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MATHEMATICAL ANALYSIS

Inequivalence of Wavelet Systems in $L_1(\mathbb{R}^d)$ and $\mathrm{BV}(\mathbb{R}^d)$

by

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Summary. Theorems stating sufficient conditions for the inequivalence of the d-variate Haar wavelet system and another wavelet system in the spaces $L_1(\mathbb{R}^d)$ and $\mathrm{BV}(\mathbb{R}^d)$ are proved. These results are used to show that the Strömberg wavelet system and the system of continuous Daubechies wavelets with minimal supports are not equivalent to the Haar system in these spaces. A theorem stating that some systems of smooth Daubechies wavelets are not equivalent to the Haar system in $L_1(\mathbb{R}^d)$ is also shown.

1. Introduction

1.1. *Statement of results.* Let $\Psi = \{\psi_i\}_{i\in\Delta}$ and $\overline{\Psi} = \{\overline{\psi}_i\}_{i\in\Delta}$ be two collections of linearly independent vectors in a normed linear space $(X, \|\ \|_X)$. We say that Ψ and $\overline{\Psi}$ are *equivalent* if the linear mapping A defined by

$$A : \overline{\psi}_i \mapsto \psi_i \quad \text{for all } i \in \Delta$$

extends to a linear isomorphism of the closed linear span of $\overline{\Psi}$ onto the closed linear span of Ψ.

In this paper we are concerned exclusively with the cases when X is either $L_1(\mathbb{R}^d)$ or $\mathrm{BV}(\mathbb{R}^d)$, $\overline{\Psi} = \mathrm{H} = (h_\lambda)_{\lambda\in\Delta}$ is the d-variate Haar system and Ψ is another d-variate wavelet system generated by a univariate mother wavelet ψ^1 and scaling function ψ^0. The definitions of BV and wavelet systems are provided in Section 1.2. Naturally, we assume that $\Psi \subset X$.

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It is well known (see [8] and [12]) that under some mild assumptions on the decay and oscillation of the wavelets that constitute Ψ the systems H and Ψ are equivalent Schauder bases in $L_p(\mathbb{R}^d)$ (with $1 < p < \infty$) and $H^1(\mathbb{R}^d)$. From wavelet characterizations of the Sobolev spaces W_p^s in [8] (again for $1 < p < \infty$) the equivalence of wavelet bases follows also for these spaces.

The ultimate goal of this paper is to show that no general equivalence theorem for wavelet systems can hold in the case of $p = 1$. The two theorems below state sufficient conditions formulated in terms of linear functionals and the mother wavelet ψ^1 for the inequivalence of Ψ and H in either $L_1(\mathbb{R}^d)$ or $\mathrm{BV}(\mathbb{R}^d)$. (Because $\mathrm{H} \not\subseteq W_1^1$, the larger space BV is considered.)

THEOREM 1. *If the mother wavelet* ψ^1 *satisfies*

$$\int_0^\infty \psi^1(t)\,dt \neq 0, \tag{1}$$

then the Haar system H *and the system* Ψ *are not equivalent in* $L_1(\mathbb{R}^d)$.

Condition (1) is satisfied for the well known Strömberg wavelet (introduced in [11]) as well as for certain integer shifts of continuous (or smoother) compactly supported Daubechies wavelets (for the simplest example see an elementary construction in [9]). These two claims are verified in Sections 2.2 and 2.3 respectively.

The system obtained from the Strömberg wavelet can be considered a real-line equivalent of the Franklin functions defined on the unit interval. An interesting context for Theorem 1 is provided by the paper of Sjölin [10], in which the inequivalence of Haar and Franklin systems in $L_1([0,1])$ is shown. These two results are compared briefly at the end of Section 2.2.

The second theorem contains a similar result for the space BV:

THEOREM 2. *If the mother wavelet* ψ^1 *satisfies*

$$\int_{[1/3,\infty)} D\psi^1(dt) + \int_{[2/3,\infty)} D\psi^1(dt) \neq 0, \tag{2}$$

then the Haar system H *and the system* Ψ *are not equivalent in* $\mathrm{BV}(\mathbb{R}^d)$. *Specifically, there exists a sequence of functions* $f_n \in \mathrm{BV}(\mathbb{R}^d)$ *such that* $\|f_n\|_{\mathrm{BV}} \le C < \infty$, *but* $|Af_n|_{\mathrm{BV}} \ge c_2 n$ *for a certain constant* $c_2 > 0$.

Note that the BV seminorm is explicitly responsible for the inequivalence.

The assumption (2) in the case of continuous ψ^1 which are 0 at infinity can be reformulated as

$$\psi^1(1/3) + \psi^1(2/3) \neq 0. \tag{3}$$

In Sections 3.2 and 3.3 it is verified for the Strömberg wavelet and a continuous Daubechies wavelet supported in the interval $[0,3]$.

Interest in the inequivalence of wavelet systems in BV is motivated by results from [4] and [13], where it is shown that the Haar coefficients of functions from $\mathrm{BV}(\mathbb{R}^d)$ with $d \geq 2$ are in the sequence space $w\ell_1$, and greedy projections with respect to H are bounded in the BV seminorm. In [3] and [1] these results are generalized to any compactly supported wavelet. Results like Theorem 2 indicate that one cannot use the equivalence of wavelet systems in BV to obtain this generalization and independent proofs are indeed required.

1.2. *Preliminaries*

BV *spaces.* We say that a distribution $f \in L_1(\mathbb{R}^d)$ belongs to the space $\mathrm{BV}(\mathbb{R}^d)$ if its distributional derivatives $D_{x_i} f$, $i = 1, \ldots, d$, are measures of finite variation. The BV *seminorm* is defined by

$$|f|_{\mathrm{BV}(\mathbb{R}^d)} := \Big(\sum_{i=1}^{d} \mathrm{Var}_{\mathbb{R}^d}(D_{x_i} f)^2\Big)^{1/2}, \tag{4}$$

where $\mathrm{Var}_\Omega(\mu)$ denotes the variation of the measure μ on the set Ω. The norm on $\mathrm{BV}(\mathbb{R}^d)$ is then defined by

$$\|f\|_{\mathrm{BV}(\mathbb{R}^d)} := \|f\|_{L_1(\mathbb{R}^d)} + |f|_{\mathrm{BV}(\mathbb{R}^d)}.$$

One can also define $\mathrm{BV}(\mathbb{R}^d)$ as the space of all $f \in L_1(\mathbb{R}^d)$ for which there is a sequence (f_n) of functions from the Sobolev space $W_1^1(\mathbb{R}^d)$ such that

$$\sup_n \|Df_n\|_{L_1(\mathbb{R}^d)} < \infty \quad \text{and} \quad \|f - f_n\|_{L_1(\mathbb{R}^d)} \to 0 \quad (n \to \infty). \tag{5}$$

The BV seminorm can be defined in this case as

$$|f|^*_{\mathrm{BV}} := \inf_{(f_n)} \liminf_{n\to\infty} \|Df_n\|_{L_1(\mathbb{R}^d)},$$

where the infimum is taken over all sequences (f_n) satisfying (5). The seminorm $|\ |^*_{\mathrm{BV}}$ is equivalent to the seminorm defined in (4). More details can be found in [14, Chapter 5].

Wavelet systems. By a *d-variate wavelet system* we mean the system obtained by taking the tensor product of a univariate multiresolution analysis associated with a univariate scaling function ψ^0 with orthogonal integer shifts and the related mother wavelet ψ^1.

For completeness, a brief outline of the construction is presented below. The reader is referred to [6] or [12] for details.

Let $E' = \{(e_1, \ldots, e_d) : e_i = 0, 1\}$ and $E = E' \setminus \{(0, \ldots, 0)\}$. For $e \in E$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ let

$$\psi^e(x_1, \ldots, x_d) := \prod_{i=1}^{d} \psi^{e_i}(x_i).$$

Now for $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$ and $\lambda := (e, j, k)$ define the functions

$$\psi_\lambda(x) := \psi^e_{j,k}(x) := 2^{dj/2}\psi^e(2^j x - k).$$

For

$$\Delta = \{(e, 0, k) : e \in E',\, k \in \mathbb{Z}^d\} \cup \{(e, j, k) : j > 0,\, e \in E,\, k \in \mathbb{Z}^d\} \tag{6a}$$

or

$$\Delta = \{(e, j, k) : e \in E,\, j \in \mathbb{Z},\, k \in \mathbb{Z}^d\} \tag{6b}$$

the system $\Psi = \{\psi_\lambda\}_{\lambda\in\Delta}$ is an orthonormal basis in $L_2(\mathbb{R}^d)$.

The orthonormal d-variate Haar system $\mathrm{H} = (h_\lambda)_{\lambda\in\Delta}$ is obtained using the same construction by taking $h^1 = \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1]}$ instead of ψ^1 and $h^0 = \mathbb{1}_{[0,1]}$ in place of ψ^0. If Δ is defined by (6a) and H is ordered so that the indices j do not decrease, then H constitutes a basis in $L_1(\mathbb{R}^d)$. It is not a basis in $\mathrm{BV}(\mathbb{R}^d)$, but nonetheless partial sum projections with respect to H are uniformly bounded in the BV norm (see [13, Corollary 11]).

2. Inequivalence in $L_1(\mathbb{R}^d)$

2.1. *Proof of Theorem 1.* For convenience, we assume that the wavelets are normalized in $L_1(\mathbb{R}^d)$, i.e. we have $\|\psi^e\|_{L_1} = 1$,

$$\psi^e_{j,k}(x) := \psi_\lambda(x) := 2^{dj}\psi^e(2^j x - k),$$

and similarly for H. Let A be a linear mapping such that $A(h_\lambda) = \psi_\lambda$ for all $\lambda \in \Delta$. We construct a bounded sequence of functions $f_n \in L_1(\mathbb{R}^d)$ such that $\|Af_n\|_{L_1} \ge c_1 n$ for a certain constant $c_1 > 0$. This implies that A is not continuous and hence the systems H and Ψ are not equivalent in $L_1(\mathbb{R}^d)$.

Let $n \in \mathbb{N}$ and $\Omega_n = [0, 2^{-n}] \times [0, 1]^{d-1}$. We define the functions

$$g_n := 2^n \mathbb{1}_{\Omega_n}.$$

Observe that $\|g_n\|_{L_1} = 1$ and $g_n = \mathbb{1}_{[0,1]^d} + f_n$, where

$$f_n = \sum_{j=0}^{n-1} 2^{-j(d-1)} \sum_{k\in K_j} h^{e_0}_{j,k},$$

with $e_0 = (1, 0, \ldots, 0)$ and

$$K_j = \{k = (k_1, \ldots, k_d) \in \mathbb{Z}^d : k_1 = 0,\, 0 \le k_2, \ldots, k_d \le 2^j - 1\}.$$

We have $\#K_j = 2^{j(d-1)}$ and

$$\|f_n\|_{L_1(\mathbb{R}^d)} \le 2.$$

Because $e_0 \ne 0$, this sequence is a good example for Δ defined by both (6a) and (6b).

Now consider the following linear functional $F \in L_1(\mathbb{R}^d)^*$:

$$F(f) := \int_\Gamma f(x)\,dx, \qquad \Gamma = [0,\infty) \times \mathbb{R}^{d-1}.$$

We calculate

$$\begin{aligned}
F(Af_n) &= \int_\Gamma \sum_{j=0}^{n-1} 2^{-j(d-1)} \sum_{l\in K_j} \psi^{e_0}_{j,k}(x)\,dx \\
&= \sum_{j=0}^{n-1} 2^{-j(d-1)} \sum_{k\in K_j} 2^{dj} \int_{[0,\infty)} \psi^1(2^j x_1)\,dx_1 \cdot \prod_{i=2}^{d} \int_{\mathbb{R}} \psi^0(2^j x_i - k_i)\,dx_i \\
&= \sum_{j=0}^{n-1} 2^{-j(d-1)} \sum_{k\in K_j} \int_{[0,\infty)} \psi^1(t)\,dt \cdot \Big(\int_{\mathbb{R}} \psi^0(t)\,dt\Big)^{d-1} \\
&= n\widetilde{c}_1,
\end{aligned}$$

where $\widetilde{c}_1 = \int_{[0,\infty)} \psi^1(t)\,dt \cdot \big(\int_{\mathbb{R}} \psi^0(t)\,dt\big)^{d-1}$ is not zero by (1) and the fact that the integral of a scaling function $\psi^0 \in L_1$ cannot vanish (see for example [12, Proposition 3.17]). Because F is bounded, this implies that $\|Af_n\|_{L_1} \ge c_1 n$ for a certain positive constant c_1. ∎

2.2. *Strömberg wavelet in L_1.* The Strömberg wavelet S, discovered by Strömberg in [11], is a continuous, piecewise linear function with knots at the points $\ldots, -3/2, -1, -1/2, 0, 1/2, 1, 2, 3, \ldots$. Its values at the knots are as follows:

$$\begin{aligned}
S(-k/2) &= S(1)(2\sqrt{3}-2)(\sqrt{3}-2)^k \quad \text{for } k = 1,2,3,\ldots, \\
S(0) &= S(1)(2\sqrt{3}-2), \qquad S(1/2) = -S(1)(\sqrt{3}+1/2), \\
S(k) &= S(1)(\sqrt{3}-2)^{k-1} \quad \text{for } k = 1,2,3,\ldots.
\end{aligned} \tag{7}$$

Obviously, $S \in L_1(\mathbb{R}^d)$, as it has exponential decay.

THEOREM 3. *The d-variate Haar and Strömberg wavelet systems are not equivalent in $L_1(\mathbb{R}^d)$.*

Proof. We need to show that $\int_0^\infty S(t)\,dt \ne 0$. Using (7) we get

$$\begin{aligned}
\int_0^\infty S(t)\,dt &= \frac{1}{4}(S(0)+S(1/2)) + \frac{1}{4}(S(1/2)+S(1)) + \frac{1}{2}\sum_{k=1}^{\infty}(S(k)+S(k+1)) \\
&= S(1)\Big(-\frac{1}{2} + \frac{1}{2}(\sqrt{3}-1)\sum_{k=0}^{\infty}(\sqrt{3}-2)^k\Big) = -\frac{3-\sqrt{3}}{6}\,S(1) \ne 0. \qquad ∎
\end{aligned}$$

Comparison with Sjölin's result. Let $\{h_n\}_{n=1}^\infty$ and $\{f_n\}_{n=0}^\infty$ be respectively the Haar and Franklin systems on the unit interval $[0,1]$, as defined in [2]. In [10] Sjölin shows that a linear mapping such that $f_{n-1} \mapsto h_n$

for $n = 1, 2, 3, \dots$ is not continuous in $L_1([0,1])$. The major difference between his result and the result proved here is the following: the mapping $h_\lambda \mapsto \psi_\lambda$ preserves the location of the wavelets (where the *location* of ψ_λ, $\lambda = (e, j, k)$, is the dyadic cube $2^{-j}([0,1]^d + k)$). On the other hand, the mapping $f_{n-1} \mapsto h_n$ shifts the location by one dyadic interval or even changes the dyadic level of the function (for $n = 2^j + k$, $k = 1, \dots, 2^j$, the functions h_n and f_n are located on the dyadic interval $[(k-1)2^{-j}, k2^{-j}]$). In particular, the sequence $\{f_{2^{l+1}-1}\}_{l=1}^\infty$ of Franklin functions with disjoint dyadic locations $[1 - 2 \cdot 2^{-l}, 1 - 2^{-l}]$ is mapped to the sequence $\{h_{2^{l+1}}\}$ of Haar functions the supports of which form a descending sequence of dyadic intervals $[1 - 2^{-l}, 1]$. A situation like this is not possible in the case of the mapping considered here.

2.3. *Daubechies wavelets in L_1.* The wavelet ψ discussed in the first part of this subsection belongs to the famous class of compactly supported Daubechies wavelets introduced in [5]. The Haar wavelet is the simplest, although lacking smoothness, wavelet of this kind. The function ψ is a minimally supported continuous wavelet from this class. Here we show the system generated by $\psi(\cdot + 1)$ or $\psi(\cdot + 2)$ is not equivalent to H in $L_1(\mathbb{R}^d)$. (Because $\psi = \psi(\cdot + 0)$ is supported in $[0,3]$ and $\int_{\mathbb{R}} \psi = 0$, it does not satisfy the assumption (1).)

Below is a list of properties of the scaling function ϕ associated with the wavelet ψ. All of these are taken from [9], where an elementary backward-engineered construction of ϕ is presented. The same material can also be found in [12, Section 5.3].

Let

$$a := \frac{1+\sqrt{3}}{4} \quad \text{and} \quad b := \frac{1-\sqrt{3}}{4}. \tag{8}$$

The scaling function ϕ associated with the wavelet ψ is supported on $[0,3]$ and satisfies the scaling equation

$$\phi(t) = a\phi(2t) + (1-b)\phi(2t-1) + (1-a)\phi(2t-2) + b\phi(2t-3). \tag{9}$$

From the general construction of wavelets we obtain a formula for ψ:

$$\psi(t) = -b\phi(2t) + (1-a)\phi(2t-1) - (1-b)\phi(2t-2) + a\phi(2t-3). \tag{10}$$

For $t \in [0,1]$ we have

$$2\phi(t) + \phi(t+1) = t + \frac{1+\sqrt{3}}{2}, \tag{11a}$$

$$\phi(t) - \phi(t+2) = t + \frac{-1+\sqrt{3}}{2}, \tag{11b}$$

as well as

$$\phi\left(\tfrac{1}{2}t\right) = a\phi(t), \tag{12a}$$

$$\phi\left(\tfrac{1}{2}(1+t)\right) = b\phi(t) + at + \frac{2+\sqrt{3}}{4}. \tag{12b}$$

THEOREM 4. *Let Ψ be the wavelet system on $\mathbb{R}^d$ generated by $\psi^0 = \phi$ and $\psi^1 = \psi(\cdot + l)$ with $l = 1$ or $l = 2$. Then Ψ is not equivalent to* H *in $L_1(\mathbb{R}^d)$.*

Proof. Because $\int_0^3 \psi\, dt = 0$, it suffices to show that $\int_0^1 \psi\, dt \neq 0$ and $\int_0^2 \psi\, dt \neq 0$. We use (10) to compute these integrals. First we will find the values of

$$I_i := \int_i^{i+1} \phi(t)\, dt, \qquad i = 0, 1, 2.$$

Integrating both sides of (12a) and (12b) over $[0,1]$ and adding the resulting equalities leads to

$$2I_0 = (a+b)I_0 + \frac{a}{2} + \frac{2+\sqrt{3}}{4},$$

which gives

$$I_0 = \frac{5+3\sqrt{3}}{12}.$$

We now integrate the identities in (11) over $[0,1]$ to obtain

$$I_1 = \frac{1}{6} \quad \text{and} \quad I_2 = \frac{5-3\sqrt{3}}{12}.$$

Using (10) and the values of I_0, I_1 and I_2 we get

$$\int_0^1 \psi(t)\, dt = \frac{1+\sqrt{3}}{12} \quad \text{and} \quad \int_0^2 \psi(t)\, dt = \frac{1-\sqrt{3}}{12}.$$

Both integrals are non-zero. ■

It would be interesting to know whether a fact similar to Theorem 4 can be shown for any smooth compactly supported wavelet. While a complete answer to this question is not known to the author, one can in fact show the following:

THEOREM 5. *Assume that the wavelet ψ^1 is compactly supported and continuous. For $k \in \mathbb{Z}$ define $\psi_k = \psi^1(\cdot - k)$ and let Ψ_k be the d-variate wavelet system generated using ψ_k as the mother wavelet. Then there exists $k \in \mathbb{Z}$ such that Ψ_k is not equivalent to* H *in $L_1(\mathbb{R}^d)$.*

The only difference between the systems Ψ_k and Ψ_0 is in how their elements are indexed. From the proof it also follows that for a given ψ^1 the same k works for all d.

Proof. By Theorem 1, it suffices to show that for a certain $k \in \mathbb{Z}$ we have

$$\int_0^\infty \psi_k^1(t)\,dt \neq 0.$$

Indeed, if this was not the case, then the wavelet expansion of the function $f = \mathbb{1}_{[0,1]}$ with respect to the system generated by ψ^1 (with Δ defined by (6a)) would be finite, which would imply that f is continuous. See [7, Lemma 3, p. 41] for an application of the same trick. ∎

3. Inequivalence in $\mathrm{BV}(\mathbb{R}^d)$

3.1. *Proof of Theorem 2.* This time we assume that the wavelets are normalized in $L_{d^*}(\mathbb{R}^d)$ with $d^* = d/(d-1)$, i.e. $\|\psi^e\|_{L_{d^*}} = 1$, $\psi_{j,k}^e(x) := 2^{(d-1)j}\psi^e(2^j x - k)$ and similarly for H. This normalization is equivalent to the normalization in the BV seminorm.

Again let A be defined by $A : h_\lambda \mapsto \psi_\lambda$ ($\lambda \in \Delta$). As in the proof of Theorem 1 we will construct a sequence $\{f_n\}$ bounded in $\mathrm{BV}(\mathbb{R}^d)$ such that the BV seminorms of the functions Af_n will be unbounded.

Let

$$\begin{aligned} g &:= \mathbb{1}_\Omega \qquad \text{with } \Omega = [0,1/3]\times[0,1]^{d-1},\\ f_n &:= \frac{1}{3}\sum_{j=0}^{2n-1} 2^{-(d-1)j} \sum_{k\in K_j} h_{k,j}^{e_0},\\ g_n &:= \frac{1}{3}\,\mathbb{1}_{[0,1]^d} + f_n, \end{aligned}$$

where again $e_0 := (1,0,\ldots,0)$ and

$$K_j := \left\{k = (k_1,\ldots,k_d) \in \mathbb{Z}^d : \frac{k_1}{2^j} < \frac{1}{3} < \frac{k_1+1}{2^j},\ 0 \le k_2,\ldots,k_d \le 2^j - 1\right\}.$$

Note again that for each $j \ge 0$ we have $\#K_j = 2^{(d-1)j}$. Moreover, all $k \in K_j$ have the same first coordinate $k_1 =: k_1(j)$. As the binary expansion of the fraction $1/3$ is 2-periodic ($1/3 = 0.010101\ldots$ in the binary notation) we also have

$$\frac{2^j}{3} - k_1(j) = \begin{cases} 1/3 & \text{for even } j,\\ 2/3 & \text{for odd } j. \end{cases} \tag{13}$$

The functions g_n are partial sums of the Haar expansion of g down to the dyadic level $2n-1$. Hence, they are uniformly bounded in the BV norm, which implies that f_n are bounded as well. As in the proof of Theorem 1, this sequence is a good example for Δ defined either by (6a) or (6b).

We now consider the following linear functional, bounded in the BV seminorm:

$$F(f) := \int_{\Gamma} D_{x_1} f(dx), \qquad \Gamma = [1/3, \infty) \times \mathbb{R}^{d-1}.$$

We will use the notation

$$V(t_0) := \int_{[t_0,\infty)} D\psi^1(dt).$$

First we observe that for $k \in K_j$,

$$\begin{aligned} F(\psi^{e_0}_{j,k}) &= V(2^j/3 - k_1(j)) \cdot \Big(\int_{\mathbb{R}} \psi^0(t)\,dt\Big)^{d-1} \\ &= \widetilde{c}_2 \cdot V\bigg(\frac{1}{3} \cdot \frac{3 + (-1)^j}{2}\bigg) \qquad \text{by (13)} \end{aligned}$$

with $\widetilde{c}_2 = (\int_{\mathbb{R}} \psi^0(t)\,dt)^{d-1}$. From this we get

$$\begin{aligned} F\bigg(A\bigg(\frac{1}{3} \cdot 2^{-(d-1)j} \sum_{k \in K_j} h^{e_0}_{j,k}\bigg)\bigg) &= \frac{1}{3} \cdot 2^{-(d-1)j} \sum_{k \in K_j} F(\psi^{e_0}_{j,k}) \\ &= \frac{1}{3} \cdot 2^{-(d-1)j} \sum_{k \in K_j} \widetilde{c}_2 V\bigg(\frac{1}{3} \cdot \frac{3 + (-1)^j}{2}\bigg) \\ &= \frac{1}{3} \cdot \widetilde{c}_2 V\bigg(\frac{1}{3} \cdot \frac{3 + (-1)^j}{2}\bigg). \end{aligned}$$

Finally, the above gives

$$F(Af_n) = \frac{1}{3} \cdot \widetilde{c}_2 \sum_{j=0}^{2n-1} V\bigg(\frac{1}{3} \cdot \frac{3 + (-1)^j}{2}\bigg) = \frac{1}{3} \cdot \widetilde{c}_2 \sum_{j=0}^{n-1} \bigg(V\Big(\frac{1}{3}\Big) + V\Big(\frac{2}{3}\Big)\bigg) = \overline{c}_2 n$$

with $\overline{c}_2 = \frac{1}{3}(\int_{\mathbb{R}} \psi^0(t)\,dt)^{d-1}(V(1/3) + V(2/3)) \neq 0$ by (2) and the already mentioned fact that the integral of ψ^0 cannot vanish. This implies that there exists a constant $c_2 > 0$ such that $|Af_n|_{\mathrm{BV}} \geq c_2 n$. ∎

The particular choice for the function g was inspired by a one-dimensional example in [3, p. 259].

3.2. *Strömberg wavelet in* BV. By (7), the distributional derivative of the Strömberg wavelet S is a piecewise constant function with exponential decay, so obviously $S \in \mathrm{BV}(\mathbb{R})$. Here we show that it generates a wavelet system not equivalent to H in $\mathrm{BV}(\mathbb{R}^d)$.

THEOREM 6. *The d-variate Haar and Strömberg systems are not equivalent in* $\mathrm{BV}(\mathbb{R}^d)$.

Proof. It suffices to check that $S(1/3)+S(2/3)\neq 0$. Using (7) we get

$$S(1/3)+S(2/3)=-S(1)-\frac{2\sqrt{3}}{3}\,S(1)\neq 0. \blacksquare$$

3.3. *Continuous Daubechies wavelet in* BV. The differentiability properties of the minimally supported continuous Daubechies wavelet ψ are analyzed in [9]. However, the question whether $\psi\in \mathrm{BV}(\mathbb{R})$ is not considered there. The answer is given below.

PROPOSITION 7. *The wavelet ψ belongs to the space* $\mathrm{BV}(\mathbb{R})$.

Proof. It suffices to show that the corresponding scaling function ϕ belongs to $\mathrm{BV}(\mathbb{R})$. In [9] the function ϕ is constructed as the L_∞-limit of a sequence (g_n) of continuous piecewise linear functions over increasingly finer dyadic meshes. We use the same construction to show that $\phi\in\mathrm{BV}(\mathbb{R})$.

Firstly, a non-linear operator K is defined (a and b are defined in (8)):

$$K(f)(t):=\begin{cases} af(2t), & t\in[0,1/2),\\ bf(2t-1)+2at+1/4, & t\in[1/2,1),\\ af(2t-1)+2bt-(2-\sqrt{3})/4, & t\in[1,3/2),\\ bf(2t-2)-2at+(4+\sqrt{3})/4, & t\in[3/2,2),\\ af(2t-2)-2bt+(7-6\sqrt{3})/4, & t\in[2,5/2),\\ bf(2t-3), & t\in[5/2,3),\\ 0, & t\in(-\infty,0)\cup[3,\infty). \end{cases} \tag{14}$$

The function g_0 is continuous and piecewise linear with knots in $\mathbb{Z}$, such that $g_0(t)=\phi(t)$ for all $t\in\mathbb{Z}$. Next,

$$g_n:=K(g_{n-1})\quad \text{for } n=1,2,3,\ldots.$$

The functions g_n are continuous with supports in $[0,3]$. Each g_n is piecewise linear with knots at the points $\{k2^{-n}\}_{k\in\mathbb{Z}}$. This in particular implies that $g_n\in W_1^1(\mathbb{R})$ for all n. Moreover, it can be shown that $\|\psi-g_n\|_{L_\infty}\to 0$, which means that $\|\psi-g_n\|_{L_1}\to 0$ as well.

LEMMA 8. *Let f be a continuous function with support in $[0,3]$ such that $f\in W_1^1(\mathbb{R})$ and $K(f)$ is also continuous. Then*

$$\|D(K(f))\|_{L_1}\le \frac{\sqrt{3}}{2}\,\|Df\|_{L_1}+\sqrt{3}. \tag{15}$$

Proof. We use (14) to estimate

$$I_k:=\int\limits_{(k)/2}^{(k+1)/2}|D(K(f))(t)|\,dt$$

for $k = 0, 1, \dots, 5$:

$$I_0 \le |a| \int_0^1 |Df(t)|\,dt, \qquad I_1 \le |b| \int_0^1 |Df(t)|\,dt + |a|,$$
$$I_2 \le |a| \int_1^2 |Df(t)|\,dt + |b|, \qquad I_3 \le |b| \int_1^2 |Df(t)|\,dt + |a|,$$
$$I_4 \le |a| \int_2^3 |Df(t)|\,dt + |b|, \qquad I_5 \le |b| \int_2^3 |Df(t)|\,dt.$$

This gives

$$\int_0^3 |D(K(f))(t)|\,dt \le (|a|+|b|)\Big(\int_0^3 |Df(t)|\,dt + 2\Big) = \frac{\sqrt{3}}{2}\int_0^3 |Df(t)|\,dt + \sqrt{3}. \ \blacksquare$$

Iterating (15) for $f = g_n, \dots, g_0$ we obtain

$$\|Dg_n\|_{L_1} \le \left(\frac{\sqrt{3}}{2}\right)^n \|Dg_0\|_{L_1} + \sqrt{3}\left(1 + \frac{\sqrt{3}}{2} + \dots + \left(\frac{\sqrt{3}}{2}\right)^{n-1}\right),$$

which implies that $\|Dg_n\|_{L_1} < M$ for a certain $M < \infty$ and all n. By the second definition of the spaces BV in Section 1.2, this implies that ϕ (i.e. the L_1-limit of g_n) belongs to $\mathrm{BV}(\mathbb{R})$. $\blacksquare$

Having established that $\psi \in \mathrm{BV}(\mathbb{R})$, we may now show the following:

THEOREM 9. *Let Ψ be the wavelet system on $\mathbb{R}^d$ generated by $\psi^0 = \phi$ and $\psi^1 = \psi$. Then Ψ and* H *are not equivalent in* $\mathrm{BV}(\mathbb{R}^d)$.

Proof. By Theorem 2, we need to show that

$$\psi(1/3) + \psi(2/3) \ne 0.$$

Using (10) and (12b) we get

$$\psi(1/3) = -b\phi(2/3) = -b\phi\left(\frac{1}{2}\left(1 + \frac{1}{3}\right)\right) = -b\left(b\phi(1/3) + \frac{a}{3} + \frac{2+\sqrt{3}}{4}\right),$$

while (10) and (11a) give

$$\begin{aligned}\psi(2/3) &= -b\phi(4/3) + (1-a)\phi(1/3)\\ &= -b\left(-2\phi(1/3) + \frac{1}{3} + \frac{1+\sqrt{3}}{2}\right) + (1-a)\phi(1/3).\end{aligned}$$

Hence

$$\psi(1/3) + \psi(2/3) = \frac{8-5\sqrt{3}}{8}\,\phi(1/3) + \frac{13+7\sqrt{3}}{48}. \tag{16}$$

Now we need to find the value of $\psi(1/3)$. By continuity of ϕ, we have

$$\phi(1/3) = \lim_{k\to\infty} \phi(x_k)$$

for $x_0 := 0$ and $x_k := \frac{1}{4}x_{k-1} + \frac{1}{4}$ for $k = 1, 2, 3, \ldots$. Observe that $x_k \nearrow 1/3$. Using the identities in (12) we get

$$\begin{aligned}
\phi(x_k) &= \phi\left(\frac{1}{2}\left(\frac{1}{2}\,x_{k-1} + \frac{1}{2}\right)\right) = a\phi\left(\frac{1}{2}\,(x_{k-1}+1)\right) \\
&= a\left(b\phi(x_{k-1}) + ax_{k-1} + \frac{2+\sqrt{3}}{4}\right) \\
&= -\frac{1}{8}\,\phi(x_{k-1}) + \frac{2+\sqrt{3}}{8}\,x_{k-1} + \frac{5+3\sqrt{3}}{16},
\end{aligned}$$

which yields

$$\phi(1/3) = \frac{19+11\sqrt{3}}{54}.$$

Substituting this in (16) leads to

$$\psi(1/3) + \psi(2/3) = \frac{13+7\sqrt{3}}{54} \neq 0. \ \blacksquare$$

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