$$f(x+y) + f(x-y) \geq 2f(x).$$

Interchanging the roles of x and y we obtain

$$f(y+x) + f(y-x) \geq 2f(y).$$

Summing up these two inequalities we derive the superadditivity of f, which together with its oddness implies that f is additive and $\phi \leq 0$. Using this and (3) we finally get $\phi = 0$. This completes the proof.

The following lemma provides sufficient conditions for the function f to satisfy the assumption (4).

Recall that a group X is called *uniquely* 2-*divisible* if the map $X \ni x \mapsto x + x \in X$ is bijective.

LEMMA 3. *Assume that X is uniquely* 2-*divisible*, *$f\colon X \to \mathbb{R}$ and $\phi\colon X \times X \to \mathbb{R}$ satisfy* (1), (2) *and*

$$\phi(2x, 2x) \leq 4\phi(x,x), \qquad x \in X. \tag{7}$$

If f is nonnegative and even, then $f(x) = \frac{1}{2}\phi(x,x)$ for $x \in X$.

Proof. By the Proposition, for every $x \in X$ and every positive integer n we have $4^n f(x/2^n) \geq 4^{n+1} f(x/2^{n+1}) \geq 0$. So, the sequence $(4^n f(x/2^n))_{n\in\mathbb{N}}$ is pointwise convergent. In particular, $\lim_{n\to\infty} 2^n f(x/2^n) = 0$ for every $x \in X$.

Now, fix an $x \in X$. Using (1) and (7), by induction, we get

$$2^k f\left(\frac{x}{2^{k-1}}\right) - 2^{k+1} f\left(\frac{x}{2^k}\right) \geq 2^k \phi\left(\frac{x}{2^k}, \frac{x}{2^k}\right) \geq \frac{1}{2^k}\,\phi(x,x)$$

for all $k \in \mathbb{N}$. Summing up these inequalities for $k \in \{1,\ldots,n\}$ we get

$$2f(x) - 2^{n+1} f\left(\frac{x}{2^n}\right) \geq \sum_{k=1}^{n} \frac{1}{2^k}\,\phi(x,x), \qquad n \in \mathbb{N}.$$

Letting n tend to $+\infty$ yields $2f(x) \geq \phi(x,x)$. Since the Proposition provides the opposite inequality, the proof is complete.

The following result yields an analogue of Corollary 1 in the paper [2] of K. Baron and Z. Kominek.

Theorem 2. *Assume X to be uniquely 2-divisible. If $f\colon X \to \mathbb{R}$ and $\phi\colon X \times X \to \mathbb{R}$ satisfy* (1), (3), (7) *and*

$$f(x) + f(-x) \geq 0, \qquad x \in X, \tag{8}$$

then there exists an additive function $a\colon X \to \mathbb{R}$ such that

$$f(x) = \frac{1}{2}\,\phi(x,x) + a(x), \qquad x \in X.$$

Moreover, ϕ is biadditive and symmetric.

Proof. Define $h, a\colon X \to \mathbb{R}$ by $h(x) := \frac{1}{2}[f(x) + f(-x)]$ and $a(x) := \frac{1}{2}[f(x) - f(-x)]$, $x \in X$. Clearly h is even whereas a is odd. Next, define $\phi_1\colon X \times X \to \mathbb{R}$ by $\phi_1(x,y) := \frac{1}{2}[\phi(x,y) + \phi(-x,-y)]$ for $x, y \in X$. It is easy to check that h and ϕ_1 satisfy the assumptions of Lemma 3. So, $h(x) = \frac{1}{2}\phi_1(x,x)$ for $x \in X$. Now, observe that the assumptions of Lemma 1 are satisfied. Therefore ϕ_1 is biadditive and symmetric and, in consequence,

$$h(x+y) - h(x) - h(y) = \phi_1(x,y), \qquad x, y \in X.$$

Define $\phi_2\colon X \times X \to \mathbb{R}$ by $\phi_2 := \phi - \phi_1$. Note that $\phi_2(x,-y) \geq -\phi_2(x,y)$ for $x, y \in X$ and

$$a(x+y) - a(x) - a(y) \geq \phi_2(x,y), \qquad x, y \in X.$$

Now, Lemma 2 applied for $f = a$ and $\phi = \phi_2$ states that a is additive and $\phi_2 = 0$, i.e. $\phi = \phi_1$. This completes the proof.

A similar reasoning allows us to derive the following corollary from Lemmas 2 and 3.

Corollary 2. *Assume X to be uniquely 2-divisible. If $f\colon X \to \mathbb{R}$ and $\phi\colon X \times X \to \mathbb{R}$ satisfy* (1), (2), (7) *and* (8), *then there exists an odd function*

$a\colon X \to \mathbb{R}$ such that $a(2x) = 2a(x)$ for $x \in X$ and

$$f(x) = \frac{1}{2}\,\phi(x,x) + a(x), \qquad x \in X.$$

Moreover, $\phi(2x,2x) = 4\phi(x,x) \geq 0$ for $x \in X$.

Proof. Define a, h, ϕ_1 and ϕ_2 as in the previous proof. Lemma 3 implies that $h(x) = \frac{1}{2}\phi_1(x,x)$ and $h(2x) = 4h(x)$ for $x \in X$. We are going to show that ϕ_2 satisfies (2). Since $\phi_2 = \phi - \phi_1$, it suffices to prove that $\phi_1(x,-x) = -\phi_1(x,x)$ for $x \in X$. But

$$\begin{aligned} -2h(x) &= h(-x+x) - h(-x) - h(x) \geq \phi_1(-x,x) \geq -\phi_1(-x,-x) \\ &= -2h(x), \end{aligned}$$

which is what we wanted. Lemma 2 implies that $a(2x) = 2a(x)$ and $\phi_2(x,x) = 0$, i.e. $h(x) = \frac{1}{2}\phi_1(x,x) = \frac{1}{2}\phi(x,x)$ for $x \in X$. This completes the proof.

We end this paper with some additional remarks.

Remark 1. If $c \in (0,\infty)$, $f : \mathbb{R} \to \mathbb{R}$ is constant and equal to $-c$, $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is constant and equal to c, then (1), (3) and (7) are satisfied. So, the assumption (4) in Lemma 1 cannot be omitted, the assumption (5) in Theorem 1 and Corollary 1 cannot be replaced by (7), the nonnegativity of f in Lemma 3 cannot be replaced by its boundedness, and the assumption (8) in Theorem 2 cannot be omitted.

Remark 2. Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a nonzero and even function which satisfies the equality

$$\varphi(2t) = 2\varphi(t), \qquad t \in \mathbb{R}$$

(see e.g. M. Kuczma, B. Choczewski and R. Ger [3] for examples of such functions). Define $f\colon \mathbb{R} \to \mathbb{R}$ and $\phi\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} f(x) &:= (\varphi(x))^2, \qquad x \in \mathbb{R}, \\ \phi(x,y) &:= f(x+y) - f(x) - f(y), \qquad x, y \in \mathbb{R}. \end{aligned}$$

Then inequality (1) is satisfied, and f is even, nonnegative and satisfies (4). Moreover, ϕ satisfies (2), (5) and (6). So, in Lemma 1, Theorems 1 and 2 and Corollary 1, (3) cannot be replaced by (2).

Remark 3. Let $(X; \|\cdot\|)$ be a normed linear space. Corollary 1 implies that the inequality

$$f(x+y) - f(x) - f(y) \geq \|x\| \cdot \|y\|, \qquad x, y \in X,$$

has no solution. In fact, the function $\phi(x,y) := \|x\| \cdot \|y\|$, $x, y \in X$, satisfies (3), (5) and (6), but ϕ fails to be biadditive.

In this inequality X may stand for an abelian group and the norm can be replaced by any real function, which is nonzero, nonnegative, even and 2-homogeneous.

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