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A Non-standard Version of the Borsuk–Ulam Theorem

by

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Summary. E. Pannwitz showed in 1952 that for any $n \geq 2$, there exist continuous maps $\varphi : S^n \to S^n$ and $f : S^n \to \mathbb{R}^2$ such that $f(x) \neq f(\varphi(x))$ for any $x \in S^n$. We prove that, under certain conditions, given continuous maps $\psi, \varphi : X \to X$ and $f : X \to \mathbb{R}^2$, although the existence of a point $x \in X$ such that $f(\psi(x)) = f(\varphi(x))$ cannot always be assured, it is possible to establish an interesting relation between the points $f(\varphi\psi(x)), f(\varphi^2(x))$ and $f(\psi^2(x))$ when $f(\varphi(x)) \neq f(\psi(x))$ for any $x \in X$, and a non-standard version of the Borsuk–Ulam theorem is obtained.

1. Introduction. Let X be a topological space. An *involution* on X is a continuous map $\varphi : X \to X$ which is its own inverse. A classical example is the antipodal map $A : S^n \to S^n$, $A(x) = -x$, where S^n denotes the n-dimensional sphere; the points x and $A(x)$ are said to be antipodal points. The classical Borsuk–Ulam theorem [1] states that every continuous map f from S^n into $\mathbb{R}^n$ collapses at least a pair of antipodal points, that is, there exists a point $x \in S^n$ such that $f(x) = f(A(x))$.

Several generalizations of this theorem, in various directions, are well known. In some of these generalizations the sphere is replaced by a more general space X and the antipodal map is replaced by an involution $T : X \to X$ which is free, that is, $T(x) \neq x$ for any $x \in X$. In this direction see, for example, the references [2, 8, 9].

Let us now replace the domain S^n by a topological space X and the identity and the antipodal map on S^n by a pair of any continuous maps ψ, φ on X. A question that naturally arises is whether or not for every continuous map $f : X \to \mathbb{R}^n$ there exists a point $x \in X$ such that $f(\psi(x)) = f(\varphi(x))$.

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We first consider the one-dimensional case. If X is a compact and connected space, then for every continuous map $f : X \to \mathbb{R}$ it is possible to show that there exists a point $x \in X$ such that

$$f(\psi(x)) = f(\varphi(x)). \tag{1.1}$$

The proof is elementary. However, for $n = 2$, $X = S^k$ and $\psi = \mathrm{Id}_{S^k}$ the answer is negative. E. Pannwitz proved in [7] that for any $k \geq 2$, there exist continuous maps $\varphi : S^k \to S^k$ and $f : S^k \to \mathbb{R}^2$ such that $f(x) \neq f(\varphi(x))$ for any $x \in S^k$.

In this paper, our objective is to show that, under certain conditions, for a given continuous map $f : X \to \mathbb{R}^2$, although the existence of a point $x \in X$ such that (1.1) holds cannot always be assured, it is possible to establish an interesting relation between the points

$$u = f(\psi\varphi(x)), \quad v = f(\varphi^2(x)), \quad w = f(\psi^2(x)) \tag{1.2}$$

when $f(\varphi(x)) \neq f(\psi(x))$ for any $x \in X$. In general, such points are vertices of a triangle in $\mathbb{R}^2$ and we prove that this triangle degenerates to a closed line segment determined by the vertices v and w for, at least, a point x in a special subset of X. The existence of such a subset is ensured when X is a complete metric space and φ is an α-contraction on X, where α is the measure of noncompactness.

When ψ is the identity map and φ is a free involution on X, we obtain a version of the Borsuk–Ulam theorem in the two-dimensional case.

We denote by $[v, w]$ the closed line segment in $\mathbb{R}^2$ joining the points v and w. We will specifically prove the following

THEOREM 1.1. *Let X be a Hausdorff space and A a compact, connected and locally pathwise connected subset of X. Let $\psi, \varphi : X \to X$ be continuous maps such that A is invariant under ψ and φ, that is, $\psi(A) \subset A$ and $\varphi(A) \subset A$. Suppose that*

(i) $\psi_* - \varphi_* : i_*(H_1(A, \mathbb{Q})) \to i_*(H_1(A, \mathbb{Q}))$ *is a surjective map*;
(ii) $(\psi \circ \varphi)(x) = (\varphi \circ \psi)(x)$ *for any* $x \in A$.

Then for every continuous map $f : X \to \mathbb{R}^2$, either there exists a point $x \in X$ such that $f(\varphi(x)) = f(\psi(x))$ or there exists a point $x \in A$ such that $f(\varphi\psi(x)) \in [f(\varphi^2(x)), f(\psi^2(x))]$.

2. Proof of Theorem 1.1. For the proof of Theorem 1.1, we need the following

LEMMA 2.1. *Let X be a connected space and $K \neq \emptyset$ a compact subset of X. Let $g_1, g_2 : X \to \mathbb{R}$ be continuous maps such that $g_1(K) \subset g_2(K)$. Then there exists a point $x \in X$ such that $g_1(x) = g_2(x)$.*

Proof. Consider the continuous map $h : X \to \mathbb{R}$ given by $h(x) = g_2(x) - g_1(x)$ for any $x \in X$. Since K is compact, there exist $x_0, x_1 \in K$ such that

$$g_2(x_0) \leq g_2(x) \leq g_2(x_1) \tag{2.1}$$

for any $x \in K$. Furthermore, $g_1(x) \in g_2(K)$ for any $x \in K$ and it follows from (2.1) that

$$g_2(x_0) \leq g_1(x) \leq g_2(x_1), \qquad \forall x \in K,$$

which implies that $h(x_0) \leq 0 \leq h(x_1)$ and consequently there is an $x \in X$ such that $h(x) = 0$, that is, $g_1(x) = g_2(x)$. ∎

As a direct consequence we obtain the following

COROLLARY 2.2. *Let X be a connected space and K a compact subset of X. Let $\psi, \varphi : X \to X$ be continuous maps such that $\psi(K) \subset \varphi(K)$. Then for every continuous map $g : X \to \mathbb{R}$ there exists a point $x \in X$ such that $g(\psi(x)) = g(\varphi(x))$.*

LEMMA 2.3. *Let X be a topological space and let $f, g : X \to S^n$ be continuous maps. Suppose that there exists $u \in H_n(X, \mathbb{Z})$ such that $f_*(u) \neq (-1)^{n+1} g_*(u)$. Then there exists $x \in X$ such that $f(x) = g(x)$.*

Proof. Suppose that $f(x) \neq g(x)$ for any $x \in X$. Then the line segment in $\mathbb{R}^{n+1}$ from $f(x)$ to $-g(x)$ does not pass through the origin, since otherwise these points would be antipodal and consequently $f(x) = g(x)$. Hence we can define a map $F : X \times I \to S^n$ by

$$F(x, t) = \frac{(1-t)(-g(x)) + t \cdot f(x)}{\|(1-t)(-g(x)) + t \cdot f(x)\|}, \qquad \forall (x, t) \in X \times I, \tag{2.2}$$

which is a homotopy between f and $-g = A \circ g$, where $A : S^n \to S^n$ denotes the antipodal map, whose degree is $(-1)^{n+1}$. It follows that for any $u \in H^n(X, \mathbb{Z})$, $f_*(u) = (-1)^{n+1} g_*(u)$. ∎

Proof of Theorem 1.1. Suppose that $f(\varphi(x)) \neq f(\psi(x))$ for any $x \in X$. Then we can define a continuous map $h : X \to S^1$ by

$$h(x) = \frac{f(\psi(x)) - f(\varphi(x))}{\|f(\psi(x)) - f(\varphi(x))\|}.$$

Let $g : A \to S^1$ be the restriction of h to A. It suffices to show the existence of a point $x \in A$ such that $g(\varphi(x)) = g(\psi(x))$ or equivalently,

$$\frac{f(\psi\varphi(x)) - f(\varphi^2(x))}{\|f(\psi\varphi(x)) - f(\varphi^2(x))\|} = \frac{f(\psi^2(x)) - f(\varphi\psi(x))}{\|f(\psi^2(x)) - f(\varphi\psi(x))\|}. \tag{2.3}$$

In fact, for any $x \in A$ set $u = f(\psi\varphi(x)) = f(\varphi\psi(x))$, $v = f(\varphi^2(x))$ and

$w = f(\psi^2(x))$. Then (2.3) is equivalent to

$$\frac{u-v}{\|u-v\|} = \frac{w-u}{\|w-u\|},$$

and so

$$u = \left(\frac{\|u-v\|}{\|u-v\|+\|w-u\|}\right)w + \left(\frac{\|w-u\|}{\|u-v\|+\|w-u\|}\right)v,$$

that is, $u = f(\psi\varphi(x))$ belongs to the line segment in $\mathbb{R}^2$ from $v = f(\varphi^2(x))$ to $w = f(\psi^2(x))$.

Let $h_* : H_1(X,\mathbb{Q}) \to H_1(S^1,\mathbb{Q})$. There are two cases to consider:

(1) there exists $v \in i_*(H_1(A,\mathbb{Q}))$ such that $h_*(v) \neq 0$,
(2) $h_*(v) = 0$ for any $v \in i_*(H_1(A,\mathbb{Q}))$.

In the first case, since $\psi_* - \varphi_*$ is surjective, there exists $u \in i_*(H_1(A,\mathbb{Q}))$ such that $v = \psi_*(u) - \varphi_*(u)$. Then

$$h_*(v) = h_*(\psi_*(u)-\varphi_*(u)) = g_*(\psi_*(u)-\varphi_*(u)) = (g\circ\psi)_*(u)-(g\circ\varphi)_*(u) \neq 0,$$

which implies that $(g\circ\psi)_*(u) \neq (g\circ\varphi)_*(u)$. It follows from Lemma 2.3 that there exists $x \in A$ such that $g(\psi(x)) = g(\varphi(x))$.

Now suppose that $h_*(v) = 0$ for any $v \in i_*(H_1(A,\mathbb{Q}))$ and let $u \in H_1(A,\mathbb{Q})$; then $i_*(u) = v \in i_*(H_1(A,\mathbb{Q}))$ and thus

$$h_*(v) = h_*(i_*(u)) = (h\circ i)_*(u) = g_*(u) = 0,$$

that is, $g_* : H_1(A,\mathbb{Q}) \to H_1(S^1,\mathbb{Q})$ is the zero map, which implies that $g_* : H_1(A,\mathbb{Z}) \to H_1(S^1,\mathbb{Z})$ is also trivial.

It follows from the commutative diagram

$$\begin{array}{ccc} \pi_1(A) & \xrightarrow{g_*} & \pi_1(S^1) \\ \downarrow & & \downarrow \\ H_1(A,\mathbb{Z}) & \xrightarrow{g_*} & H_1(S^1,\mathbb{Z}) \end{array} \tag{2.4}$$

where the vertical arrows denotes the Hurewicz homomorphism, that $g_* : \pi_1(A) \to \pi_1(S^1)$ is the zero map. Since A is Hausdorff and locally pathwise connected, by the lifting theorem (see, for example, [5, p. 89] and [4, p. 26, Theorem 6.1]) there exists $\tilde{g} : A \to \mathbb{R}$ such that the diagram

$$\begin{array}{ccc} & & \mathbb{R} \\ & \overset{\tilde{g}}{\nearrow} & \downarrow p \\ A & \xrightarrow[g]{} & S^1 \end{array} \tag{2.5}$$

is commutative, where $p : \mathbb{R} \to S^1$ is the universal covering. On the other hand, since A is invariant under φ, we obtain the sequence $\{\varphi^n(A)\}_{n\in\mathbb{N}}$ of subsets of A such that

$$\cdots \subset \varphi^n(A) \subset \varphi^{n-1}(A) \subset \cdots \subset \varphi^2(A) \subset \varphi(A) \subset A.$$

We consider the following compact subset of A:

$$K = \bigcap_{n\in\mathbb{N}} \varphi^n(A), \tag{2.6}$$

and we observe that $\psi(K) \subset K = \varphi(K)$. In fact, by hypothesis A is invariant under ψ and φ. Furthermore, $\varphi \circ \psi = \psi \circ \varphi$ on A. Thus

$$\psi(K) = \psi\Big(\bigcap_{n\in\mathbb{N}} \varphi^n(A)\Big) \subset \bigcap_{n\in\mathbb{N}} \psi(\varphi^n(A)) \subset \bigcap_{n\in\mathbb{N}} \varphi^n(\psi(A)) \subset \bigcap_{n\in\mathbb{N}} \varphi^n(A) = K.$$

It follows from Corollary 2.2 that there exists a point $x \in A$ such that $\tilde{g}(\varphi(x)) = \tilde{g}(\psi(x))$. Then $p \circ \tilde{g}(\varphi(x)) = p \circ \tilde{g}(\psi(x))$, which implies that $g(\varphi(x)) = g(\psi(x))$, and the result follows. ∎

We have the following immediate corollary:

Corollary 2.4. *Let X be a Hausdorff space and A a compact, connected and locally pathwise connected subset of X. Let $\varphi : X \to X$ be a free involution such that $\varphi(A) \subset A$. Suppose that $\mathrm{Id}_* - \varphi_* : i_*(H_1(A,\mathbb{Q})) \to i_*(H_1(A,\mathbb{Q}))$ is a surjective map. Then for every continuous map $f : X \to \mathbb{R}^2$ there exists $x \in X$ such that $f(x) = f(\varphi(x))$.*

Remark 2.5. When $A = X = S^2$ and φ is the antipodal map, we obtain the classical Borsuk–Ulam theorem in the two-dimensional case.

We observe that when $i_*(H_1(A,\mathbb{Q}))$ is the trivial group, the homomorphism $\psi_* - \varphi_*$ must be surjective. Example 2.6 illustrates this case.

Example 2.6. Let $T_n = T\sharp\cdots\sharp T$ be the n-fold connected sum of tori, which is embedded in $\mathbb{R}^3$ symmetrically with respect to the origin. Let $\varphi : T_n \to T_n$ be the antipodal map. If n is even, there exists a loop A in T_n, homologous to zero, which separates T_n into two components symmetrical with respect to the origin such that $\varphi(A) = A$, as indicated in Figure 1.

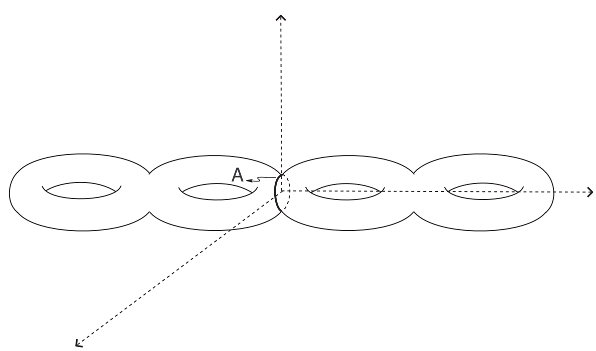

Fig. 1

The group $i_*(H_1(A,\mathbb{Q}))$ is trivial, and so by Corollary 2.4, for every continuous map $f : T_n \to \mathbb{R}^2$ there exists a point $x \in T_n$ such that $f(x) = f(\varphi(x))$. If n is odd, one can show that this is not true.

REMARK 2.7. The referee remarked that it is possible to show the existence of a point $x \in T_n$ such that $f(x) = f(\varphi(x))$ by using the Yang–Smith index.

REMARK 2.8. In [8, Theorem A], we prove that if (X, T) is a free involution and X is pathwise connected such that $H_r(X, \mathbb{Z}_2) = 0$ for $1 \le r \le n-1$, then for every continuous map $f : X \to \mathbb{R}^k$ with $k \le n$ there exists a point $x \in X$ such that $f(x) = f(T(x))$. We observe that the above example cannot be obtained from that theorem, since $H_1(T_n, \mathbb{Z}_2) \neq 0$.

THEOREM 2.9. *Let X be a Hausdorff space and A a compact, connected and locally pathwise connected subset of X. Let $\varphi : X \to X$ be a continuous map such that $\varphi(A) \subset A$. Suppose that* $\mathrm{Id}_* - \varphi_* : i_*(H_1(A,\mathbb{Q})) \to i_*(H_1(A,\mathbb{Q}))$ *is a surjective map. Then for every continuous map $g : X \to \mathbb{R}$ there exists $x \in X$ such that*

$$g(x) \le g(\varphi(x)) \le g(\varphi^2(x)) \le g(\varphi^3(x)) \quad \textit{or}$$
$$g(x) \ge g(\varphi(x)) \ge g(\varphi^2(x)) \ge g(\varphi^3(x)).$$

Proof. Consider the continuous map $f : X \to \mathbb{R}^2$ given by

$$f(x) = (g(x), g(\varphi(x))), \quad \forall x \in X.$$

By Theorem 1.1, there exists $x \in X$ such that $f(\varphi(x))$ belongs to the closed line segment in $\mathbb{R}^2$ from $f(\varphi^2(x))$ to $f(x)$. Suppose that $f(\varphi(x)) = f(x)$; this implies that

$$g(x) = g(\varphi(x)) = g(\varphi^2(x)).$$

Since $g(\varphi^2(x)) \le g(\varphi^3(x))$ or $g(\varphi^2(x)) \ge g(\varphi^3(x))$, the result follows. The proof remains the same when $f(\varphi(x)) = f(\varphi^2(x))$.

Now, suppose that $f(\varphi(x)) \neq f(x)$ and $f(\varphi(x)) \neq f(\varphi^2(x))$. Then $f(\varphi(x))$ belongs to the open line segment in $\mathbb{R}^2$ from $f(\varphi^2(x))$ to $f(x)$, that is, there exists $0 < \lambda < 1$ such that $f(\varphi(x)) = f(x) + \lambda(f(\varphi^2(x)) - f(x))$. Thus,

$$g\varphi(x) = g(x) + \lambda(g\varphi^2(x) - g(x)),$$
$$g\varphi^2(x) = g\varphi(x) + \lambda(g\varphi^3(x) - g\varphi(x)),$$

which implies the required alternative of inequalities. ∎

We have the following immediate corollary:

COROLLARY 2.10. *Let X be a Hausdorff space and A a compact, connected and locally pathwise connected subset of X. Let $\varphi : X \to X$ be a* *continuous map such that* $\varphi(A) \subset A$ *and* $\varphi^3 = \mathrm{Id}_X$. *Suppose that*

$$\mathrm{Id}_* - \varphi_* : i_*(H_1(A,\mathbb{Q})) \to i_*(H_1(A,\mathbb{Q}))$$

is a surjective map. Then for every continuous map $g : X \to \mathbb{R}$ *there exists a point* $x \in X$ *such that* $g(x) = g(\varphi(x)) = g(\varphi^2(x))$.

EXAMPLE 2.11. Let S^3 be the 3-dimensional standard sphere in complex 2-space $\mathbb{C}^2$. Let $\varphi : S^3 \to S^3$ be the transformation defined by

$$\varphi(z_0, z_1) = (e^{2\pi i/3} z_0, e^{2\pi i/3} z_1),$$

where z_0, z_1 are complex numbers with $\sum_{i=0}^{1} |z_i| = 1$. Then φ acts freely on S^3 and generates the cyclic group $\mathbb{Z}_3$.

Since $H_1(S^3,\mathbb{Q}) = 0$, we see that $\mathrm{Id}_* - \varphi_*$ is surjective. It follows from Corollary 2.10 that for every continuous map $g : S^3 \to \mathbb{R}$ there exists $x \in S^3$ such that $g(x) = g(\varphi(x)) = g(\varphi^2(x))$.

3. The particular case that φ is an α-contraction. In the proof of Theorem 1.1, since A is a compact subset of X, it was possible to construct a compact subset K of A such that $\psi(K) \subset \varphi(K)$ (see (2.6)). In Lemma 3.4, we prove that even if A is not compact, it is possible to ensure the existence of such a subset, provided X is a metric space, A is complete and φ is an α-contraction. Consider the following

DEFINITION 3.1. Let X be a normed linear space. For any bounded subset $A \subset X$, we define the *measure* $\alpha(A)$ *of noncompactness* of A to be

$$\alpha(A) = \inf\{k > 0 : A \text{ has a finite covering by sets of diameter} \le k\}.$$

Some important properties of α are given in the following proposition (for more details see, for example, [3] and [6]).

PROPOSITION 3.2. *Suppose* A, B *are bounded subsets of* X *and* $k \in \mathbb{R}$. *Then*:

(1) $A \subset B$ *implies* $\alpha(A) \le \alpha(B)$;
(2) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$;
(3) $\alpha(A + B) \le \alpha(A) + \alpha(B)$;
(4) $\alpha(kA) = |k|\alpha(A)$;
(5) $\alpha(\mathrm{Co}\, A) = \alpha(A)$, *where* $\mathrm{Co}\, A$ *denotes the convex hull of* A;
(6) $\alpha(\overline{A}) = \alpha(A)$, *where* $\overline{A}$ *denotes the closure of* A;
(7) $\alpha(A) = 0$ *if and only if* A *is totally bounded.*

DEFINITION 3.3. Suppose A is a subset of X and $\varphi : A \to X$ is a continuous map. The map φ is said to be an *α-contraction* if there exists an r, $0 \le r < 1$, such that $\alpha(\varphi(B)) \le r\alpha(B)$ for any bounded subset B of A.

Lemma 3.4. *Let M be a metric space and A a bounded and complete subset of M. Let $\psi,\varphi : M \to M$ be continuous maps such that A is invariant under ψ and φ and $(\psi\circ\varphi)(a) = (\varphi\circ\psi)(a)$ for any $a \in A$. Then if φ is an α-contraction on A, there exists a compact subset $K \subset A$ such that $\psi(K) \subset \varphi(K) = K$.*

Proof. Let K be the intersection of subsets K_n of A inductively defined by $K_1 = \overline{\varphi(A)}$ and $K_{n+1} = \overline{\varphi(K_n)}$. We will show that $\alpha(K) = 0$, which implies by Proposition 3.2(7) that K is totally bounded, and since A is complete we conclude that K is compact. In fact, for any $n \in \mathbb{N}$, since φ is an α-contraction, from Proposition 3.2(1) and (6) we have

$$\begin{aligned}(3.1)\qquad \alpha(K_n) &= \alpha(\overline{\varphi(K_{n-1})}) = \alpha(\varphi(K_{n-1})) \le r\alpha(K_{n-1})\\ &\le r^2\alpha(K_{n-2}) \le \cdots \le r^{n-1}\alpha(K_1) \le r^n\alpha(A).\end{aligned}$$

Since $K = \bigcap K_n$, we have $K \subset K_n$ for any $n \in \mathbb{N}$. It follows from Proposition 3.2(1) and from (3.1) that

$$(3.2)\qquad \alpha(K) \le \alpha(K_n) \le r^n\alpha(A), \quad \forall n \in \mathbb{N}.$$

Since $0 \le r < 1$, we have $\lim_{n\to\infty} r^n = 0$ and from (3.2) we conclude that $\alpha(K) = 0$.

Now, we will show that $K = \varphi(K)$. It is easy to see that $\varphi(K) \subset K$. On the other hand, $K \subset \varphi(K_n)$ for any $n \in \mathbb{N}$. Let $x \in K$. Then $x = \varphi(x_n)$ for some $x_n \in K_n$. Let $S = \{x_1, x_2, \ldots\}$ and observe that $\alpha(S) = 0$; thus S is compact and so $(x_n)_{n\in\mathbb{N}}$ has a subsequence converging to some $y \in K$. Then $x = \varphi(y)$ and thus $K \subset \varphi(K)$. The condition $\psi(K) \subset K = \varphi(K)$ follows from the commutativity of the maps φ and ψ on A. ∎

As a consequence of Lemma 3.4 we have the following version of Theorem 1.1 in the case that φ is an α-contraction.

Theorem 3.5. *Let M be a metric space and let A be a bounded, complete, connected and locally pathwise connected subset of M. Let $\psi,\varphi : M \to M$ be continuous maps such that A is invariant under ψ and φ. Suppose that*

(i) $\psi_* - \varphi_* : i_*(H_1(A,\mathbb{Q})) \to i_*(H_1(A,\mathbb{Q}))$ *is a surjective map*;
(ii) $(\psi\circ\varphi)(x) = (\varphi\circ\psi)(x)$ *for any $x \in A$.*

Then for every continuous map $f : X \to \mathbb{R}^2$, either there exists a point $x \in X$ such that $f(\varphi(x)) = f(\psi(x))$ or there exists a point $x \in X$ such that $f(\varphi\psi(x)) \in [f(\varphi^2(x)), f(\psi^2(x))]$.

Proof. The arguments are similar to those used in the proof of Theorem 1.1: just observe that the existence of a compact subset K of A such that $\psi(K) \subset \varphi(K)$, as in (2.6), is ensured by Lemma 3.4. ∎

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