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A Note on a Theorem of Lion

by

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Summary. In this note we bind together Wilkie's complement theorem with Lion's theorem on geometric, regular and 0-regular families of functions.

0. Introduction. In [W] Wilkie proved that every weak o-minimal structure which has the DSF property (is defined by its smooth functions) is o-minimal. Karpinski and Macintyre [KM] gave a generalization of this result and weakened the assumptions on smoothness for functions, which determine a weak o-minimal structure. Lion [L] proved that a geometric, regular and 0-regular family has the uniform finiteness property. He mentioned without proof that, by a modification of Wilkie's theorem, such a family generates an o-minimal structure. The aim of our note is to check this by proving

THEOREM. *Let $\mathfrak{F} = \{\mathfrak{F}_n\}_{n\in\mathbb{N}}$ be a regular, geometric and* 0*-regular family. Then there exists an o-minimal structure $\mathfrak{S}$ such that every $f \in \mathfrak{F}$ is definable in $\mathfrak{S}$.*

This paper is organized as follows. In the first section we recall Wilkie's and Lion's theorems. The second section is devoted to showing that a geometric, regular family which has the uniform fibre finiteness property satisfies the DC^N condition for all N (Def. 1.7). Then, the above theorem is an immediate consequence of Lion's theorem together with Proposition 2.5.

1. Theorems of Lion and Wilkie. Firstly we recall the following definitions.

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Definition 1.1 (see [L]). We say that a family $\mathfrak{G} = \{\mathfrak{G}_n\}_{n\in\mathbb{N}}$, where each $\mathfrak{G}_n$ is a set of real valued functions on $\mathbb{R}^n$, is a *geometric family* if the following conditions hold:

(G1) if $f, g \in \mathfrak{G}_n$, then fg and $f + g \in \mathfrak{G}_n$,
(G2) if $f \in \mathfrak{G}_n$, and $f(x) \neq 0$ for every $x \in \mathbb{R}^n$, then $1/f \in \mathfrak{G}_n$,
(G3) $\mathbb{R}[X_1, \ldots, X_n] \subset \mathfrak{G}_n$,
(G4) if $f \in \mathfrak{G}_n$, and $L : \mathbb{R}^m \to \mathbb{R}^n$ is an affine map, then $f \circ L \in \mathfrak{G}_m$.

Definition 1.2 (see [L]). A geometric family $\mathfrak{G} = \{\mathfrak{G}_n\}_{n\in\mathbb{N}}$ is called *regular* if for every $n \in \mathbb{N}$ and every $g \in \mathfrak{G}_n$, there exist a finite number of affine hyperplanes $H_1, \ldots, H_l$ and n functions $g_1, \ldots, g_n \in \mathfrak{G}_n$ such that for $U = \mathbb{R}^n \setminus (H_1 \cup \cdots \cup H_l)$ the following conditions are satisfied:

(1) $g|_U$ is of class $\mathcal{C}^1$,
(2) $\frac{\partial}{\partial x_i}(g|_U) = g_i|_U$, $i = 1, \ldots, n$.

Let $g : \mathbb{R}^n \to \mathbb{R}^m$ and $t \in \mathbb{R}^m$. By $\operatorname{reg} g^{-1}(t)$ we denote (after Lion [L]) the set of all $x \in g^{-1}(t)$ for which there exists an open neighbourhood $U \subset \mathbb{R}^n$ of x such that $g|_U$ is a submersion of class $\mathcal{C}^1$.

Definition 1.3 (see [L]). We say that a geometric family $\mathfrak{G} = \{\mathfrak{G}_n\}_{n\in\mathbb{N}}$ is 0-*regular* if for every $n \in \mathbb{N}$, every mapping $g = (g_1, \ldots, g_n) : \mathbb{R}^n \to \mathbb{R}^n$, where $g_i \in \mathfrak{G}_n$ $(i = 1, \ldots, n)$, and for each $t \in \mathbb{R}^n$, the set $\operatorname{reg} g^{-1}(t)$ is finite.

Definition 1.4 (see [L]). We say that a geometric family $\mathfrak{G} = \{\mathfrak{G}_n\}_{n\in\mathbb{N}}$ has the *uniform fibre finiteness* (UFF) *property* if for every $n, p \in \mathbb{N}$ and $g = (g_1, \ldots, g_p) : \mathbb{R}^n \to \mathbb{R}^p$, where $g_i \in \mathfrak{G}_n$ $(i = 1, \ldots, p)$, there exists $N \in \mathbb{N}$ such that for each $t \in \mathbb{R}^p$,

$$\sharp\{A \subset \mathbb{R}^n \mid A \text{ is a connected component of } g^{-1}(t)\} < N.$$

Theorem 1.5 (Lion [L]). *Let $\mathfrak{G} = \{\mathfrak{G}_n\}_{n\in\mathbb{N}}$ be a geometric regular family. If it is 0-regular, then it has the uniform fibre finiteness property.*

Now, we turn to the modification of Wilkie's theorem by Karpinski and Macintyre. Let $AG(\mathbb{R}^n)$ denote the set of all affine subspaces of $\mathbb{R}^n$. Let $A \subset \mathbb{R}^n$. Then we put

$$\gamma(A) := \min\{N \in \mathbb{N} : \text{for all } V \in AG(\mathbb{R}^n),\ A \cap V \text{ has at most } N \text{ connected components}\}.$$

If such an N does not exist, then we put $\gamma(A) = \infty$.

Definition 1.6. A sequence $\mathcal{S} = \{\mathcal{S}_n\}_{n\in\mathbb{N}}$, where $\mathcal{S}_n \subset \mathcal{P}(\mathbb{R}^n)$ for each $n \in \mathbb{N}$, is called a *weak o-minimal structure* if for every $n, m \in \mathbb{N}$, the following conditions are satisfied:

(W1) if $A, B \in \mathcal{S}_n$, then $A \cap B \in \mathcal{S}_n$,
(W2) $\mathcal{S}_n$ contains all semialgebraic subsets of $\mathbb{R}^n$,

(W3) if $A \in \mathcal{S}_n$ and $B \in \mathcal{S}_m$, then $A \times B \in \mathcal{S}_{n+m}$,
(W4) if $A \in \mathcal{S}_n$ and σ is a permutation of coordinates, then $\sigma(A) \in \mathcal{S}_n$,
(W5) if $A \in \mathcal{S}_n$, then $\gamma(A) < \infty$,
(W6) if $A \in \mathcal{S}_n$, then there exist $m \geq n$ and a closed set $B \in \mathcal{S}_m$ such that $A = \Pi_{m,n}(B)$, where $\Pi_{m,n} : \mathbb{R}^m \ni (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_n) \in \mathbb{R}^n$.

DEFINITION 1.7. Let $N \in \mathbb{N}$. A weak o-minimal structure $\mathcal{S} = \{\mathcal{S}_n\}_{n\in\mathbb{N}}$ satisfies the *DC^N condition* for all N if for each $A \in \mathcal{S}_n$ there exists $p \geq n$, such that for each $N \in \mathbb{N}$, A is equal to $\Pi_{p,n}(\{f_N = 0\})$, where

(1) $f_N : \mathbb{R}^p \to \mathbb{R}$ is of class $\mathcal{C}^N$,
(2) graph $f_N \in \mathcal{S}_{p+1}$.

THEOREM 1.8 (Wilkie, Karpinski, Macintyre). *Suppose $\mathcal{S} = \{\mathcal{S}_n\}_{n\in\mathbb{N}}$ is a weak o-minimal structure satisfying DC^N for all N. Then there exists an o-minimal structure $\widetilde{\mathcal{S}} = \{\widetilde{\mathcal{S}}_n\}_{n\in\mathbb{N}}$ which contains $\mathcal{S}$.*

It is not difficult to check that if $\mathfrak{G} = \{\mathfrak{G}_n\}_{n\in\mathbb{N}}$ is a regular geometric family with the uniform fibre finiteness property, then defining $\mathcal{S}_n$ to be the family of all subsets of $\mathbb{R}^n$ of the form $f^{-1}(0)$, where $f \in \mathfrak{G}_n$, we obtain a weak o-minimal structure. It is less obvious that this structure satisfies the DC^N condition for all $N \in \mathbb{N}$. We will check this in detail.

2. DC^N condition

LEMMA 2.1. *Let $\mathfrak{F} = \{\mathfrak{F}_n\}_{n\in\mathbb{N}}$ be a geometric family with the uniform fibre finiteness property. Then*

(1) *for every $k \in \mathbb{N}$,*

$$\widetilde{\mathfrak{S}}_k := \{g : \mathbb{R}^k \to \mathbb{R} \mid \text{there exist } n \in \mathbb{N},\ f \in \mathfrak{F}_n \text{ and a semialgebraic map } \psi : \mathbb{R}^k \to \mathbb{R}^n \text{ such that } g = f \circ \psi\}$$

is a ring,
(2) *the family $\mathfrak{S} = \{\mathfrak{S}_k\}_{k\in\mathbb{N}}$, where $\mathfrak{S}_k$ is the ring of fractions of $\widetilde{\mathfrak{S}}_k$ with respect to the multiplicative set of nowhere vanishing functions, is a geometric family with the UFF property.*

Proof. (1) Let $f : \mathbb{R}^p \to \mathbb{R}$, $g : \mathbb{R}^s \to \mathbb{R}$, $\psi : \mathbb{R}^k \to \mathbb{R}^p$, $\phi : \mathbb{R}^k \to \mathbb{R}^s$, where $f \in \mathfrak{F}_p$, $g \in \mathfrak{F}_s$, and ψ, ϕ are semialgebraic maps. Then the function

$$h : \mathbb{R}^p \times \mathbb{R}^q \ni (u, v) \mapsto f(u)g(v) \in \mathbb{R}$$

belongs to $\mathfrak{S}_{p+q}$, by (G1) and (G4). Consequently,

$$(f \circ \psi) \cdot (g \circ \phi) = h \circ (\psi, \phi) \in \widetilde{\mathfrak{S}}_k.$$

In a similar way we can show that $f \circ \psi + g \circ \phi \in \widetilde{\mathfrak{S}}_k$.

(2) The family $\mathfrak{S} = \{\mathfrak{S}_k\}_{k\in\mathbb{N}}$ satisfies conditions (G1)–(G4) in the obvious way. To check the UFF property, take

$$F_i : \mathbb{R}^{j_i} \to \mathbb{R}, \quad G_i : \mathbb{R}^{l_i} \to \mathbb{R}, \quad G_i, F_i \in \mathfrak{F}, \quad i = 1, \dots, k,$$
$$\psi_i : \mathbb{R}^n \to \mathbb{R}^{j_i}, \quad \phi_i : \mathbb{R}^n \to \mathbb{R}^{l_i}, \quad \psi_i, \phi_i \text{ semialgebraic maps, } i = 1, \dots, k,$$

and

$$H = \left(\frac{F_1 \circ \psi_1}{G_1 \circ \phi_1}, \dots, \frac{F_k \circ \psi_k}{G_k \circ \phi_k}\right),$$

where $G_1 \circ \phi_1(x) \neq 0, \dots, G_k \circ \phi_k(x) \neq 0$, for every $x \in \mathbb{R}^n$. By (G4) we may assume that $s = j_i = l_i$ and $\xi = \psi_i = \phi_i$ for $i = 1, \dots, k$. There exist (see [BCR, 2.2]) $m \in \mathbb{N}$ and a polynomial $P : \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^m \to \mathbb{R}$ such that

$$\operatorname{graph} \xi = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^s \mid \exists z \in \mathbb{R}^m : P(x, y, z) = 0\}.$$

Define

$$\Theta : \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^s \times \mathbb{R}^m \to \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R},$$

by

$$\begin{aligned}\Theta\big(x, (a_1, \dots, a_k), (b_1, \dots, b_k), y, z\big) \\ = \big((a_1 b_1, \dots, a_k b_k), (F_1(y) - a_1, \dots, F_k(y) - a_k), \\ (b_1 G_1(y) - 1, \dots, b_k G_k(y) - 1), P(x, y, z)\big).\end{aligned}$$

There exists $N \in \mathbb{N}$ such that the number of connected components of $\Theta^{-1}(t, 0, 0, 0,)$, for every $t \in \mathbb{R}^k$, is not greater than N. It is easy to see that

$$H^{-1}(t) = \Pi(\Theta^{-1}(t, 0, 0, 0)),$$

where Π is the projection on the first n coordinates. Since the image of a connected set under a continuous map is connected, the map H has the UFF property. ∎

Definition 2.2. We say that a geometric family $\mathfrak{G} = \{\mathfrak{G}_n\}_{n\in\mathbb{N}}$ is *semialgebraically regular* if for every $n \in \mathbb{N}$ and $g \in \mathfrak{G}_n$, there exists a semialgebraic, closed, nowhere dense subset $A \subset \mathbb{R}^n$ and functions $g_1, \dots, g_n \in \mathfrak{G}_n$ such that, for $U = \mathbb{R}^n \setminus A$:

(1) $g|_U$ is of class $\mathcal{C}^1$,
(2) $\frac{\partial}{\partial x_i}(g|_U) = g_i|_U$, $i = 1, \dots, n$.

This is a generalization of the notion of a regular geometric family. Lemma 2.1 easily implies

Lemma 2.3. *Any geometric regular family $\mathfrak{F} = \{\mathfrak{F}_n\}_{n\in\mathbb{N}}$ with the UFF property generates a semialgebraically regular geometric family with the UFF property, closed with respect to compositions on the right with semialgebraic maps.*

Proof. It is enough to show that every composition $f \circ \phi$ of $f \in \mathfrak{F}_n$ and a semialgebraic map $\phi : \mathbb{R}^m \to \mathbb{R}^n$ is of class $\mathcal{C}^1$ except on a closed, nowhere dense semialgebraic set $A \subset \mathbb{R}^m$.

We prove this by induction on n. For $n = 1$ it is obvious. Let $n > 1$, and assume that the statement is true for every $m < n$. There exist hyperplanes $H_1, \ldots, H_k$ such that $f|_U$ is of class $\mathcal{C}^1$, where $U = \mathbb{R}^n \setminus (H_1 \cup \cdots \cup H_k)$. Also ϕ is of class $\mathcal{C}^1$ outside a closed, nowhere dense semialgebraic set $D \subset \mathbb{R}^m$. Let $B = \phi^{-1}(H_1 \cup \cdots \cup H_k)$. If $\dim B < n$, there is nothing to prove. When $\operatorname{int} B \neq \emptyset$, then it suffices to consider the maps $g_i = f|_{H_i} \circ \widetilde{\phi}_i$, where

$$\widetilde{\phi}_i(x) = \begin{cases} \phi(x), & x \in \phi^{-1}(H_i), \\ a_i, & x \in \mathbb{R}^n \setminus \phi^{-1}(H_i), \end{cases}$$

and a_i is arbirarily chosen from H_i. By the inductive hypothesis g_i is of class $\mathcal{C}^1$ except a closed, nowhere dense semialgebraic set C_i. It follows that $f \circ \phi$ is of class $\mathcal{C}^1$ outside the set $C = C_1 \cup \cdots \cup C_k \cup D$. ∎

LEMMA 2.4. *If C is a semialgebraic cell in $\mathbb{R}^n$ of dimension k, then there exists a semialgebraic $\mathcal{C}^\infty$-mapping $\phi_C : \mathbb{R}^k \to \mathbb{R}^n$ such that $C = \operatorname{im} \phi_C$.*

Proof. Use Proposition 2.9.10 from [BCR] and the $\mathcal{C}^\infty$-diffeomorphism

$$\Phi_k : \mathbb{R}^k \ni (x_1, \ldots, x_k) \to \left(\frac{1}{2}\left(\frac{x_1}{\sqrt{1+x_1^2}}\right), \ldots, \frac{1}{2}\left(\frac{x_k}{\sqrt{1+x_k^2}}\right)\right)$$

onto $(0,1)^k$. ∎

Now we can state

PROPOSITION 2.5. *Let $\mathfrak{G} = \{\mathfrak{G}_n\}_{n\in\mathbb{N}}$ be a semialgebraically regular geometric family with the uniform fibre finiteness property, closed with respect to compositions on the right with semialgebraic maps. Then, for each $n \in \mathbb{N}$, there exists $l \in \mathbb{N}$ such that if $F = (F_1, \ldots, F_k) : \mathbb{R}^n \to \mathbb{R}^k$, where $F_i \in \mathfrak{G}_n$, $i = 1, \ldots, k$, $A := F^{-1}(0)$, then for every $N \in \mathbb{N}$ there exists $\widetilde{F} : \mathbb{R}^{n+l} \to \mathbb{R}$ of class $\mathcal{C}^N$ such that $\widetilde{F}_i \in \mathfrak{G}_{n+l}$ for every $i = 1, \ldots, n+l$ and $A = \Pi_{n+l,n}(\widetilde{F}^{-1}(0))$.*

Proof. We will prove the proposition by induction on n. For $n = 1$ it is obvious, because sets on the real line are finite sums of points and intervals.

Now assume the conclusion is true for every $m < n+1$. Take $F : \mathbb{R}^{n+1} \to \mathbb{R}^k$, where $F_i \in \mathfrak{G}_{n+1}$, and let $A = F^{-1}(0)$. Let $V \subset \mathbb{R}^{n+1}$ be a closed, nowhere dense semialgebraic set such that $F|_{\mathbb{R}^{n+1}\setminus V}$ is of class $\mathcal{C}^N$. Take a cell decomposition $\mathcal{B}$ of $\mathbb{R}^{n+1}$ compatible with V. Then $\mathcal{B} = \mathcal{B}_0 \cup \cdots \cup \mathcal{B}_{n+1}$, where

$$\mathcal{B}_i = \{B \in \mathcal{B}_i \mid \dim B = i\}, \qquad i = 0, 1, \ldots, n+1.$$

Let $B \in \mathcal{B}$. Consider two cases:

(1) $B \in \mathcal{B}_{n+1}$. There exists a semialgebraic diffeomorphism

$$\varphi_B = (\varphi_B^1, \ldots, \varphi_B^{n+1}) : \mathbb{R}^{n+1} \to B$$

of class $\mathcal{C}^\infty$. Then

$$A \cap B = \{x \in \mathbb{R}^{n+1} \mid \exists z \in \mathbb{R}^{n+1} : \psi_B(x, z) = 0\},$$

where $\psi_B(x, z) = (F \circ \varphi_B)^2(z) + \sum_{i=1}^{n+1} (\varphi_B^i(z) - x_i)^2$ is a function of class $\mathcal{C}^N$ and $\psi_B \in \mathfrak{G}_{2n+2}$.

(2) $B \in \mathcal{B}_j$ for some $j = 1, \ldots, n$. There exists a semialgebraic diffeomorphism $\varphi_B = (\varphi_B^1, \ldots, \varphi_B^{n+1}) : \mathbb{R}^j \to B$ of class $\mathcal{C}^\infty$. By induction hypothesis there exist $l_j \in \mathbb{N}$ and $\mathcal{C}^N$-maps $\widehat{F}_B : \mathbb{R}^{j+l_j} \to \mathbb{R}$ such that $\Pi_{j+l_j,j}(\widehat{F}_B^{-1}(0)) = (F \circ \varphi_B)^{-1}(0)$. Now

$$A \cap B = \{x \in \mathbb{R}^{n+1} \mid \exists t^j \in \mathbb{R}^j \ \exists u^j \in \mathbb{R}^{l_j} : \psi_B(x, t^j, u^j) = 0\},$$

where $\psi_B(x, t^j, u^j) = \widehat{F}^2(t^j, u^j) + \sum_{i=1}^{n+1} (\varphi_B^i(t^j) - x_i)^2$ is a function of class $\mathcal{C}^N$ and $\psi_B \in \mathfrak{G}_{n+j+l_j+1}$.

Define $B_0 = \bigcup_{B \in \mathcal{B}_0} B$ and $A_0 = B_0 \cap A$. Consider the $\mathcal{C}^N$-function

$$\Psi : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^n \times \mathbb{R}^{l_n} \times \cdots \times \mathbb{R}^1 \times \mathbb{R}^{l_1} \longrightarrow \mathbb{R}$$

given by

$$\Psi(x, z, t^n, u^{l_n}, \ldots, t^1, u^{l_1}) = \prod_{B \in \mathcal{B}_{n+1}} \psi_B(x, z) \prod_{j=1}^{n} \prod_{B \in \mathcal{B}_j} \psi_B(x, t^j, u^j) \cdot \prod_{a \in A_0} \Big(\sum_{i=1}^{n+1} (a_i - x_i)^2 \Big).$$

Let $l = 2n + 2 + n(n+1)/2 + \sum_{j=1}^{n} l_j$. It is easy to see that $A = \Pi(\Psi^{-1}(0))$, where $\Pi : \mathbb{R}^l \to \mathbb{R}^{n+1}$ is the projection onto the first $n+1$ coordinates. ∎

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