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*FUNCTIONAL ANALYSIS*

# On Property $\boldsymbol{\beta}$ of Rolewicz in Köthe–Bochner Function Spaces

by

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**Summary.** It is proved that the Köthe–Bochner function space $E(X)$ has property $\boldsymbol{\beta}$ if and only if $X$ is uniformly convex and $E$ has property $\boldsymbol{\beta}$. In particular, property $\boldsymbol{\beta}$ does not lift from $X$ to $E(X)$ in contrast to the case of Köthe–Bochner sequence spaces.

**1. Introduction.** The geometry of Köthe–Bochner spaces $E(X)$ of vector-valued functions has been intensively developed during the last years (see for example [2], [3], [9], [14], [16], [21], [22] and [27]). A survey of geometry in Köthe–Bochner spaces can be found in [23]. $E(X)$ are generalizations of Lebesgue–Bochner and Orlicz–Bochner spaces. One of the principal problems in these spaces is the question whether or not a geometric property lifts from $X$ and $E$ to $E(X)$. The answer is often the same in the case of function and sequence Köthe–Bochner spaces. However, the really peculiar situation is when the relevant criteria are different. This is the case for the Kadec–Klee property (**KK** for short), uniform Kadec–Klee property (**UKK**) and nearly uniform convexity (**NUC**).

Property **KK** is also known as the Radon–Riesz property ([10]). It has been intensively studied in Köthe–Bochner spaces, and shown to lift from $X$ to $E(X)$ when $E$ is a Köthe sequence space, but not necessarily if $E$ is a Köthe function space ([2], [16], [22] and [27]).

Properties **UKK** and **NUC** have been introduced by Huff in [10]. He proved that a Banach space is nearly uniformly convex if and only if it has the uniform Kadec–Klee property and is reflexive. The criteria for **UKK** and **NUC** of Köthe–Bochner sequence spaces have been proved in [14] and [21].

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It turns out that the function and sequence cases are essentially different with regard to these properties (see [23] and also Theorem 1 below).

In the whole paper, considering a property $A$ in the class of Banach spaces, we denote by $(A)$ the class of spaces with property $A$.

In this paper we study property $\boldsymbol{\beta}$ in Köthe–Bochner function spaces. This property was introduced by Rolewicz in [26]. He proved the implications $\mathbf{UC} \Rightarrow \boldsymbol{\beta} \Rightarrow \mathbf{NUC}$, where $\mathbf{UC}$ denotes uniform convexity. Moreover, the class of spaces with an equivalent norm with property $\boldsymbol{\beta}$ coincides neither with that of superreflexive spaces ([18] and [25]) nor with the class of nearly uniformly convexifiable spaces ([17]). It is known that in Orlicz sequence spaces property $\boldsymbol{\beta}$ coincides with reflexivity, and in Orlicz–Lorentz function spaces property $\boldsymbol{\beta}$ and uniform convexity are equivalent ([4] and [13]). Moreover, if a Banach space $X$ has property $\boldsymbol{\beta}$, then both $X$ and $X^*$ have the fixed point property (**FPP** for short) for nonexpansive self-maps on closed, bounded, convex and nonempty sets. For $X$, this follows from the theorem that if $X \in (\mathbf{NUC})$, then $X \in (\mathbf{FPP})$ ([11]). Moreover, property $\boldsymbol{\beta}$ implies normal structure of the dual space ([20]). Since normal structure implies weak normal structure and they coincide in the class of reflexive spaces, and property $\boldsymbol{\beta}$ implies reflexivity, it follows that property $\boldsymbol{\beta}$ implies the fixed point property for the dual space.

Denote by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$ the sets of natural, real and non-negative real numbers, respectively. We will let $(T, \Sigma, \mu)$ be a $\sigma$-finite, complete measure space. By $L^0 = L^0(T)$ we denote the set of all $\mu$-equivalence classes of real-valued measurable functions defined on $T$.

A Banach space $E = (E, \|\cdot\|_E)$ is said to be a *Köthe space* if $E$ is linear subspace of $L^0$ and:

(i) if $x \in E, y \in L^0, |y| \le |x|$ $\mu$-a.e. in $T$, then $y \in E$ and $\|y\|_E \le \|x\|_E$,
(ii) there exists a function $x$ in $E$ that is positive on the whole $T$ (see [12] and [24]).

Every Köthe space is a Banach lattice under the obvious order ($x \ge 0$ if $x(t) \ge 0$ for $\mu$-a.e. $t \in T$). In particular, if we consider the space $E$ over the non-atomic measure space $(T, \Sigma, \mu)$, then we shall say that $E$ is a *Köthe function space.* If we replace the measure space $(T, \Sigma, \mu)$ by the counting measure space $(\mathbb{N}, 2^{\mathbb{N}}, m)$, then $E$ is a *Köthe sequence space.*

A Köthe space $E$ is said to be *uniformly monotone* ($E \in (\mathbf{UM})$) if for every $q \in (0,1)$ there exists $p \in (0,1)$ such that for all $0 \le y \le x$ satisfying $\|x\|_E = 1$ and $\|y\|_E \ge q$, we have

$$\|x - y\|_E \le 1 - p$$

(see [7]). Then the *modulus* $p(\cdot)$ *of uniform monotonicity* of $E$ is defined as follows:

$$p(q) = \inf\{1 - \|x - y\|_E : \|x\|_E = 1,\ \|y\|_E \ge q,\ 0 \le y \le x\}.$$

A Köthe space $E$ is called *order continuous* ($E \in (\mathbf{OC})$) if for every $x \in E$ and every sequence $(x_m) \subset E$ such that $0 \le x_m \le |x|$ and $x_m \to 0$ $\mu$-a.e. in $T$, we have $\|x_m\|_E \to 0$ (see [12] and [24]).

For a real Banach space $(X, \|\cdot\|_X)$, $B(X)$ and $S(X)$ stand for the closed unit ball and the unit sphere of $X$, respectively. For any subset $A$ of $X$, we denote by $\operatorname{conv}(A)$ the convex hull of $A$.

A Banach space $(X, \|\cdot\|_X)$ is said to be *uniformly convex* ($X \in (\mathbf{UC})$) if for each $\varepsilon > 0$ there is $\delta > 0$ such that for any $x, y \in S(X)$ the inequality $\|x - y\|_X > \varepsilon$ implies $\|x + y\|_X < 2(1 - \delta)$.

We say that for a given $\varepsilon > 0$ a sequence $\{x_n\} \subset X$ is *$\varepsilon$-separated* if

$$\operatorname{sep}\{x_n\}_X = \inf\{\|x_n - x_m\|_X : n \ne m\} > \varepsilon.$$

Although the original definition of property $\beta$ uses the Kuratowski measure of noncompactness (see [26]), the following equivalent formulation given by Kutzarova in [19] is more convenient for our considerations.

LEMMA 1. *A Banach space $X$ has property $\beta$ if and only if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for each $x \in B(X)$ and each sequence $(x_n)$ in $B(X)$ with* $\operatorname{sep}\{x_n\}_X \ge \varepsilon$ *there is an index $k$ for which $\|(x + x_k)/2\|_X \le 1 - \delta$.*

A Banach space $X$ is said to be *nearly uniformly convex* ($X \in (\mathbf{NUC})$) if for every $\varepsilon > 0$ there exists $\delta \in (0, 1)$ such that for every sequence $\{x_n\} \subseteq B(X)$ with $\operatorname{sep}\{x_n\}_X \ge \varepsilon$, we have $\operatorname{conv}(\{x_n\}) \cap (1 - \delta)B(X) \ne \emptyset$. This property has been independently introduced and studied by Goebel and Sękowski using the Kuratowski measure of noncompactness ([11]).

A Banach space $X$ is said to have the *uniform Kadec-Klee property* ($X \in (\mathbf{UKK})$ for short) if for every $\varepsilon > 0$ there exists $\delta \in (0, 1)$ such that $\|x\|_X < 1 - \delta$ whenever $(x_n) \subset B(X)$, $x_n \xrightarrow{w} x$ and $\operatorname{sep}\{x_n\}_X \ge \varepsilon$.

Now, let us define the type of spaces to be considered in this paper. For a real Banach space $(X, \|\cdot\|_X)$, denote by $M(T, X)$, or just $M(X)$, the family of strongly measurable functions $x : T \to X$, where functions which are equal $\mu$-almost everywhere are identified. Define

$$\widetilde{x}(\cdot) = \|x(\cdot)\|_X \quad \text{and} \quad E(X) = \{x \in M(X) : \widetilde{x} \in E\}.$$

Then $E(X)$ equipped with the norm $\|x\| = \|\widetilde{x}\|_E$ becomes a Banach space that is called a *Köthe–Bochner space.*

**2. Auxiliary lemmas.** Define $r \wedge s = \min\{r, s\}$ and $r \vee s = \max\{r, s\}$ for $r, s \in \mathbb{R}$.

LEMMA 2 ([9, Lemma 1]). *Let $x, y \in X \setminus \{0\}$. Set $\widehat{x} = x/\|x\|_X$.*

(i) *If $\|\widehat{x} - \widehat{y}\|_X \ge \varepsilon$ and $\|x\|_X \wedge \|y\|_X \ge \eta\{\|x\|_X \vee \|y\|_X\}$, then*

$$\|x + y\|_X \le (1 - \eta\delta_X(\varepsilon))(\|x\|_X + \|y\|_X),$$

*where $\delta_X(\cdot)$ is the modulus of convexity of the space $X$, i.e.*

$$\delta_X(\varepsilon) = \inf\left\{1 - \tfrac{1}{2}\|x+y\|_X : x, y \in S(X), \|x-y\|_X \geq \varepsilon\right\}.$$

(ii) *We have*

$$\|x+y\|_X \leq |\|x\|_X - \|y\|_X| + \{\|x\|_X \wedge \|y\|_X\}(\|\widehat{x} + \widehat{y}\|_X).$$

A geometric property, called *orthogonal uniform convexity*, is essential in studying property $\boldsymbol{\beta}$ in Köthe–Bochner spaces ([8]). It has been introduced in [13] in order to investigate property $\boldsymbol{\beta}$ in Banach lattices.

Definition 1. We say that a Köthe space $(E, \|\cdot\|_E)$ is *orthogonally uniformly convex* ($E \in (\mathbf{UC}^{\perp})$) if for each $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that for $x, y \in B(E)$ the inequality $\|x\chi_{A_{xy}}\|_E \vee \|y\chi_{A_{xy}}\|_E \geq \varepsilon$ implies that $\|(x+y)/2\|_E \leq 1 - \delta$, where $A_{xy} = \operatorname{supp} x \div \operatorname{supp} y$ and $A \div B = (A \setminus B) \cup (B \setminus A)$.

Obviously, if $E \in (\mathbf{UC})$, then $E \in (\mathbf{UC}^{\perp})$. It is known that any uniformly convex Köthe space is uniformly monotone ([7]). Moreover, the following stronger result is true.

Lemma 3 ([13, Lemma 3]). *Let $E$ be a Köthe space. If $E \in (\mathbf{UC}^{\perp})$, then $E \in (\mathbf{UM})$.*

It is known that in Köthe sequence spaces one has the implications $\mathbf{UC} \Rightarrow \mathbf{UC}^{\perp} \Rightarrow \boldsymbol{\beta}$ and none of them can be reversed in general ([15]). On the other hand, the implications

$$\mathbf{UC} \Rightarrow \boldsymbol{\beta} \Rightarrow \mathbf{UC}^{\perp} \tag{1}$$

hold in Köthe function spaces and the last one cannot be reversed ([13], [15] and [26]).

## 3. Results

Theorem 1. *Suppose that $(T, \Sigma, \mu)$ is a measure space which is not purely atomic. Let $X$ be a real Banach space. Assume that $X$ is separable or $X^*$ has the Radon–Nikodym property. If $E(X)$ has the uniform Kadec–Klee property, then $X$ is uniformly convex.*

*Proof.* We apply some techniques from the proof of Theorem 3.5 in [2] and Theorem 3.4.9 in [23]. Since $(T, \Sigma, \mu)$ is not purely atomic, there exists $A \in \Sigma$ such that $0 < \mu(A) < \infty$ and $A$ has no atoms. Define the family of sets $A(j, 2^k)$, for $j = 1, 2, \ldots, 2^k$ and $k = 1, 2, \ldots$, by the following iteration. We divide $A$ into two disjoint subsets $A(1,2)$ and $A(2,2)$ such that $\mu(A(1,2)) = \mu(A(2,2))$. Suppose that for a fixed $k$ the sets $A(j, 2^k)$ $(1 \leq j \leq 2^k)$ are already defined. To obtain $A(j, 2^{k+1})$ $(1 \leq j \leq 2^{k+1})$ we divide every set $A(j, 2^k)$ $(1 \leq j \leq 2^k)$ into two disjoint subsets $A(2j-1, 2^{k+1})$ and $A(2j, 2^{k+1})$ such that $\mu(A(2j-1, 2^{k+1})) = \mu(A(2j, 2^{k+1}))$. Define

$$A_k^1 = \bigcup_{i=1}^{2^k} A(2i-1, 2^k) \quad \text{and} \quad A_k^2 = \bigcup_{i=1}^{2^k} A(2i, 2^k) \quad \text{for } k = 1, 2, \ldots.$$

Define the $k$th Rademacher function on $A$ by

$$r_k(t) = \begin{cases} 1 & \text{for } t \in A_k^1, \\ -1 & \text{for } t \in A_k^2. \end{cases}$$

Suppose for contradiction that $E(X) \in (\mathbf{UKK})$ and $X \notin (\mathbf{UC})$. Then there exists a number $\varepsilon > 0$ and sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in $B(X)$ satisfying

$$\|x_n - y_n\|_X \ge \varepsilon \quad \text{and} \quad \|x_n + y_n\|_X \ge 2(1 - 1/n) \tag{2}$$

for each $n \in \mathbb{N}$. Let

$$f_k^n(t) = \frac{1}{\|\chi_A\|_E} (x_n \chi_{A_k^1}(t) + y_n \chi_{A_k^2}(t)) \quad \text{for } k = 1, 2, \ldots,$$
$$f^n(t) = \frac{1}{2\|\chi_A\|_E} (x_n + y_n)\chi_A(t)$$

for every $n \in \mathbb{N}$. Then

$$\|f_k^n\| = \frac{1}{\|\chi_A\|_E} \big\| \|x_n\|_X \chi_{A_k^1} + \|y_n\|_X \chi_{A_k^2} \big\|_E \le \frac{1}{\|\chi_A\|_E} \|\chi_{A_k^1} + \chi_{A_k^2}\|_E = 1$$

for all $n, k \in \mathbb{N}$. Analogously, we conclude that $f^n \in B(E(X))$ for every $n \in \mathbb{N}$. We will show that for each $n \in \mathbb{N}$ we have $f_k^n \xrightarrow{w} f^n$ as $k \to \infty$ in $E(X)$. Let $f^*$ be a continuous linear functional on $E(X)$. Note that from the implications $E(X) \in (\mathbf{UKK}) \Rightarrow E \in (\mathbf{UKK}) \Rightarrow E \in (\mathbf{KK})$ it follows that $E \in (\mathbf{OC})$, because $\mathbf{KK} \Rightarrow \mathbf{OC}$ in any Köthe space ([5]). Hence, applying Theorem 5.3 from [6] or the corresponding result from [1], we can write this functional in the form

$$f^*(f) = \int_T \langle f(t), g(t) \rangle \, d\mu \quad \text{for any } f \in E(X),$$

where $\langle x, x^* \rangle$ stands for the value of $x^* \in X^*$ at $x \in X$, and $g \in E'(X^*)$, that is, $g$ is a strongly measurable function from $T$ to $X^*$ and $E'$ is the Köthe dual of $E$ equipped with the norm defined by

$$\|h\|_{E'} = \sup\Big\{ \int_T |f(t)h(t)| \, d\mu : f \in E, \, \|f\|_E \le 1 \Big\}$$

for any $h \in M(\mathbb{R})$. Hence for any fixed $n \in \mathbb{N}$ and $f^* \in (E(X))^*$, we have

$$\begin{aligned} \lim_{k\to\infty} f^*(f_k^n - f^n) &= \lim_{k\to\infty} \int_T \langle (f_k^n - f^n)(t), g(t) \rangle \, d\mu \\ &= \frac{1}{2\|\chi_A\|_E} \lim_{k\to\infty} \int_A r_k(t) \langle (x_n - y_n), g(t) \rangle \, d\mu = 0. \end{aligned} \tag{3}$$

This follows from the facts that $\langle (x_n - y_n), g(\cdot)\rangle \chi_A$ is a real integrable function, the set of simple functions is dense in $L^1$, and $\lim_{k\to\infty} \int_A r_k(t) b(t)\, d\mu = 0$ for every simple function $b$ defined on $A$.

Define $h_k^n = f_k^n - f^n$ for all $n, k \in \mathbb{N}$. By (3) we conclude that $h_k^n \xrightarrow{w} 0$ as $k \to \infty$ in $E(X)$ for every $n \in \mathbb{N}$. Moreover, by (2), we get

$$\|h_k^n\| = \left\| \frac{\|x_n - y_n\|_X}{2\|\chi_A\|_E}\, \chi_{A_k^1} + \frac{\|y_n - x_n\|_X}{2\|\chi_A\|_E}\, \chi_{A_k^2} \right\|_E \geq \frac{\varepsilon}{2}$$

for all $n, k \in \mathbb{N}$. Then, applying the Hahn–Banach theorem, it is easy to prove that for each $n \in \mathbb{N}$ there exists a subsequence $(z_k^n)_{k=1}^\infty$ of $(h_k^n)_{k=1}^\infty$ such that $\operatorname{sep}\{z_k^n\}_{E(X)} \geq \varepsilon/4$. Denote this subsequence still by $(h_k^n)_{k=1}^\infty$. Since $\operatorname{sep}\{h_k^n\}_{E(X)} = \operatorname{sep}\{f_k^n\}_{E(X)}$, for every $n \in \mathbb{N}$ we can find an element $f^n \in B(E(X))$ and a sequence $(f_k^n)_{k=1}^\infty \subset B(E(X))$ such that $\operatorname{sep}\{f_k^n\}_{E(X)} \geq \varepsilon/4$ and $f_k^n \xrightarrow{w} f^n$ as $k \to \infty$ in $E(X)$. On the other hand, $\|f^n\| \geq 1 - 1/n$ for every $n \in \mathbb{N}$. This means that $E(X)$ does not have the uniform Kadec–Klee property.

THEOREM 2. *Let $E$ be a Köthe function space and $X$ be a Banach space. If $X$ is uniformly convex and $E$ has property $\boldsymbol{\beta}$, then $E(X)$ has property $\boldsymbol{\beta}$.*

*Proof.* Let $\varepsilon \in (0, 2)$. Note that property $\boldsymbol{\beta}$ can be equivalently considered on the unit sphere instead of the unit ball ([8]). Take $x, x_n \in S(E(X))$, $n = 1, 2, \ldots$, such that $\operatorname{sep}\{x_n\}_{E(X)} \geq \varepsilon$. By Lemma 3 and (1) we conclude that $E \in (\mathbf{UC}^\perp)$ and $E \in (\mathbf{UM})$. Denote by $p(\cdot)$ the modulus of uniform monotonicity of $E$, by $\delta_X(\cdot)$ the modulus of convexity of $X$ defined in Lemma 2, by $\delta_E(\cdot)$ the function $\delta(\cdot)$ used for $E$ in Lemma 1, and by $\delta_E^\perp(\cdot)$ the function $\delta(\cdot)$ from Definition 1 for $E$. We define some constants:

$$\begin{aligned}
&\delta_1 = \delta_E^\perp(\varepsilon/32) > 0, && 0 < \alpha < \delta_1/8 \wedge \varepsilon/224,\\
&0 < b < \alpha, && \\
&(1 - \varepsilon/16) \vee (1 - \alpha b/4) < u < 1, && \delta_2 = \delta_E\left(\frac{\alpha b(1-u)}{2(1+u)}\right) > 0,\\
&p_1 = p(4\alpha b \delta_X(\varepsilon/32)) > 0, && p_2 = p(\alpha^2 b^2 \delta_X(\varepsilon/32)/2) > 0.
\end{aligned} \tag{4}$$

For any $n \neq m$ set

$$A_{nm}^1 = \{t \in T : \|x_n(t)\|_X \wedge \|x_m(t)\|_X < u(\|x_n(t)\|_X \vee \|x_m(t)\|_X)\}$$

and $A_{nm}^2 = T \setminus A_{nm}^1$. We divide the proof into two parts.

**I.** Suppose that for any $n \neq m$ we have $\|(x_n - x_m)\chi_{A_{nm}^1}\| \geq \alpha b/2$. Notice that for any $x, y \in X$ satisfying $\|x\|_X \wedge \|y\|_X < u(\|x\|_X \vee \|y\|_X)$ we have

$$\|x - y\|_X \leq |\, \|x\|_X - \|y\|_X| \left(1 + \frac{2u}{1-u}\right). \tag{5}$$

If $\|x\|_X \geq \|y\|_X$, then $\|x\|_X - \|y\|_X \geq (1/u-1)\|y\|_X$. Hence

$$\begin{aligned}\|x-y\|_X &\leq \|x\|_X + \|y\|_X = \|x\|_X - \|y\|_X + 2\|y\|_X \\ &\leq \|x\|_X - \|y\|_X + 2u\,\frac{\|x\|_X - \|y\|_X}{1-u} \\ &= (\|x\|_X - \|y\|_X)\left(1+\frac{2u}{1-u}\right).\end{aligned}$$

In the case when $\|x\|_X < \|y\|_X$, the proof is analogous. Applying (5) and the definition of the set $A^1_{nm}$, we get

$$\alpha b/2 \leq \big\|\|(x_n - x_m)(\cdot)\|_X \chi_{A^1_{nm}}\big\|_E \leq \left(1+\frac{2u}{1-u}\right)\big\|\|x_n(\cdot)\|_X - \|x_m(\cdot)\|_X\big\|_E$$

for any $n \neq m$. Set $g(\cdot) = \|x(\cdot)\|_X$ and $g_n(\cdot) = \|x_n(\cdot)\|_X$. Then $\|g\|_E = \|g_n\|_E = 1$ and $\operatorname{sep}\{g_n\}_E \geq \alpha b(1-u)/2(1+u)$. By property $\boldsymbol{\beta}$ of $E$ we conclude that there exists $k \in \mathbb{N}$ such that $\|g+g_k\|_E \leq 2(1-\delta_2)$, where $\delta_2$ is defined in (4). Finally, $\|x+x_k\| = \big\|\|(x+x_k)(\cdot)\|_X\big\|_E \leq \|g+g_k\|_E \leq 2(1-\delta_2)$.

**II.** Assume that for some $n \neq m$ we have

$$\|(x_n - x_m)\chi_{A^1_{nm}}\| < \alpha b/2. \tag{6}$$

Set $A^1_{nm} = A^1$, $A^2_{nm} = A^2$, i.e.

$$A^2 = \{t \in T : \|x_n(t)\|_X \wedge \|x_m(t)\|_X \geq u(\|x_n(t)\|_X \vee \|x_m(t)\|_X)\}.$$

Then $\|(x_n - x_m)\chi_{A^2}\| \geq \varepsilon - \alpha b/2 \geq \varepsilon/2$. Let

$$\begin{aligned}A^{21} &= \{t \in A^2 : \|x_n(t) - x_m(t)\|_X \geq \varepsilon/8(\|x_n(t)\|_X \vee \|x_m(t)\|_X)\},\\ A^{22} &= \{t \in A^2 : \|x_n(t) - x_m(t)\|_X < \varepsilon/8(\|x_n(t)\|_X \vee \|x_m(t)\|_X)\}.\end{aligned}$$

It is easy to see that

$$\|(x_n - x_m)\chi_{A^{21}}\| \geq \varepsilon/4. \tag{7}$$

Indeed, if not, then $\|(x_n - x_m)\chi_{A^{22}}\| \geq \varepsilon/4$. Hence, applying strict monotonicity of $E$, we get $\varepsilon/4 \leq \|(x_n - x_m)\chi_{A^{22}}\| < 2\varepsilon/8$, which is a contradiction. For $x \in X\setminus\{0\}$ set $\widehat{x} = x/\|x\|_X$. We will prove that

$$\|\widehat{x_n(t)} - \widehat{x_m(t)}\|_X \geq \varepsilon/16 \tag{8}$$

for every $t \in A^{21}$. We claim that for any $y, z \in B(X)$ satisfying $\|y\|_X \wedge \|z\|_X \geq u(\|y\|_X \vee \|z\|_X)$ and $\|y-z\|_X \geq \varepsilon/8$, we have

$$\|\widehat{y} - \widehat{z}\|_X \geq \varepsilon/16. \tag{9}$$

By Lemma 2(ii), we get

$$\begin{aligned}\varepsilon/8 \leq \|y-z\|_X &\leq |\,\|y\|_X - \|z\|_X| + (\|y\|_X \wedge \|z\|_X)(\|\widehat{y} - \widehat{z}\|_X)\\ &\leq 1-u+\|\widehat{y} - \widehat{z}\|_X,\end{aligned}$$

which proves the claim in view of (4). Then, to deduce (8) it is enough to apply the definition of the sets $A^2, A^{21}$ and (9) with

$$y(t) = \frac{x_n(t)}{\|x_n(t)\|_X \vee \|x_m(t)\|_X} \quad \text{and} \quad z(t) = \frac{x_m(t)}{\|x_n(t)\|_X \vee \|x_m(t)\|_X}$$

for each $t \in A^{21}$. Moreover, by (4), (7) and the definition of the set $A^2$, it follows that

$$\|x_i \chi_{A^{21}}\|_E \geq \varepsilon/16 \quad \text{for } i = n, m. \tag{10}$$

Define

$$\begin{aligned} &B = \{t \in \operatorname{supp} x : \|x(t)\|_X \wedge \|x_n(t)\|_X \geq b(\|x(t)\|_X \vee \|x_n(t)\|_X)\}, \\ &C = \operatorname{supp} x \setminus B, \\ &B^1 = \{t \in B : \|\widehat{x_n(t)} - \widehat{x(t)}\|_X \geq \varepsilon/32\}, \qquad B^2 = B \setminus B^1, \\ &C^1 = \{t \in C : \|x(t)\|_X = \|x(t)\|_X \wedge \|x_n(t)\|_X\}, \qquad C^2 = C \setminus C^1 \end{aligned}$$

and

$$\begin{aligned} &D^1 = \{t \in B^2 : \|x_m(t)\|_X \wedge \|x_n(t)\|_X \geq \alpha b(\|x_m(t)\|_X \vee \|x_n(t)\|_X)\}, \\ &D^2 = B^2 \setminus D^1, \\ &E^1 = \{t \in D^1 : \|\widehat{x_n(t)} - \widehat{x_m(t)}\|_X \geq \varepsilon/16\}, \\ &E^2 = D^1 \setminus E^1. \end{aligned}$$

**II.1.** Suppose that $\|x\chi_{B^1}\| \geq 8\alpha$. Applying Lemma 2(i), we get

$$\|(x + x_n)(\cdot)\|_X \chi_{B^1} \leq (1 - b\delta_X(\varepsilon/32))(\|x(\cdot)\|_X + \|x_n(\cdot)\|_X)\chi_{B^1}.$$

Consequently,

$$\left\|\frac{x + x_n}{2}(\cdot)\right\|_X \leq \frac{\|x(\cdot)\|_X + \|x_n(\cdot)\|_X}{2} - \frac{b\delta_X(\varepsilon/32)}{2}(\|x(\cdot)\|_X + \|x_n(\cdot)\|_X)\chi_{B^1}.$$

Hence, applying uniform monotonicity of $E$, we get $\|(x + x_n)/2\| \leq 1 - p_1$, where $p_1$ is defined in (4).

**II.2.** Let

$$\|x\chi_{B^1}\| < 8\alpha. \tag{11}$$

We divide the proof into two parts.

**a.** Assume that $\|x_m \chi_{E^1}\| \geq \alpha$. For every $t \in E^1$ we have

$$\|x_m(t)\|_X \wedge \|x(t)\|_X \geq \alpha b^2(\|x_m(t)\|_X \vee \|x(t)\|_X).$$

It follows by the definition of $E^1$ and $B^2$ that

$$\|\widehat{x_m(t)} - \widehat{x(t)}\|_X = \|\widehat{x_m(t)} - \widehat{x_n(t)} - (\widehat{x(t)} - \widehat{x_n(t)})\|_X \geq \varepsilon/32$$

for every $t \in E^1$. Applying Lemma 2(i) we get

$$\|(x + x_m)(\cdot)\|_X \chi_{E^1} \leq (1 - \alpha b^2 \delta_X(\varepsilon/32))(\|x(\cdot)\|_X + \|x_m(\cdot)\|_X)\chi_{E^1}.$$

Consequently,

$$\left\|\frac{x+x_m}{2}(\cdot)\right\|_X \le \frac{\|x(\cdot)\|_X + \|x_m(\cdot)\|_X}{2} - \frac{\alpha b^2 \delta_X(\varepsilon/32)}{2}(\|x(\cdot)\|_X + \|x_m(\cdot)\|_X)\chi_{E^1}.$$

Hence, similarly to case II.1, we conclude that $\|(x+x_m)/2\| \le 1-p_2$, where $p_2$ is defined in (4).

**b.** Suppose that $\|x_m\chi_{E^1}\| < \alpha$. First we will show that

$$\left\| \big|\|x_n(\cdot)\|_X - \|x_m(\cdot)\|_X\big| \right\|_E < \alpha b. \tag{12}$$

In view of (6), we get $\left\| \big|\|x_n(\cdot)\|_X - \|x_m(\cdot)\|_X\big|\chi_{A^1}\right\|_E < \alpha b/2$. Moreover, by the definition of the set $A^2$, we get $\left\| \big|\|x_n(\cdot)\|_X - \|x_m(\cdot)\|_X\big|\chi_{A^2}\right\|_E < 2(1-u)$. Consequently, by (4),

$$\left\| \big|\|x_n(\cdot)\|_X - \|x_m(\cdot)\|_X\big| \right\|_E < \alpha b/2 + 2(1-u) < \alpha b.$$

Note that if $\|x\chi_{C^1}\| \ge \alpha$, then $\|x_n\| \ge \|x_n\chi_{C^1}\| \ge (1/b)\alpha > 1$. Hence

$$\|x\chi_{C^1}\| < \alpha. \tag{13}$$

Furthermore $\|x_n\chi_{C^2}\| < b < \alpha$. Consequently, if $\|x_m\chi_{C^2}\| \ge 2\alpha$, then

$$\left\| \big|\|x_n(\cdot)\|_X - \|x_m(\cdot)\|_X\big|\chi_{C^2}\right\|_E \ge \alpha > \alpha b,$$

but this contradicts inequality (12). Thus

$$\|x_m\chi_{C^2}\| < 2\alpha. \tag{14}$$

Moreover, we will show that

$$\|x_m\chi_{D^2}\| < 4\alpha b. \tag{15}$$

Suppose conversely that $\|x_m\chi_{D^2}\| \ge 4\alpha b$ and let

$$D^{21} = \{t \in D^2 : \|x_m(t)\|_X = \|x_m(t)\|_X \wedge \|x_n(t)\|_X\}, \qquad D^{22} = D^2 \setminus D^{21}.$$

If $\|x_m\chi_{D^{21}}\| \ge 2\alpha b$, then $\|x_n\chi_{D^{21}}\| \ge 2$. But $x_n \in B(E(X))$. Hence $\|x_m\chi_{D^{22}}\| \ge 2\alpha b$. On the other hand, $\|x_n\chi_{D^{22}}\| < \alpha b$. Consequently, $\left\|\big|\|x_n(\cdot)\|_X - \|x_m(\cdot)\|_X\big|\right\|_E \ge \alpha b$, which contradicts (12), so inequality (15) is proved.

Then, by (14) and (15), we get

$$\|x_m\chi_{C^2\cup D^2\cup E^1}\| < 3\alpha + 4\alpha b < 7\alpha. \tag{16}$$

Notice that, in view of inequality (8) and the definition of $E^2$, we get $A^{21} \cap E^2 = \emptyset$. Furthermore, inequality (10) yields $\|x_m\chi_{A^{21}}\|_E \ge \varepsilon/16$. Consequently, by (4) and (16), we obtain

$$\|x_m\chi_{A^{21}\setminus(B^2\cup C^2)}\| = \|x_m\chi_{A^{21}\setminus(D^2\cup E^1\cup E^2\cup C^2)}\| \ge \frac{\varepsilon}{16} - 7\alpha \ge \frac{\varepsilon}{32}. \tag{17}$$

Let $z_1 = \|x(\cdot)\|_X\chi_{B^2\cup C^2}$ and $z_2 = \|x_m(\cdot)\|_X$. Define $G = \operatorname{supp} z_1 \div \operatorname{supp} z_2$. Then, by (17), we get $\|z_2\chi_G\|_E \ge \varepsilon/32$. Since $E \in (\mathbf{UC}^{\perp})$, so $\|z_1+z_2\|_E \le 2(1-\delta_1)$, where $\delta_1$ is defined in (4). Thus, by (4), (11) and (13), we obtain

$$\left\|\frac{x+x_m}{2}\right\| \leq \left\|\frac{\|x(\cdot)\|_X\chi_{B^2\cup C^2}+\|x(\cdot)\|_X\chi_{T\setminus(B^2\cup C^2)}+\|x_m(\cdot)\|_X}{2}\right\|_E$$
$$\leq \frac{9\alpha}{2}+\left\|\frac{z_1+z_2}{2}\right\|_E \leq 5\alpha+1-\delta_1 \leq 1-3\delta_1/8.$$

Combining all of the cases, we get $\|(x+x_k)/2\|_E \leq 1-\lambda$ for some $k\in\mathbb{N}$, where $\lambda=\min\{\delta_2,p_1,p_2,3\delta_1/8\}$, which finishes the proof. ■

It is known that if $X$ is reflexive, then $X^*$ has the Radon–Nikodym property. Moreover, $E$ is embedded isometrically into $E(X)$ and property $\boldsymbol{\beta}$ is inherited by subspaces. Consequently, as an immediate consequence of Theorems 1 and 2, we get

Corollary 1. *Let $X$ be a real Banach space and $E$ be a Köthe function space. Then $E(X)$ has property $\boldsymbol{\beta}$ if and only if $X$ is uniformly convex and $E$ has property $\boldsymbol{\beta}$.*

Let us collect results concerning property $\boldsymbol{\beta}$ in Köthe–Bochner sequence spaces. If $X$ is an infinite-dimensional Banach space and $E$ is a Köthe sequence space, then $E(X)$ has property $\boldsymbol{\beta}$ if and only if $X$ has property $\boldsymbol{\beta}$ and $E$ is orthogonally uniformly convex ([8]). If $X$ is a finite-dimensional Banach space, then $E(X)\in(\boldsymbol{\beta})$ if and only if $E\in(\boldsymbol{\beta})$ ([8]).

The Orlicz–Lorentz function space $\Lambda_{\Phi,\omega}$ is a generalization of Orlicz function space. On the other hand, $\Lambda_{\Phi,\omega}$ is a special Calderón–Lozanowskiĭ space (see [7] and [13] for the definition and bibliography). Applying the results from [9], [13] and Corollary 1, we get the following

Corollary 2. *Let $\Lambda_{\Phi,\omega}$ be an Orlicz–Lorentz function space over the finite or infinite non-atomic measure space. Let $X$ be a real Banach space. Then $\Lambda_{\Phi,\omega}(X)\in(\mathbf{UC})$ if and only if $\Lambda_{\Phi,\omega}(X)\in(\boldsymbol{\beta})$.*

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