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FOURIER ANALYSIS

On Billard's Theorem for Random Fourier Series

by

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Summary. We show that Billard's theorem on a.s. uniform convergence of random Fourier series with independent symmetric coefficients is not true when the coefficients are only assumed to be centered independent. We give some necessary or sufficient conditions to ensure the validity of Billard's theorem in the centered case.

1. Introduction. In this paper we deal with a.s. uniform convergence of random Fourier series. Let Γ be the unit circle, and denote by $C(\Gamma)$ the Banach space of continuous functions on Γ, with the sup-norm that we denote by $\|\cdot\|_{C(\Gamma)}$. Given a sequence $\{X_n\}$ of independent centered random variables defined on a probability space $(\Omega, \mathbb{P})$, one wonders whether, for $\mathbb{P}$-almost every $x \in \Omega$, the series $\sum_{n=1}^{\infty} X_n(x)\lambda^n$ is uniformly convergent on Γ.

Since the early study of Paley and Zygmund [14], this matter benefited from many works by Salem and Zygmund [16], Billard [1], Kahane [8], Marcus [11], Cuzick and Lai [2], Marcus and Pisier [12], [13], Talagrand [17], Fernique [4], and Weber [18].

Most of the time, the sequence $\{X_n\}$ is first assumed to be symmetric, so that, by a theorem of Billard [1], it suffices to prove that, a.s., $\sum_{n=1}^{\infty} X_n(x)\lambda^n$ represents a continuous function. Then, one may hope to reach the general centered case by symmetrization.

Our purpose is to prove that this procedure may fail sometimes. We show that Billard's theorem is no longer true when the independent sequence $\{X_n\}$ is only assumed to be centered. To illustrate our result, we recall some

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more or less known moment conditions which ensure the validity of Billard's theorem, and give also necessary or sufficient conditions.

2. Conditions for validity of Billard's theorem. Let us first recall Billard's theorem, as it appears in Kahane [8, Theorem 3, p. 58].

THEOREM 2.1 (Billard). *Let $\{X_n\}$ be a sequence of independent* symmetric *complex-valued random variables. Then the following conditions are equivalent for the random Fourier series $\sum_{n=1}^{\infty} X_n\lambda^n$:*

(i) *Almost surely the series represents a bounded function.*
(ii) *Almost surely the series represents a continuous function.*
(iii) *Almost surely the series is convergent at every point.*
(iv) *Almost surely the series is uniformly convergent.*

We would like to know what happens in the centered case. We discuss conditions which allow one to proceed by symmetrization. We denote by $L_1(\mathbb{P}, C(\Gamma))$ the Banach space of all $C(\Gamma)$-valued random variables ξ on Ω with the norm $\int_\Omega \|\xi\|_{C(\Gamma)}\, d\mathbb{P}$. The following result is a simple combination of a theorem of Itô and Nisio [6] (see also Ledoux and Talagrand [9, Theorem 6.1]), and a theorem of Hoffman-Jørgensen [5] (see also Jain and Marcus [7]).

THEOREM 2.2. *Let $\{X_n\} \subset L_1(\mathbb{P})$ be a sequence of independent complex-valued centered random variables on $(\Omega, \mathbb{P})$. Assume that $\mathbb{E}(\sup_{n\ge 1}|X_n|) < \infty$. Let $\{X_n'\}$ be an independent copy of $\{X_n\}$, defined on $(\Omega', \mathbb{P}')$. The following conditions are equivalent*:

(i) *Almost surely the series $\sum_{n=1}^{\infty} X_n\lambda^n$ converges in $C(\Gamma)$.*
(ii) *Almost surely in $\Omega\times\Omega'$ the series $\sum_{n=1}^{\infty}(X_n - X_n')\lambda^n$ converges in $C(\Gamma)$.*
(iii) *The series $\sum_{n=1}^{\infty}(X_n - X_n')\lambda^n$ converges in $L_1(\mathbb{P}\otimes\mathbb{P}', C(\Gamma))$.*
(iv) *The series $\sum_{n=1}^{\infty} X_n\lambda^n$ converges in $L_1(\mathbb{P}, C(\Gamma))$.*

Moreover, *if any of the above holds*, *then* $\sup_{n\ge1}\max_{|\lambda|=1}|\sum_{k=1}^{n}(X_k - X_k')\lambda^k|$ *and* $\sup_{n\ge1}\max_{|\lambda|=1}|\sum_{k=1}^{n} X_k\lambda^k|$ *are integrable.*

Proof. Clearly (i)⇒(ii). By assumption $\mathbb{E}(\sup_{n\ge1}|X_n - X_n'|) < \infty$, hence (ii)⇒(iii) by [5, Corollary 3.3]. The implication (iii)⇒(iv) follows by independence and convexity (see [12, Lemma 4.2, p. 42]). The implication (iv)⇒(i) follows from the Itô and Nisio theorem [6]. If one of the conditions (i)–(iv) holds, then the integrability of the maximal functions follows by [5, Corollary 3.3]. ∎

REMARK. In the above theorem, if we assume that $\{X_n\} \subset L_p(\mathbb{P})$ for some $1 \le p < \infty$, and $\mathbb{E}(\sup_{n\ge1} |X_n|^p) < \infty$, then we have, in fact, convergence in $L_p(\mathbb{P}, C(\Gamma))$ and $L_p(\mathbb{P}, C(\Gamma))$-integrability of the maximal functions (see [5, Corollary 3.3]).

The next lemma is a simple case of Lemma 3.1 in [7].

LEMMA 2.3. *Let $\{X_n\} \subset L_1(\mathbb{P})$ be a sequence of independent random variables. Then $\mathbb{E}(\sup_{n\ge1} |X_n|) < \infty$ if and only if $\sum_{n=1}^\infty \mathbb{E}(|X_n| \mathbf{1}_{\{|X_n|\ge a\}}) < \infty$ for some $a > 0$.*

COROLLARY 2.4. *Let $1 < p < 2$, and let $\{a_n\}$ be a sequence of complex numbers such that*

$$\sum_{n=1}^\infty \frac{(\sum_{k=n}^\infty |a_k|^p)^{1/p}}{n(\log n)^{1/p}} < \infty.$$

Let $\{X_n\} \subset L_1(\mathbb{P})$ be a sequence of centered independent random variables. Assume that for some $M > 0$ we have $\mathbb{P}(|X_n| \ge u) \le M/u^p$ for every $u \ge 1$ and $n \ge 1$. Then the series $\sum_{n=1}^\infty a_n X_n \lambda^n$ is a.s. uniformly convergent.

Proof. It follows from Fernique [4] (see also Marcus and Pisier [13, Theorem B and the discussion in Remark 4.4]) that the result is true for *symmetric* independent $\{X_n\}$. Now, by the condition on the tails of the variables $\{X_n\}$, we have, for $a_n \ne 0$,

$$\begin{aligned}\mathbb{E}(|a_n X_n| \mathbf{1}_{\{|a_n X_n|\ge1\}}) &= |a_n| \int_{1/|a_n|}^\infty \mathbb{P}(|X_n| > u)\, du + \mathbb{P}(|a_n X_n| \ge 1)\\ &\le \frac{p}{p-1} M |a_n|^p.\end{aligned}$$

Hence, by Lemma 2.3 we have $\mathbb{E}(\sup_{n\ge1} |a_n X_n|) < \infty$. So, Theorem 2.2 yields the result. ∎

REMARK. In a similar way, the results of Fernique for $2 \le p < \infty$ (see [4, Examples (d) and (e)]) can be extended to the centered case.

COROLLARY 2.5. *Let $1 < p \le 2$, and let $\{X_n\} \subset L_p(\mathbb{P})$ be a sequence of centered independent random variables. If*

$$\sum_{n=2}^\infty \frac{(\sum_{k=n}^\infty \mathbb{E}|X_k|^p)^{1/p}}{n(\log n)^{1/p}} < \infty,$$

in particular, if for some $\varepsilon > 0$ the series $\sum_{n=2}^\infty \mathbb{E}|X_n|^p (\log n)^{p-1} (\log\log n)^{p+\varepsilon}$ converges, then the random Fourier series $\sum_{n=1}^\infty X_n \lambda^n$ is a.s. convergent in $C(\Gamma)$.

Proof. First we prove the case $p = 2$. By Theorem 5.1.5 of Salem and Zygmund [16], and by the *principle of reduction* [8, p. 9] (see also

[12, Ch. VII, §1]), we obtain the assertion for the case where $\{X_n\}$ are symmetric random variables. Since the condition of the theorem implies $\mathbb{E}(\sup_{n\ge1}|X_n|^p)<\infty$, the non-symmetric case follows from the symmetrization argument of Theorem 2.2.

In the case $1<p<2$ we do the following: for $X_n\neq 0$ put $a_n=(\mathbb{E}|X_n|^p)^{1/p}$ and $Y_n=X_n/(\mathbb{E}|X_n|^p)^{1/p}$. Otherwise put $a_n=0$ and $Y_n=0$. Clearly, we have $\mathbb{P}(|Y_n|>u)\le 1/u^p$. By application of Corollary 2.4 to $\{a_n\}$ and $\{Y_n\}$, we obtain the result. ∎

Remark. Under the condition of the corollary above, $\mathbb{E}(\sup_{n\ge1}|X_n|^p)<\infty$. Hence, Theorem 2.2 also yields convergence in $L_p(\mathbb{P},C(\Gamma))$ and L_p-integrability of the maximal function (see the remark after Theorem 2.2).

Corollary 2.6 (Cuzick and Lai, [2]). *Let $\{Z_n\}$ be centered i.i.d. random variables with $\int_\Omega |Z_1|\log^+|Z_1|\,d\mathbb{P}<\infty$. Then the series $\sum_{n=1}^\infty (Z_n/n)\lambda^n$ is a.s. uniformly convergent.*

Proof. It was proved in Cuzick and Lai [2] and Talagrand [17] that for a symmetric i.i.d. sequence $\{Z_n\}$ with $\int_\Omega |Z_1|\log^+\log^+|Z_1|\,d\mathbb{P}<\infty$, the conclusion of the corollary holds. Now, since $\int_\Omega |Z_1|\log^+|Z_1|\,d\mathbb{P}<\infty$ we have

$$\sum_{n=1}^\infty \mathbb{E}\Big(\frac{|Z_n|}{n}\mathbf{1}_{\{|Z_n|\ge n\}}\Big)\le \int_\Omega |Z_1|\sum_{n=1}^\infty \frac{\mathbf{1}_{\{|Z_1|\ge n\}}}{n}\,d\mathbb{P}\le \int_\Omega |Z_1|\sum_{n=1}^{[|Z_1|]+1}\frac{1}{n}\,d\mathbb{P}$$
$$\le \int_\Omega |Z_1|\log(|Z_1|+1)\,d\mathbb{P}<\infty,$$

where $[|Z_1|]$ denotes the integral part of $|Z_1|$. Hence, Lemma 2.3 and Theorem 2.2 yield the result. ∎

Remarks 1. Corollary 2.6 is already proven in [2]. We gave it to illustrate the use of Theorem 2.2, and also because the statement of this corollary is very close to the situation of the counterexample given in Theorem 3.1 below. Indeed, modulo a rearrangement, the sequence $\{a_n\}$ defined in the proof of Theorem 3.1 is essentially the sequence $\{1/n\}$, and by Lemma 2.3, the condition $\mathbb{E}(\sup|Z_n|/n)<\infty$ is actually equivalent to $\int|Z_1|\log^+|Z_1|\,d\mathbb{P}<\infty$. In Theorem 3.1 we have $\int|Z_1|(\log^+|Z_1|)^{1-\varepsilon}\,d\mathbb{P}<\infty$ for every $\varepsilon>0$, while $\int|Z_1|\log^+|Z_1|\,d\mathbb{P}=\infty$.

2. The optimality of the condition $\int|Z_1|\log^+|Z_1|\,d\mathbb{P}<\infty$ is also justified by the discussion in Marcinkiewicz and Zygmund [10, p. 78].

3. When $\{Z_n\}$ are symmetric i.i.d., Talagrand [17] proved that the condition $\int|Z_1|\log^+\log^+|Z_1|\,d\mathbb{P}<\infty$ is necessary and sufficient for the a.s. uniform convergence of $\sum_{n=1}^\infty Z_n\lambda^n/n$.

Using Billard's theorem and the symmetrization argument of Theorem 2.2, we obtain the following extension of Billard's theorem in the centered case.

THEOREM 2.7. *Let* $\{X_n\} \subset L_1(\mathbb{P})$ *be a complex sequence of centered independent random variables. If* $\mathbb{E}(\sup_{n\ge 1}|X_n|) < \infty$, *then the following conditions are equivalent for the random Fourier series* $\sum_{n=1}^{\infty} X_n\lambda^n$:

(i) *Almost surely the series represents a bounded function.*
(ii) *Almost surely the series represents a continuous function.*
(iii) *Almost surely the series converges everywhere.*
(iv) *Almost surely the series converges uniformly.*

DEFINITION 2.1. We say that the random Fourier series $\sum_{n=1}^{\infty} X_n\lambda^n$ has *property* $\mathcal{B}$ if conditions (i)–(iv) of Theorem 2.7 are all equivalent, i.e., either all of them hold or all of them are false.

REMARK. The condition $\sup_{n\ge 1}\|\sum_{k=1}^{n} X_k\lambda^k\|_{C(\varGamma)} < \infty$ a.s. implies that, almost surely, $\sum_{k=1}^{n} X_n\lambda^n$ represents a bounded function (see [19, Theorem 4.2(ii), p. 136]). Hence, under the assumption $\mathbb{E}(\sup_{n\ge 1}|X_n|) < \infty$ assertions (i)–(iv) are all equivalent to $\sup_{n\ge 1}\|\sum_{k=1}^{n} X_k\lambda^k\|_{C(\varGamma)} < \infty$ a.s., for centered independent variables. In the symmetric case, this remains true without the assumption $\mathbb{E}(\sup_{n\ge 1}|X_n|) < \infty$, by Billard's theorem (see also the discusssion in [8, Theorem 1, p. 13]).

Here we give a different type of moment condition.

COROLLARY 2.8. *Let* $\{X_n\} \subset L_4(\mathbb{P})$ *be a sequence of independent centered random variables, and assume that* $\sup_{n\ge 1}\mathbb{E}(|X_n|^4)/(\mathbb{E}|X_n|^2)^2 < \infty$. *Then the series* $\sum_{n=1}^{\infty} X_n\lambda^n$ *has property* $\mathcal{B}$.

Proof. Let $\{X'_n\}$ be an independent copy of $\{X_n\}$, and denote by $\mathbb{E}'$ the corresponding expectation. If one of the above conditions (i)–(iv) holds, then it also holds for $\sum_{n=1}^{\infty}(X_n - X'_n)\lambda^n$. Billard's theorem shows that a.s. $\sum_{n=1}^{\infty}(X_n - X'_n)\lambda^n$ converges uniformly. By the Riesz–Fischer theorem, $\sum_{n=1}^{\infty}|X_n - X'_n|^2$ converges a.s. Since $\mathbb{E}\mathbb{E}'|X_n - X'_n|^4 \le 16\mathbb{E}|X_n|^4$ and $\mathbb{E}\mathbb{E}'|X_n - X'_n|^2 = 2\mathbb{E}|X_n|^2$, we find that

$$\sup_{n\ge 1}\mathbb{E}\mathbb{E}'(|X_n - X'_n|^4)/(\mathbb{E}\mathbb{E}'|X_n - X'_n|^2)^2 < \infty.$$

Theorem 5 in [8, Ch. 3, p. 33], applied to $\{|X_n - X'_n|^2\}$, shows that $\sum_{n=1}^{\infty}\mathbb{E}\mathbb{E}'|X_n - X'_n|^2 < \infty$, and that yields $\mathbb{E}(\sup_{n\ge 1}|X_n|^2) \le \sum_{n=1}^{\infty}\mathbb{E}|X_n|^2 < \infty$. Theorem 2.7 yields the result. ∎

COROLLARY 2.9. *Let* $\{a_n\}$ *be a sequence of complex numbers, and let* $\{X_n\} \subset L_1(\mathbb{P})$ *be a sequence of centered independent random variables. Assume that there exists* $C > 0$ *such that* $\int_u^{\infty}\mathbb{P}(|X_n| > v)\,dv \le Cu\mathbb{P}(|X_n| > u)$

for every $n \geq 1$ and $u \geq 1$. Then the random Fourier series $\sum_{n=1}^{\infty} a_n X_n \lambda^n$ has property $\mathcal{B}$.

REMARKS. 1. If there exists $p > 1$ such that for every $n \geq 1$ the function $u \mapsto u^p \mathbb{P}(|X_n| > u)$ is non-increasing, then the condition of the corollary is satisfied.

2. The condition is also satisfied in the case where the $\{X_n\}$ are centered i.i.d., and the tail $\mathbb{P}(|X_1| > u)$ is regularly varying with exponent $\varrho < -1$ (see the definition in Feller [3, p. 276], and use [3, Theorem 1(a), p. 281]).

Proof of Corollary 2.9. Assume that the series $\sum_{n=1}^{\infty} a_n X_n \lambda^n$ satisfies one of the conditions (i)–(iv) above. Conditions (iii) or (iv) yield $a_n X_n \to 0$ a.s. Conditions (i) or (ii) and the Riemann–Lebesgue lemma imply that $a_n X_n \to 0$ a.s. Hence, the events $\{|a_n X_n| \geq 1\}$ take place a finite number of times a.s., and by the Borel–Cantelli lemma, the series $\sum_{n=1}^{\infty} \mathbb{P}(|a_n X_n| \geq 1)$ converges. By the assumption on the tail, applied with $u = 1/|a_n|$ (when $a_n \neq 0$), we obtain

$$\begin{aligned}\mathbb{E}(|a_n X_n| \mathbf{1}_{\{|a_n X_n| \geq 1\}}) &= |a_n| \int_{1/|a_n|}^{\infty} \mathbb{P}(|X_n| > v)\, dv + \mathbb{P}(|a_n X_n| \geq 1) \\ &\leq (C+1)\mathbb{P}(|a_n X_n| \geq 1).\end{aligned}$$

Hence by Lemma 2.3 we have $\mathbb{E}(\sup_{n \geq 1} |a_n X_n|) < \infty$, and the result follows by Theorem 2.7. ■

THEOREM 2.10. *Let $\{X_n\}$ be a sequence of centered independent random variables. If the deterministic series $\sum_{n=1}^{\infty} \mathbb{E}(X_n \mathbf{1}_{\{|X_n| \leq 1\}}) \lambda^n$ is uniformly convergent, then the random Fourier series $\sum_{n=1}^{\infty} X_n \lambda^n$ has property $\mathcal{B}$.*

Proof. Assume the series $\sum_{n=1}^{\infty} X_n \lambda^n$ satisfies one of the conditions (i)–(iv) above, say $(j) \in \{(\mathrm{i}), \ldots, (\mathrm{iv})\}$. As shown in the proof of Corollary 2.9, we have $X_n \to 0$ a.s., so $\sum_{n=1}^{\infty} X_n \mathbf{1}_{\{|X_n| \leq 1\}} \lambda^n$ satisfies condition (j). Since $\sum_{n=1}^{\infty} \mathbb{E}(X_n \mathbf{1}_{\{|X_n| \leq 1\}}) \lambda^n$ is uniformly convergent, the series $\sum_{n=1}^{\infty} [X_n \mathbf{1}_{\{|X_n| \leq 1\}} - \mathbb{E}(X_n \mathbf{1}_{\{|X_n| \leq 1\}})] \lambda^n$ satisfies condition (j). Theorem 2.7 shows that it is a.s. uniformly convergent, so the series $\sum_{n=1}^{\infty} X_n \mathbf{1}_{\{|X_n| \leq 1\}} \lambda^n$ is a.s. uniformly convergent. Since $X_n \to 0$ a.s., the series $\sum_{n=1}^{\infty} X_n \lambda^n$ satisfies (i)–(iv). ■

THEOREM 2.11. *Let $\{X_n\}$ be a sequence of centered independent random variables. If the random Fourier series $\sum_{n=1}^{\infty} X_n \lambda^n$ satisfies one of the above conditions* (i)–(iv), *then also the deterministic series $\sum_{n=1}^{\infty} \mathbb{E}(X_n \mathbf{1}_{\{|X_n| \leq 1\}}) \lambda^n$ satisfies this condition.*

Proof. Assume the series $\sum_{n=1}^{\infty} X_n \lambda^n$ satisfies one of the conditions (i)–(iv) above, say $(j) \in \{(\mathrm{i}), \ldots, (\mathrm{iv})\}$. So, $X_n \to 0$ a.s., and the series

$\sum_{n=1}^{\infty} X_n \mathbf{1}_{\{|X_n|\le 1\}}\lambda^n$ satisfies condition (j). Let $\{X'_n\}$ be an independent copy of $\{X_n\}$, and let $\mathbb{E}'$ be the corresponding expectation in the probability space of $\{X'_n\}$. For every $n \ge 1$, define the *symmetric* random variables $Y_n = X_n \mathbf{1}_{\{|X_n|\le 1\}} - X'_n \mathbf{1}_{\{|X'_n|\le 1\}}$. Now, the series $\sum_{n=1}^{\infty} Y_n\lambda^n$ satisfies condition (j), so by Billard's theorem it is a.s. uniformly convergent. Hence, by Theorem 2.2, the series $\sum_{n=1}^{\infty}[X_n \mathbf{1}_{\{|X_n|\le 1\}} - \mathbb{E}(X_n \mathbf{1}_{\{|X_n|\le 1\}})]\lambda^n$ is a.s. uniformly convergent, and the result follows. ∎

3. A counterexample in the centered case

THEOREM 3.1. *There exist a sequence $\{Z_n\}$ of independent identically distributed centered* (*hence in L_1*) *real random variables and a sequence $\{a_n\} \in \ell_2$ of complex numbers such that almost surely the Fourier series $\sum_{n=1}^{\infty} a_n Z_n \lambda^n$ converges at every $\lambda \in \Gamma$ and represents a continuous function, while $\sum_{n=1}^{\infty} a_n Z_n \lambda^n$ is almost surely not uniformly convergent. Moreover, almost surely, the partial sums of the series are uniformly bounded on the circle.*

REMARKS. 1. Cuzick and Lai [2, p. 10] constructed an example of a random Fourier series for which a.s. there is convergence at each point, *except one* for which, in fact, the random Fourier series diverges to infinity. So, the limit function is not continuous.

2. The sequence $\{a_n Z_n\}$ in the theorem is complex. We do not know of any counterexample with $\{a_n Z_n\}$ being real-valued.

Theorem 3.1 is based on the following basic result.

PROPOSITION 3.2. *There exists $C > 0$ such that for all $l \ge 2$, we have*

$$\sup_{n\ge 1}\sup_{x\in\mathbb{R}} \left| \sum_{k=1}^{n} \frac{\sin kx}{(k+l)\log(k+l)} \right| \le \frac{C}{\log l}.$$

Proof. Define

$$S_n(x) = \sum_{k=1}^{n} \frac{\sin kx}{k}$$

for every $n \ge 1$. It is well known (see e.g. Zygmund [19, Vol. I, p. 61]) that $\{S_n\}$ is uniformly bounded, i.e., $K := \sup_{n\ge 1}\sup_{x\in\mathbb{R}} |S_n(x)| < \infty$. Define

$$R_{l,n}(x) = \sum_{k=1}^{n} \frac{\sin kx}{(k+l)\log(k+l)}$$

for every $n \ge 1$, and put $S_0(x) \equiv 0$. Also, for any $l \ge 2$ and $k \ge 1$ define

$$u_l(k) = \frac{1}{\log(k+l)} \quad \text{and} \quad v_l(k) = \frac{l}{(k+l)\log(k+l)}.$$

By Abel's summation by parts we have

$$\begin{aligned} R_{l,n}(x) &= \sum_{k=1}^{n}[S_k(x) - S_{k-1}(x)]\,\frac{k}{(k+l)\log(k+l)} \\ &= \sum_{k=1}^{n}[S_k(x) - S_{k-1}(x)][u_l(k) - v_l(k)] \\ &= \sum_{k=1}^{n} S_k(x)[u_l(k) - v_l(k) - u_l(k+1) + v_l(k+1)] \\ &\quad + S_n(x)[u_l(n+1) - v_l(n+1)]. \end{aligned}$$

Hence, by monotonicity (in k) of $\{u_l(k)\}$ and $\{v_l(k)\}$ we have

$$\begin{aligned} |R_{l,n}(x)| &\le \sum_{k=1}^{n}|S_k(x)|[u_l(k) - u_l(k+1)] + \sum_{k=1}^{n}|S_k(x)|[v_l(k) - v_l(k+1)] \\ &\qquad + |S_n(x)||u_l(n+1) - v_l(n+1)| \\ &\le K[u_l(1) - u_l(n+1)] + K[v_l(1) - v_l(n+1)] + K[u_l(n+1) + v_l(n+1)] \\ &\le K[u_l(1) + v_l(1)] \le K\left[\frac{1}{\log(l+1)} + \frac{l}{(l+1)\log(l+1)}\right] \le \frac{2K}{\log(l+1)}. \ \blacksquare \end{aligned}$$

Proof of Theorem 3.1. For any $n \ge 1$ put $u_n = 2^{2^n}$ and $v_n = (u_n + u_{n+1})/2$. In addition put

$$a_k = \begin{cases} \dfrac{1}{u_n + v_n - k}\, e^{ik/n} & \text{if } u_n + 1 \le k \le v_n - 1, \\ -\dfrac{1}{u_n - v_n + k}\, e^{ik/n} & \text{if } v_n + 1 \le k \le u_{n+1} - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let Z be a real random variable with distribution law given by

$$\mathbb{P}(Z = -1) = \alpha, \qquad \mathbb{P}(Z \ge t) = \beta \int_t^\infty f(x)\,dx \qquad \text{for } t \ge 2,$$

where $f(x) = \frac{1}{x^2(\log\, x)^2}$, and α and β are chosen such that Z be a centered random variable.

Let $\{Z_n\}$ be a sequence of independent copies of Z. We want to prove that the series $\sum_{n=1}^\infty a_n Z_n \lambda^n$ satisfies the assertions of the theorem.

As suggested by Theorems 2.10 and 2.11, we will proceed by truncation in order to reduce our proof to the case of a deterministic Fourier series.

For any $n \ge 1$ put

$$b_k = \begin{cases} \dfrac{u_n + v_n - k}{\log(u_n + v_n - k)} & \text{if } u_n \le k \le v_n - 1, \\ \dfrac{u_n - v_n + k}{\log(u_n - v_n + k)} & \text{if } v_n \le k \le u_{n+1} - 1, \\ 0 & \text{otherwise.} \end{cases}$$

We define $Y_n = Z_n \mathbf{1}_{\{Z_n \le b_n\}}$, and we show that the series $\sum_{n=1}^{\infty} |a_n|\,|Z_n - Y_n|$ converges a.s.

(i) Since $\mathbb{E}|Z| < \infty$,

$$\sum_{k=5}^{\infty} \mathbb{P}(|a_k|\,|Z_k - Y_k| \ge 1)$$

$$= \sum_{n=1}^{\infty} \Bigg(\sum_{k=u_n+1}^{v_n-1} \mathbb{P}(|Z_k - Y_k| \ge u_n + v_n - k) + \sum_{k=v_n+1}^{u_{n+1}-1} \mathbb{P}(|Z_k - Y_k| \ge u_n - v_n + k) \Bigg)$$

$$= \sum_{n=1}^{\infty} \Bigg(\sum_{k=u_n+1}^{v_n-1} \mathbb{P}(Z_k \ge u_n + v_n - k) + \sum_{k=v_n+1}^{u_{n+1}-1} \mathbb{P}(Z_k \ge u_n - v_n + k) \Bigg)$$

$$= 2 \sum_{n=1}^{\infty} \sum_{k=u_n+1}^{v_n-1} \mathbb{P}(Z \ge k) \le 2 \sum_{n=5}^{\infty} \mathbb{P}(Z \ge n) < \infty.$$

(ii) In a similar way, breaking the sums, we have

$$\sum_{k=5}^{\infty} \mathbb{E}(|a_k|\,|Z_k - Y_k| \mathbf{1}_{\{|a_k|\,|Z_k - Y_k| \le 1\}})$$

$$= \sum_{n=1}^{\infty} \Bigg(\sum_{k=u_n+1}^{v_n-1} |a_k| \mathbb{E}(Z_k \mathbf{1}_{\{b_k < Z_k \le 1/|a_k|\}}) + \sum_{k=v_n+1}^{u_{n+1}-1} |a_k| \mathbb{E}(Z_k \mathbf{1}_{\{b_k < Z_k \le 1/|a_k|\}}) \Bigg)$$

$$= \sum_{n=1}^{\infty} \sum_{k=u_n+1}^{v_n-1} \frac{1}{u_n + v_n - k} \mathbb{E}(Z \mathbf{1}_{\{\frac{u_n+v_n-k}{\log(u_n+v_n-k)} \le Z \le u_n+v_n-k\}})$$

$$+ \sum_{n=1}^{\infty} \sum_{k=v_n+1}^{u_{n+1}-1} \frac{1}{u_n - v_n + k} \mathbb{E}(Z \mathbf{1}_{\{\frac{u_n-v_n+k}{\log(u_n-v_n+k)} \le Z \le u_n-v_n+k\}})$$

$$= 2 \sum_{n=1}^{\infty} \sum_{k=u_n+1}^{v_n-1} \frac{1}{k} \mathbb{E}(Z \mathbf{1}_{\{k/\log k \le Z \le k\}}) \le 2 \sum_{n=5}^{\infty} \frac{1}{n} \mathbb{E}(Z \mathbf{1}_{\{n/\log n \le Z \le n\}})$$

$$= 2\beta \sum_{n=5}^{\infty} \frac{1}{n} \int_{n/\log n}^{n} x f(x)\, dx = 2\beta \sum_{n=5}^{\infty} \frac{1}{n} \left(\frac{1}{\log n - \log\log n} - \frac{1}{\log n} \right)$$

$$\le C \sum_{n=5}^{\infty} \frac{\log\log n}{n(\log n)^2} < \infty.$$

Combining (i) and (ii) shows that $\sum_{n=1}^{\infty} |a_n|\,|Z_n - Y_n|$ converges a.s.

We have

$$\sum_{n=1}^{\infty} 2^{n/2}\Big(\sum_{k=u_n+1}^{u_{n+1}} \mathbb{E}|a_k Y_k|^2\Big)^{1/2}$$
$$\le \sum_{n=1}^{\infty} 2^{n/2}\bigg(2\sum_{k=u_n+1}^{v_n-1}\frac{1}{k^2}\,\mathbb{E}(Z^2\mathbf{1}_{\{|Z|\le k/\log k\}})\bigg)^{1/2}$$
$$\le \sqrt{2}\sum_{n=1}^{\infty} 2^{n/2}\bigg(\sum_{k=u_n+1}^{\infty}\frac{1}{k^2}\Big(\alpha+\int_2^{k/\log k} x^2 f(x)\,dx\Big)\bigg)^{1/2}$$
$$\le C\sum_{n=1}^{\infty} 2^{n/2}\bigg(\sum_{k\ge u_n+1}\frac{1}{k(\log k)^3}\bigg)^{1/2} \le C'\sum_{n=1}^{\infty} 2^{-n/2} < \infty,$$

for some positive constants C and C'.

Let $\{Y_n'\}$ be an independent copy of $\{Y_n\}$ and write $\widehat{Y}_n = Y_n - Y_n'$. Then

$$\sum_{n\ge 1} 2^{n/2}\Big(\sum_{k=u_n+1}^{u_{n+1}} \mathbb{E}|a_k\widehat{Y}_k|^2\Big)^{1/2} < \infty.$$

By the proof of Theorem 1 in [8, Ch. 7, p. 84] (see also [18, Theorem 10]), almost surely, the random Fourier series $\sum_{n=1}^{\infty} a_n\widehat{Y}_n\lambda^n$ is uniformly convergent. Since by construction we have $\sup_{n\ge 1}|a_nY_n| < \infty$ everywhere, Theorem 2.2 shows that the random Fourier series $\sum_{n=1}^{\infty} a_n(Y_n - \mathbb{E}Y_n)\lambda^n$ is a.s. uniformly convergent. Since by (i) and (ii) above the series $\sum_{n=1}^{\infty} a_n(Y_n - Z_n)\lambda^n$ is a.s. absolutely convergent, the random Fourier series $\sum_{n=1}^{\infty} a_nZ_n\lambda^n$ a.s. converges everywhere (or uniformly), or represents a continuous function if and only if the *deterministic* series $\sum_{n=1}^{\infty} a_n\mathbb{E}Y_n\lambda^n$ converges everywhere (or uniformly), or represents a continuous function.

For any $k\ge 5$ we have

$$\mathbb{E}(Z\mathbf{1}_{\{Z\le k/\log k\}}) = -\mathbb{E}(Z\mathbf{1}_{\{Z>k/\log k\}})$$
$$= -\beta\int_{k/\log k}^{\infty} xf(x)\,dx = -\frac{\beta}{\log k - \log\log k}.$$

Hence,

$$\mathbb{E}Y_k = -\frac{\beta}{\log|a_k| - \log\log|a_k|},$$

and we obtain

$$(*)\qquad \mathbb{E}Y_k = -\frac{\beta}{\log|a_k|} + O\bigg(\frac{\log\log|a_k|}{\log^2|a_k|}\bigg).$$

Using the convention $0/\log 0 = 0$, define

$$T(\lambda) = \sum_{k=1}^{\infty} \frac{a_k}{\log|a_k|}\,\lambda^k$$

in $L_2(d\lambda)$. By $(*)$ we have

$$\sum_{k=1}^{\infty} \left| a_k \mathbb{E} Y_k + \frac{\beta a_k}{\log|a_k|} \right| < \infty.$$

Hence, it is enough for our purpose to study the behavior of $T(\lambda)$.

For any $n \geq 1$, define

$$Q_n(\lambda) = \sum_{k=u_n+1}^{v_n-1} \frac{\lambda^k - \lambda^{2v_n-k}}{(u_n+v_n-k)\log(u_n+v_n-k)}.$$

We have

$$T(\lambda) = \sum_{n=1}^{\infty} \sum_{k=u_n+1}^{u_{n+1}-1} \frac{a_k}{\log|a_k|}\,\lambda^k = \sum_{n=1}^{\infty} \sum_{k=u_n+1}^{v_n-1} \frac{(e^{i/n}\lambda)^k - (e^{i/n}\lambda)^{2v_n-k}}{(u_n+v_n-k)\log(u_n+v_n-k)}$$
$$= \sum_{n=1}^{\infty} Q_n(e^{i/n}\lambda).$$

Since

$$Q_n(\lambda) = \sum_{k=u_n+1}^{v_n-1} \frac{\lambda^k - \lambda^{2v_n-k}}{(u_n+v_n-k)\log(u_n+v_n-k)}$$
$$= \lambda^{v_n} \sum_{k=1}^{v_n-u_n-1} \frac{\lambda^{-k} - \lambda^k}{(k+u_n)\log(k+u_n)},$$

Proposition 3.2 yields

$$\|Q_n\|_{C(\Gamma)} \leq \frac{2C}{\log u_n} \leq \frac{2C}{2^n}.$$

Hence the partial sums $\sum_{k=1}^{u_{n+1}} a_k\lambda^k/\log|a_k| = \sum_{k=1}^{n} Q_k(e^{i/k}\lambda)$ converge uniformly, necessarily to $T(\lambda)$, so T represents a continuous function.

Now we show that $\sum_{k=1}^{\infty} a_k\lambda^k/\log|a_k|$ converges pointwise to $T(\lambda)$. For any $u_n < m \leq u_{n+1}$ we have

$$\left| \sum_{k=1}^{m} \frac{a_k\lambda^k}{\log|a_k|} - T(\lambda) \right| \leq \left| \sum_{k=1}^{u_n} \frac{a_k\lambda^k}{\log|a_k|} - T(\lambda) \right| + \left| \sum_{k=u_n+1}^{m} \frac{a_k\lambda^k}{\log|a_k|} \right|.$$

As already shown, the first term on the right hand side converges uniformly to zero, so it is enough to show that pointwise

$$\max_{u_n<m\leq u_{n+1}} \left| \sum_{k=u_n+1}^{m} \frac{a_k\lambda^k}{\log|a_k|} \right| \to_n 0.$$

Since we clearly have

$$\max_{u_n<m\le u_{n+1}}\Big|\sum_{k=u_n+1}^{m}\frac{a_k\lambda^k}{\log|a_k|}\Big| \le \max_{u_n<m\le v_n}\Big|\sum_{k=u_n+1}^{m}\frac{a_k\lambda^k}{\log|a_k|}\Big| + \max_{v_n<m\le u_{n+1}}\Big|\sum_{k=v_n+1}^{m}\frac{a_k\lambda^k}{\log|a_k|}\Big|,$$

it is enough to show that each of the terms on the right hand side goes to zero pointwise. Put $D_n(t) := \sum_{k=1}^{n} e^{int}$. Simple computation yields $|D_n(t)| \le 4/|t|$ for $0<|t|\le\pi$. Fix $\lambda_0 = e^{it_0}$. Using Abel's summation by parts we have

$$\begin{aligned}\sum_{k=u_n+1}^{m}\frac{a_k\lambda_0^k}{\log|a_k|} &= \sum_{k=u_n+1}^{m}\frac{|a_k|e^{ik(1/n+t_0)}}{\log|a_k|}\\ &= \sum_{k=u_n+1}^{m}\frac{|a_k|[D_k(1/n+t_0)-D_{k-1}(1/n+t_0)]}{\log|a_k|}\\ &= \sum_{k=u_n+1}^{m}\Big(\frac{|a_k|}{\log|a_k|}-\frac{|a_{k+1}|}{\log|a_{k+1}|}\Big)D_k(1/n+t_0)\\ &\quad -\frac{|a_{u_n+1}|D_{u_n}(1/n+t_0)}{\log|a_{u_n+1}|}+\frac{|a_{m+1}|D_m(1/n+t_0)}{\log|a_{m+1}|}.\end{aligned}$$

For any t_0 we have eventually $|1/n+t_0|>0$. Using monotonicity of $\{-|a_k|/\log|a_k|\}$ we have

$$\Big|\sum_{k=u_n+1}^{m}\frac{a_k\lambda_0^k}{\log|a_k|}\Big| \le \frac{8}{|1/n+t_0|}\Big(\frac{1}{u_n\log u_n}-\frac{1}{v_n\log v_n}\Big).$$

Hence

$$\max_{u_n<m\le v_n}\Big|\sum_{k=u_n+1}^{m}\frac{a_k\lambda_0^k}{\log|a_k|}\Big| \to_n 0.$$

Similarly

$$\max_{v_n<m\le u_{n+1}}\Big|\sum_{k=v_n+1}^{m}\frac{a_k\lambda^k}{\log|a_k|}\Big| \to_n 0$$

pointwise.

Now, by evaluating certain blocks of the series $\sum_{k=1}^{\infty} a_k\lambda^k/\log|a_k|$ along the sequence $\{e^{-i/n}\}$, we show that $\sum_{k=1}^{n} a_k\lambda^k/\log|a_k|$ is not Cauchy in $C(\Gamma)$, hence does not converge uniformly. This is established by the following:

$$\left|\sum_{k=u_n+1}^{v_n-1} \frac{a_k e^{-ik/n}}{\log|a_k|}\right| = \sum_{k=u_n+1}^{v_n-1} \frac{1}{(u_n+v_n-k)\log(u_n+v_n-k)} = \sum_{k=u_n+1}^{v_n-1} \frac{1}{k\log k}$$
$$\geq \log\log(v_n-1) - \log\log(u_n+1) \geq \log\log(u_{n+1}/2) - \log\log(2u_n)$$
$$\geq \log((2^{n+1}-1)\log 2) - \log((2^n+1)\log 2) \geq \log 2 + o(1/2^n).$$

On the other hand, since

$$\sup_{m\geq 1}\max_{|\lambda|=1}\left|\sum_{k=1}^{m} \frac{a_k\lambda^k}{\log|a_k|}\right| \leq \sup_{n\geq 1}\max_{|\lambda|=1}\left|\sum_{k=1}^{u_n} \frac{a_k\lambda^k}{\log|a_k|}\right| + \sup_{n\geq 1}\left\{\sum_{k=u_n+1}^{u_{n+1}} \frac{|a_k|}{-\log|a_k|}\right\}$$
$$\leq \sum_{k=1}^{\infty} \|Q_k(e^{i/k}\lambda)\|_{C(\Gamma)} + 2\sup_{n\geq 1}\left\{\sum_{k=u_n+1}^{v_n-1} \frac{1}{k\log k}\right\}$$
$$\leq 4C + 2\sup_{n\geq 1}(\log\log(v_n) - \log\log(u_n))$$
$$\leq 4C + 2\sup_{n\geq 1}(\log\log(2^{2^{n+1}}) - \log\log(2^{2^n})) = 4C + 2\log 2,$$

the partial sums $\sum_{k=1}^{n} a_k\lambda^k/\log|a_k|$ are uniformly bounded. ∎

Remark. The construction in Theorem 3.1 is inspired by the counterexample of Fejér as presented in Zygmund [19, Vol. I, p. 299].

Corollary 3.3. *There exist a sequence $\{Z_n\}$ of independent identically distributed centered real random variables and a sequence $\{a_n\} \in \ell_2$ of complex numbers such that almost surely, for any contraction T on a Hilbert space $\mathcal{H}$ the series $\sum_{n=1}^{\infty} a_n Z_n T^n h$ converges in norm for every $h \in \mathcal{H}$, but the Fourier series $\sum_{n=1}^{\infty} a_n Z_n \lambda^n$ does not converge uniformly.*

Proof. For the sequence constructed in the previous theorem, a.s. the partial sums $\sum_{k=1}^{n} a_k Z_k \lambda^k$ are uniformly bounded. We use the spectral theorem and the dominated convergence theorem to establish the corollary for unitary operators. Then we apply the unitary dilation theorem to pass to the contraction case (see [15, Appendix 4, §153]). ∎

Remark. Let $\{b_n\}$ be a series of complex numbers, and let $\{\gamma_n\}$ be a dense sequence in Γ. On ℓ_2 we define a unitary operator U by the following rule: $U\mathbf{c} = \{\gamma_n c_n\}$ for every $\mathbf{c} := \{c_n\} \in \ell_2$. It is easy to check that uniform convergence of $\sum_{n=1}^{\infty} b_n\lambda^n$ is equivalent to operator norm convergence of $\sum_{n=1}^{\infty} b_n U^n$. Hence, in the above corollary we cannot have operator norm convergence for all contractions T (even only all unitary operators) on some Hilbert space.

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