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OPERATOR THEORY

Unbounded Jacobi Matrices with Empty Absolutely Continuous Spectrum

by

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Summary. Sufficient conditions for the absence of absolutely continuous spectrum for unbounded Jacobi operators are given. A class of unbounded Jacobi operators with purely singular continuous spectrum is constructed as well.

1. Introduction. In this paper we will discuss self-adjoint unbounded Jacobi operators in $l^2 = l^2(\mathbb{N})$ without absolutely continuous part. Let $\{e_n\}_{n\in\mathbb{N}}$ be the canonical orthonormal basis in l^2. A *Jacobi operator* H is densely defined in l^2 by the formula

$$He_n = a_{n-1}e_{n-1} + b_ne_n + a_ne_{n+1}, \quad n = 1, 2, \ldots,$$

where the *weights* $\{a_n\}_{n\in\mathbb{N}}$ and the *diagonal* $\{b_n\}_{n\in\mathbb{N}}$ are sequences of real numbers with $a_0 = 0$. More precisely, the Jacobi operator H associated to $\{a_n\}$ and $\{b_n\}$ is defined on its maximal domain in l^2 by

$$(Hf)_n = a_{n-1}f_{n-1} + b_nf_n + a_nf_{n+1}, \quad n = 1, 2, \ldots.$$

ASSUMPTION. In what follows we will always assume that $H^* = H$ and $a_n > 0$ for all $n \in \mathbb{N}$.

The spectral theory of Jacobi operators is a large field far from being completed. We mention here only a few papers concerning spectral analysis of H in the case $\lim_n a_n = \infty$ [1–3, 6, 8–10]. Most of these papers contain suf-

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ficient conditions for existence of nontrivial absolutely continuous spectrum of H.

The problem of the absence of absolutely continuous spectrum of Schrödinger operators has been studied by Simon and Spencer [17] and by Simon and Stolz [18]. They studied the cases of continuous and discrete versions (i.e. Jacobi operators with $a_n \equiv 1$). Following their ideas we will find general criteria for the absence of absolutely continuous spectrum of H, i.e. for $\sigma_{\mathrm{ac}}(H) = \emptyset$. For $\lambda \in \mathbb{R}$ consider the infinite system of equations

$$a_{n-1}u_{n-1} + b_n u_n + a_n u_{n+1} = \lambda u_n, \qquad n > 1. \tag{1.1}$$

It is well known that spectral analysis of H is strongly related to the asymptotic behaviour of solutions of (1.1) (Gilbert–Pearson subordination theory [12]). In particular, using this theory one can formulate the following general observation which can be used to verify that $\sigma_{\mathrm{ac}}(H) = \emptyset$.

Proposition 1.1. *If for almost all $\lambda \in \mathbb{R}$ (with respect to the Lebesgue measure) there exists a nontrivial solution u of* (1.1) *such that*

$$\sum_{i=1}^{N} |u_i|^2 = o\Big(\sum_{i=1}^{N-1} \frac{1}{a_i}\Big), \tag{1.2}$$

then $\sigma_{\mathrm{ac}}(H) = \emptyset$.

Remark 1.2. The above $o(\cdot)$ term in general depends on λ.

Proof. We claim that any solution of (1.1) satisfying (1.2) must be subordinated, [12]. Suppose that there exists v linearly independent of u and a constant $c_0 > 0$ such that

$$\sum_{s=1}^{N_i} |u_s|^2 \Big(\sum_{s=1}^{N_i} |v_s|^2\Big)^{-1} \ge c_0$$

for a sequence $N_i \to \infty$. Let $c = W(u,v)$ be the Wrońskian of u and v. By the Schwarz inequality we have (cf. [8, p. 222])

$$\begin{aligned}\sum_{s=1}^{N_i-1} \frac{|c|}{a_s} &\le 2\Big[\sum_{s=1}^{N_i} |u_s|^2\Big]^{1/2}\Big[\sum_{s=1}^{N_i} |v_s|^2\Big]^{1/2} \le \frac{2}{c_0^{1/2}} \sum_{s=1}^{N_i} |u_s|^2 \\ &= \frac{2}{c_0^{1/2}}\, o\Big(\sum_{s=1}^{N_i-1} \frac{1}{a_s}\Big),\end{aligned}$$

a contradiction. This proves our claim and completes the proof. ■

Despite its simplicity, Proposition 1.1 is, in general, not easy to apply. Therefore we look for more efficient criteria for the absence of absolutely continuous spectrum of H. In Section 2 we shall formulate sufficient conditions (in terms of $\{a_n\}$ and $\{b_n\}$) which guarantee that $\sigma_{\mathrm{ac}}(H) = \emptyset$. Moreover, in

Section 3 a class of Jacobi operators with purely singular continuous spectrum will be constructed.

2. The absence of absolutely continuous spectrum. Due to the results of [5] sufficient conditions are known for compactness of the resolvent $R(H,\cdot)$ of H, which are given in terms of asymptotic behaviour of the weights and the diagonal. We want to find sufficient conditions which guarantee that H does not have absolutely continuous spectrum. It turns out that the method used by Simon and Spencer [15] for Schrödinger operators (continuous or discrete) can be easily adapted to our case. We shall consider the following three cases:

(a) $\{a_n\}$ is unbounded and $\{b_n\} \in l^\infty$,
(b) $\{a_n\} \in l^\infty$ and $\{b_n\}$ is unbounded,
(c) both $\{a_n\}$ and $\{b_n\}$ are unbounded.

Before formulating the results concerning the above three cases we introduce the following definition.

Let a and b be the weight sequence and the diagonal, respectively, of the original Jacobi operator H. Suppose that $\{L_k\}_{k\geq 1}$ is a sequence of mutually disjoint finite subsets of $\mathbb{N}$, and define $Z_k = \bigcup_{s=1}^k L_s$ for $k \in \mathbb{N}\cup\{\infty\}$, $Z_0 = \emptyset$.

Define a new sequence $a^{(k)}$ of weights by

$$a_n^{(k)} := \begin{cases} 0, & n \in Z_k, \\ a_n, & n \notin Z_k, \end{cases}$$

for $k \in \{0\} \cup \mathbb{N} \cup \{\infty\}$ and $n \in \mathbb{N}$.

Definition 2.1. We define H_k to be the Jacobi operator given by the weight sequence $a^{(k)}$ and the diagonal sequence b.

Note that $H_0 = H$ and H_∞ is the direct sum of finite-dimensional Jacobi matrices provided all $L_k \neq \emptyset$. Observe that all H_k are self-adjoint. Indeed, if k is finite, then H_k is a finite-dimensional perturbation of a self-adjoint operator H. For $k = \infty$ the restriction of H_∞ to the linear space D of finite linear combinations of base vectors $e_n, n \in \mathbb{N}$, is essentially self-adjoint because the space D is equal to the space of finite linear combinations of all the eigenvectors of $H_\infty|_D$.

Theorem 2.2. *Let H be a self-adjoint Jacobi operator corresponding to a positive sequence of weights $\{a_k\}$ and a diagonal $\{b_n\}$. Suppose that one of the following conditions holds*:

(a) $\lim_k a_k = \infty$, $\{b_n\} \in l^\infty$ *and*

$$\liminf_n \, [a_{n-1}^2 + a_{n+1}^2] a_n^{-1} = 0. \tag{2.1}$$

(b) $\{a_n\} \in l^\infty$ *and* $\{b_n\}$ *is unbounded*, *i.e.* $\sup_n |b_n| = \infty$.

(c) $\{a_n\}$ *and* $\{b_n\}$ *are unbounded*, $\liminf_n a_n > 0$, (2.1) *holds and*

$$\liminf_n \left(1 - \frac{b_n b_{n+1}}{a_n^2}\right)^2 = \varrho^2 > 0, \tag{2.2}$$

$$\sup_n \,(b_n^2 + b_{n+1}^2) a_n^{-2} < \infty. \tag{2.3}$$

Then $\sigma_{\mathrm{ac}}(H) = \emptyset$.

Proof. The proof is based on the (folklore) result formulated below. In what follows, for an operator A the notation $A \pm i$ means $A \pm iI$, where I is the identity operator.

LEMMA 2.3. *Let* H_k *be the sequence of Jacobi operators given in Definition* 2.1 *and such that* $(H_{k-1} - i)^{-1} - (H_k - i)^{-1}$ *is of trace class for* $k = 1, 2, \ldots$ *with*

$$\sum_{k=1}^{\infty} \|(H_{k-1} - i)^{-1} - (H_k - i)^{-1}\|_1 < \infty, \tag{2.4}$$

where $\|T\|_1$ *denotes the trace norm of* T. *Then* $\sigma_{\mathrm{ac}}(H) = \emptyset$.

Proof. First note that for any $n \in \mathbb{N}$, $(H_0 - i)^{-1} - (H_n - i)^{-1}$ is of trace class. Moreover, from (2.4) we know that

$$(H_0 - i)^{-1} - (H_n - i)^{-1} = \sum_{k=1}^{n} [(H_{k-1} - i)^{-1} - (H_k - i)^{-1}]$$

is convergent in trace norm to a trace class operator C as $n \to \infty$. On the other hand, $(H_n - i)^{-1}$ converges strongly to $(H_\infty - i)^{-1}$. Indeed, by definitions of H_n and H_∞ we have

$$H_n f \to H_\infty f \quad \text{ for any } f \in D.$$

But $\overline{H_n|_D} = H_n$ and $\overline{H_\infty|_D} = H_\infty$ (see [3]). Applying Theorem VIII.25 of [14] we get the desired strong convergence of the resolvents. Hence $(H_0 - i)^{-1} - (H_\infty - i)^{-1} = C$ is of trace class. Finally, the Birman–Kuroda theorem (see [11]) completes the proof. ∎

(c) Choose $n_k \to \infty$ such that $n_{k+1} - n_k > 2$ for all k, $n_1 > 1$, and

$$\sum_k [a_{n_k-1}^2 + a_{n_k+1}^2] a_{n_k}^{-1} < \infty.$$

Let $L_k = \{n_k - 1, n_k + 1\}$ and let H_k be as in Definition 2.1. We claim that

$$\sum_{k=1}^{\infty} \|(H_{k-1} - i)^{-1} - (H_k - i)^{-1}\|_1 < \infty. \tag{2.5}$$

Since the rank of $H_k - H_{k-1}$ equals 4, we have

$$\|(H_k - i)^{-1} - (H_{k-1} - i)^{-1}\|_1 \ \le\ 4\|(H_k - i)^{-1} - (H_{k-1} - i)^{-1}\|.$$

Below we will estimate

$$\|(H_k - i)^{-1} - (H_{k-1} - i)^{-1}\| \;=\; \|(H_k - i)^{-1}(H_{k-1} - H_k)(H_{k-1} - i)^{-1}\|.$$

Note that the domains of H_k and H_{k-1} are equal and $H_{k-1} = H_k + T_k + T_k^*$, where

$$(T_k f)(n) = \begin{cases} 0, & n \notin \{n_k, n_k+1\}, \\ a_{n_k-1} f(n_k - 1), & n = n_k, \\ a_{n_k+1} f(n_k + 2), & n = n_k + 1. \end{cases}$$

Now

$$(H_k - i)^{-1} = \begin{pmatrix} X & 0 & 0 \\ 0 & \begin{pmatrix} b_{n_k} - i & a_{n_k} \\ a_{n_k} & b_{n_k+1} - i \end{pmatrix}^{-1} & 0 \\ 0 & 0 & Y \end{pmatrix},$$

where $X = (A_k - i)^{-1}$ is the resolvent of the obvious finite-dimensional Jacobi matrix A_k and $Y = (B_k - i)^{-1}$ is the resolvent of the infinite-dimensional Jacobi matrix B_k having the weights $c_n = a_{n_k+1+n}$ and the diagonal $d_n = b_{n_k+1+n}$.

Hence

$$[(H_k - i)^{-1} T_k f](n) = \begin{cases} 0, & n \notin \{n_k, n_k+1\}, \\ c(f,k), & n = n_k, \\ c(f, k+1), & n = n_k + 1, \end{cases}$$

where

$$c(f,k) := [(b_{n_k+1} - i) a_{n_k-1} f(n_k - 1) - a_{n_k} a_{n_k+1} f(n_k + 2)] s_{n_k}^{-1},$$
$$c(f,k+1) := [(b_{n_k} - i) a_{n_k+1} f(n_k + 2) - a_{n_k} a_{n_k-1} f(n_k - 1)] s_{n_k}^{-1},$$

with

$$s_{n_k} := \det\begin{pmatrix} b_{n_k} - i & a_{n_k} \\ a_{n_k} & b_{n_k+1} - i \end{pmatrix}.$$

Let $\|\cdot\|_2$ denote the Hilbert–Schmidt norm. Using the above relations we compute

$$\|(H_k - i)^{-1} T_k\| \le \|(H_k - i)^{-1} T_k\|_2 \tag{2.6}$$
$$\le \left[\frac{(b_{n_k+1}^2 + 1)a_{n_k-1}^2 + a_{n_k+1}^2 a_{n_k}^2 + a_{n_k-1}^2 a_{n_k}^2 + (b_{n_k}^2 + 1) a_{n_k+1}^2}{(a_{n_k}^2 - b_{n_k} b_{n_k+1})^2 + b_{n_k}^2 + b_{n_k+1}^2 + 2a_{n_k}^2 + 1}\right]^{1/2} =: W_{n_k}.$$

Obviously a similar estimate holds for $\|(H_k + i)^{-1} T_k\|$. On the other hand,

$$
\begin{aligned}
(2.7)\quad & \|(H_k - i)^{-1}T_k^*(H_{k-1}-i)^{-1}\| \\
&\le \|T_k^*(H_{k-1}-i)^{-1}\| \\
&\le \|T_k^*(H_k - i)^{-1}\| + \|T_k^*(H_k-i)^{-1}(H_k - H_{k-1})(H_{k-1}-i)^{-1}\| \\
&\le \|(H_k + i)^{-1}T_k\| + \|{T_k}^*(H_k - i)^{-1}\|\,\|(H_k - H_{k-1})(H_{k-1}-i)^{-1}\| \\
&\le \; W_{n_k} + \sqrt{2}\, W_{n_k}(a_{n_k-1}^2 + a_{n_k+1}^2)^{1/2}.
\end{aligned}
$$

Note that the denominator of W_{n_k} is equal to

$$
\begin{aligned}
|s_{n_k}| &:= a_{n_k}^2\left[\left(\frac{b_{n_k}b_{n_{k+1}}}{a_{n_k}^2} - 1\right)^2 + \frac{b_{n_k}^2 + b_{n_{k+1}}^2 + 1}{a_{n_k}^4} + \frac{2}{a_{n_k}^2}\right]^{1/2} \\
&\ge a_{n_k}^2\left|\frac{b_{n_k}b_{n_{k+1}}}{a_{n_k}^2} - 1\right| \ge \frac{\varrho}{2}\, a_{n_k}^2
\end{aligned}
$$

for k sufficiently large. In the last inequality we used (2.2). Now the numerator of W_{n_k} is equal to

$$
(2.8)\quad (a_{n_k-1}^2 + a_{n_k+1}^2)^{1/2} a_{n_k}\left[1 + \frac{a_{n_k-1}^2(b_{n_k+1}^2 + 1) + a_{n_k+1}^2(b_{n_k}^2+1)}{(a_{n_k-1}^2 + a_{n_k+1}^2)a_{n_k}^2}\right]^{1/2}.
$$

Using (2.3) it is clear that the expression in square brackets is uniformly bounded by a constant M. Combining (2.6) and (2.8) we have

$$
(2.9)\qquad W_{n_k} \le \frac{2M}{\varrho}\,\frac{(a_{n_k-1}^2 + a_{n_k+1}^2)^{1/2}}{a_{n_k}}
$$

for k sufficiently large.

Combining all the above estimates we have

$$
\begin{aligned}
\|(H_{k-1}-i)^{-1} - (H_k - i)^{-1}\| &\le W_{n_k}[2 + \sqrt{2}(a_{n_k-1}^2 + a_{n_k+1}^2)^{1/2}] \\
&\le C\,\frac{a_{n_k-1}^2 + a_{n_k+1}^2}{a_{n_k}}
\end{aligned}
$$

for some positive constant C independent of n_k. Due to our choice of n_k it follows that

$$
(2.10)\qquad \sum_{k=0}^{\infty} \|(H_{k+1}-i)^{-1} - (H_k - i)^{-1}\| < \infty.
$$

Now it is enough to apply Lemma 2.3. This completes the proof of (c).

(b) We follow the proof given by Simon–Spencer in [17]. Let $n_{k+1} > n_k+1$ be a sequence of natural numbers such that

$$
\sum_{k=1}^{\infty}\frac{1}{|b_{n_k}|} < \infty.
$$

Define $L_s := \{n_s - 1, n_s\}$ and the sets Z_k and weights $a_n^{(k)}$ as above.

By definition of H_k we can write, for $f \in D(H_k)$,

$$(H_{k-1} - H_k)f = (R_k + R_k^*)f, \tag{2.11}$$

where

$$(R_k f)(n) = \begin{cases} a_{n_k-1} f(n_k - 1) + a_{n_k} f(n_k + 1), & n = n_k, \\ 0, & n \neq n_k. \end{cases}$$

Using again the definition of H_k we have

$$(H_k - i)^{-1} = \begin{pmatrix} (A-i)^{-1} & 0 & 0 \\ 0 & (b_{n_k} - i)^{-1} & 0 \\ 0 & 0 & (B-i)^{-1} \end{pmatrix}, \tag{2.12}$$

where A and B are suitable Jacobi matrices coming from the original matrix H (see the corresponding part of the proof of (c)). Assume that $\sup_n a_n = M$.

Repeating computations given in case (c) we have

$$\begin{aligned} \|(H_{k-1} - i)^{-1} - (H_k - i)^{-1}\| &= \|(H_k - i)^{-1}(H_{k-1} - H_k)(H_{k-1} - i)^{-1}\| \\ &\le \|(H_k - i)^{-1} R_k (H_{k-1} - i)^{-1}\| + \|(H_k - i)^{-1} R_k^* (H_{k-1} - i)^{-1}\|. \end{aligned}$$

Using (2.11) and (2.12) we estimate

$$\|(H_k - i)^{-1} R_k (H_{k-1} - i)^{-1}\| \le \|(H_k - i)^{-1} R_k\| \le \frac{2M}{|b_{n_k} - i|} < \frac{2M}{|b_{n_k}|}, \tag{2.13}$$

$$\begin{aligned} \|(H_k - i)^{-1} R_k^* (H_{k-1} - i)^{-1}\| &\le \|R_k^* (H_{k-1} - i)^{-1}\| \\ &\le \|R_k^* (H_k - i)^{-1}\| + \|R_k^* (H_k - i)^{-1}\| \, \|(H_k - H_{k-1})(H_{k-1} - i)^{-1}\|. \end{aligned} \tag{2.14}$$

But

$$\|R_k^* (H_k - i)^{-1}\| = \|(H_k + i)^{-1} R_k\| \le 2M/|b_{n_k}|, \tag{2.15}$$

and

$$\|(H_k - H_{k-1})(H_{k-1} - i)^{-1}\| \le 4M. \tag{2.16}$$

Thus the above four inequalities imply that

$$\|(H_{k-1} - i)^{-1} - (H_k - i)^{-1}\| \le C/|b_{n_k}|, \qquad k = 1, 2, \ldots,$$

for some positive C. Since all the estimated operators have rank less than or equal to 3 similar inequalities also hold for the trace norm, and again Lemma 2.3 ends the proof of (b).

(a) Let H_k be defined as in (c), i.e. $L_s = \{n_s - 1, n_s + 1\}$, where $\{n_k\}$ is chosen in such a way that

$$\sum_{k=1}^{\infty} (a_{n_k-1}^2 + a_{n_k+1}^2) a_{n_k}^{-1} < \infty.$$

In this case estimates of W_{n_k} (see case (c)) are immediate. In fact, applying the formula for s_{n_k} we have, for any $\varepsilon \in (0,1)$ and k sufficiently large,

$$|s_{n_k}| \geq {a_{n_k}}^2 \left| \frac{b_{n_k} b_{n_k+1}}{a_{n_k}^2} - 1 \right| \geq a_{n_k}^2 (1-\varepsilon). \tag{2.17}$$

This is clear because $\lim_k a_{n_k} = \infty$. On the other hand, the numerator of W_{n_k} can be estimated from above by

$$(a_{n_k-1}^2 + a_{n_k+1}^2)^{1/2} a_{n_k} (1+\varepsilon), \tag{2.18}$$

provided k is sufficiently large. This is obvious because $\{b_n\} \in l^\infty$.

Using (2.17) and (2.18) we obtain the desired estimate

$$W_{n_k} \leq (a_{n_k-1}^2 + a_{n_k+1}^2)^{1/2} a_{n_k}^{-1} (1+\varepsilon)(1-\varepsilon)^{-1}$$

for k sufficiently large. The rest of the proof is the same as the final part of the proof in (c).

This ends the proof of the theorem. ∎

3. Construction of an unbounded Jacobi matrix with singular continuous spectrum. By the well known general result of Simon [15, Theorem 4.1] the set X of bounded self-adjoint Jacobi operators A in $l^2(\mathbb{Z})$ with $\sigma(A) = [-a-2, a+2]$ and purely singular spectrum is Baire typical. This means that for a suitable metric on the space of all bounded and self-adjoint Jacobi operators, the above set X is dense and G_δ. Later in a joint paper with Stolz they found explicit examples of Jacobi operators (with $a_n = 1$) having purely singular continuous spectrum in $(-2,2)$ [18]. Using Theorem 2.2 of Section 2 we shall construct an unbounded Jacobi operator (with $b_n \equiv 0$) having the same property. The idea of our construction is similar to the one presented by Simon and Stolz (which in turn resembles the classical one given by Pearson for the Schrödinger operator [13]).

Before we start the construction let us recall some notation. For $\lambda \in \mathbb{R}$ consider the system (1.1). Using the transfer matrix

$$B_\lambda(n) := \begin{pmatrix} 0 & 1 \\ -a_{n-1}/a_n & \lambda/a_n \end{pmatrix},$$

and setting $\vec{u}(n) = \binom{u_{n-1}}{u_n}$ one can rewrite (1.1) in the form

$$\vec{u}(n+1) = B_\lambda(n)\vec{u}(n), \qquad n \geq 2.$$

For given sequences $\{c_s\}$ and $\{w_s\}$ of positive numbers and a sequence $\{k_s\}$ of natural numbers with $k_{s+1} - k_s > 2$ we define the sequence $\{a_n\}$ by

$$a_n = \begin{cases} c_s, & n \in [k_s+1, k_{s+1}), \\ w_s, & n = k_s, \end{cases} \tag{3.1}$$

and $a_n = 1$ for $n < k_1$.

Below we shall impose some conditions on the above sequences but at this point it is enough to know that $\{1/c_s\} \in l^1$.

Fix $\lambda > 0$ and choose $s_0 \in \mathbb{N}$ such that $\lambda^2 < 4c_s^2$ for $s \geq s_0$.

Define $C_s := B_\lambda(k_{s+1} - 1)$. Then for $s \geq s_0$, $\sigma(C_s) = \{\varrho_s, \overline{\varrho}_s\}$, where

$$\varrho_s = \frac{1}{2}\,(\lambda c_s^{-1} + i\sqrt{4 - \lambda^2 c_s^{-2}}\,)$$

and $\overline{\varrho}_s$ denotes the complex conjugate of ϱ_s. The matrix

$$X_s := \begin{pmatrix} 1 & 1 \\ \varrho_s & \overline{\varrho}_s \end{pmatrix}$$

diagonalizes C_s, i.e.

$$X_s^{-1} C_s X_s = \begin{pmatrix} \varrho_s & 0 \\ 0 & \overline{\varrho}_s \end{pmatrix}.$$

Using $\|A\|^2 = \|A^*A\| = \|A^*A\|_{\mathrm{s}}$ (the spectral norm of A^*A), we compute

$$\|X_s\|^2 = 2 + \frac{\lambda}{c_s}, \qquad \|X_s^{-1}\|^2 = \left(2 - \frac{\lambda}{c_s}\right)^{-1}.$$

Hence

$$\|X_s\|^2 \cdot \|X_s^{-1}\|^2 = 1 + \frac{\lambda}{c_s} + O\Big(\Big(\frac{\lambda}{c_s}\Big)^2\Big), \qquad s \to \infty. \tag{3.2}$$

Suppose that the above sequences $\{c_s\}, \{w_s\}$ and $\{k_s\}$ also satisfy

(i) $(c_{s-1}^2 + c_s^2) w_s^{-1} \to 0$ as $s \to \infty$,
(ii) $\sum_s (k_{s+1} - k_s) c_s^{-2} \prod_{p=1}^{s} [c_p w_p^{-1}]^2 = \infty$.

In particular, (i) implies that $c_s w_s^{-1} < 1$ for large s. This fact will be used a few times.

Let J_0 be the Jacobi operator defined by $\{a_n\}$ given in (3.1) and $b_n \equiv 0$ with sequences $\{c_s\}$ and $\{w_s\}$ obeying (i) and (ii).

From the above assumptions, we have

$$\sum_n a_n^{-1} > \sum_s (k_{s+1} - k_s) c_s^{-1} \geq \sum_s (k_{s+1} - k_s) c_s^{-2} \prod_{p=1}^{s} \left(\frac{c_p}{w_p}\right)^2,$$

and the Carleman condition $\sum_n a_n^{-1} = \infty$ guarantees that J_0 is self-adjoint. We claim that J_0 has purely singular continuous spectrum in $\mathbb{R} \setminus \{0\}$. First note that $\sigma_{\mathrm{ac}}(J_0) = \emptyset$ by Theorem 2.2(a). Indeed, by (3.1) and (i) the condition (a) of Theorem 2.2 is satisfied. Therefore it remains to prove that J_0 does not have point spectrum in $\mathbb{R} \setminus \{0\}$. This can be proved by applying the following general result (which is also implicitly contained in [18]) of [16, Theorem 10.5.3].

LEMMA 3.1. *Let J be a self-adjoint Jacobi operator defined by the sequences $\{a_n\}$ and $\{b_n\}$. For $\lambda \in \mathbb{R}$ define*

$$M_\lambda(n) := B_\lambda(n) \ldots B_\lambda(2).$$

If

$$\sum_n a_n^{-2} \|M_\lambda(n)\|^{-2} = \infty, \tag{3.3}$$

then (1.1) *has no solution in* l^2, *i.e.* $\lambda \notin \sigma_{\mathrm{p}}(J)$.

Below we check that J_0 satisfies (3.3). Let $A_s := B_\lambda(k_s+1)B_\lambda(k_s)$. Then for $l \in [k_s+2, k_{s+1}-1]$ we have

$$\begin{aligned} M_\lambda(l) = {} & B_\lambda(l) \ldots B_\lambda(k_s+2) A_s B_\lambda(k_s-1)^{k_s - k_{s-1}-2} \\ & \cdot A_{s-1} B_\lambda(k_{s-1}-1)^{k_{s-1}-k_{s-2}-2} \ldots A_{s_0} B_\lambda(k_{s_0}-1) \ldots B_\lambda(2). \end{aligned} \tag{3.4}$$

Write

$$A_s = \begin{pmatrix} -c_{s-1} w_s{}^{-1} & \lambda w_s{}^{-1} \\ -\lambda c_{s-1}(c_s w_s)^{-1} & \lambda^2 (c_s w_s)^{-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -w_s c_s{}^{-1} \end{pmatrix}.$$

Note that all entries of the first matrix belong to l^1. This is obvious by using the above assumption (i) and the identities

$$c_{s-1} w_s^{-1} = c_{s-1}^{-1}(c_{s-1}^2 w_s^{-1}), \qquad w_s^{-1} = c_s^{-1}(c_s w_s^{-1}).$$

Thus

$$\|A_s\| \le \frac{w_s}{c_s} + r_s, \tag{3.5}$$

where $\{r_s\} \in l^1$.

Combining (3.2), (3.4), (3.5) and using the diagonalization of C_s and $|\varrho_s| = 1$ we can write, for $l \in [k_s+2, k_{s+1}-1]$,

$$\begin{aligned} \|M_\lambda(l)\|^2 & \le \|X_s\|^2 \|X_s^{-1}\|^2 \|A_s\|^2 \cdot \|X_{s-1}\|^2 \|X_{s-1}^{-1}\|^2 \|A_{s-1}\|^2 \\ & \quad \ldots \|A_{s_0}\|^2 \|B_\lambda(k_{s_0}-1) \ldots B_\lambda(2)\|^2 \\ & \le M(\lambda) \prod_{p=s_0}^{s} \left[\left(1 + \frac{\lambda}{c_p} + O\left(\left(\frac{\lambda}{c_p}\right)^2\right)\right) \left(\frac{w_p}{c_p} + r_p\right)^2 \right] \\ & \le C(\lambda) \prod_{p=s_0}^{s} \left(\frac{w_p}{c_p}\right)^2, \end{aligned} \tag{3.6}$$

where $M(\lambda)$ and $C(\lambda)$ are some finite positive constants.

Note that

$$\sum_n a_n^{-2} \|M_\lambda(n)\|^{-2} \ge \sum_s c_s^{-2} \sum_{l=k_s+2}^{k_{s+1}-1} \|M_\lambda(l)\|^{-2}. \tag{3.7}$$

Using (3.6) and (ii) we have

$$\sum_{s\ge s_0} c_s^{-2} \sum_{l=k_s+2}^{k_{s+1}-1} \|M_\lambda(l)\|^{-2} \ge C(\lambda)^{-1} \sum_{s\ge s_0} c_s^{-2}(k_{s+1}-k_s-2) \prod_{p=s_0}^{s} \left(\frac{c_p}{w_p}\right)^2 = \infty.$$

Therefore (3.3) of Lemma 3.1 is satisfied and so $\lambda \notin \sigma_{\mathrm{p}}(J_0)$. Since the spectrum of J_0 is symmetric (because the diagonal of J_0 vanishes) (see [7]), we do not have to repeat the proof for $\lambda < 0$. It follows that $\sigma_{\mathrm{p}}(J_0) \setminus \{0\} = \emptyset$.

Consequently, we have proved

THEOREM 3.2. *Let $\{c_s\}$ be a sequence of positive numbers with $\{c_s^{-1}\} \in l^1$, and let $\{w_s\}$ be a sequence of positive numbers such that*

$$(c_{s-1}^2 + c_s^2) w_s^{-1} \to 0.$$

For any sequence $\{k_n\}$ of integers with $k_{n+1} > k_n + 2$ satisfying the condition

$$\sum_s (k_{s+1}-k_s) c_s^{-2} \prod_{p=1}^{s} \left(\frac{c_p}{w_p}\right)^2 = \infty \tag{3.8}$$

and the weights defined by

$$a_n = \begin{cases} c_s, & n \in [k_s+1, k_{s+1}), \\ w_s, & n = k_s, \\ 1, & n < k_1, \end{cases}$$

the Jacobi operator J_0 with the above weights and zero diagonal has purely singular continuous spectrum in $\mathbb{R} \setminus \{0\}$.

COROLLARY 3.3. *Let $\{a_n\}$, $\{c_s\}$, $\{k_s\}$ and $\{w_s\}$ be as in Theorem 3.2. If each k_s is odd, then the spectrum of the Jacobi operator J_0 defined in that theorem has purely singular continuous spectrum in $\mathbb{R}$.*

Proof. Direct computation shows that $0 \in \sigma_{\mathrm{p}}(J_0)$ if and only if

$$\sum_n \prod_{l=1}^{n} (a_{2l-1}/a_{2l})^2 < \infty.$$

But

$$\sum_n \prod_{l=1}^{n} (a_{2l-1}/a_{2l})^2 > \sum_s \prod_{l=1}^{(k_s+1)/2} (a_{2l-1}/a_{2l})^2 = \sum_s \prod_{p=1}^{s} (w_p/c_p)^2 = \infty.$$

The last equality holds because $w_p c_p^{-1} > 1$ for large p and so the third product is increasing for large s. This completes the proof. ∎

The following two examples satisfy the assumptions of Theorems 2.2 and 3.2, respectively.

EXAMPLE 3.4. Let $\{c_n\} \in l^\infty$. Take $\alpha \in (0,1]$ and define $a_{3n-1} = a_{3n+1} = n^\alpha$, $a_1 = 1$, and $a_{3n} = n^\beta$, where $\beta > 2\alpha$. The diagonal b_n is given by $b_{2n} = c_n$ and $b_{2n+1} = n^\gamma$ with $0 < \gamma < \alpha$. It is easy to check that for these weights and diagonal all assumptions of Theorem 2.2(c) are satisfied.

EXAMPLE 3.5. Take $\alpha > 1$ and define $c_s = s^\alpha$ and $w_s = s^{2\alpha+\varepsilon}$, $\varepsilon > 0$. Choose a sequence $\{k_s\} \subset \mathbb{N}$ such that

$$\sum_s (k_{s+1} - k_s) s^{-2\alpha}\ (s!)^{-2(\alpha+\varepsilon)} = \infty. \tag{3.9}$$

Then the condition (3.8) of Theorem 3.2 is satisfied.

REMARK 3.6. One can construct examples of unbounded Jacobi operators J with $\sigma_{\mathrm{ac}}(J) = \mathbb{R}$ and nonempty $\sigma_{\mathrm{p}}(J)$ (see [8]). However, we do not know explicit examples of unbounded weights $\{a_n\}$ which define a Jacobi operator (with zero diagonal) having mixed absolutely continuous and singular continuous spectrum.

We conclude this paper with the following question. Let J be a Jacobi operator with purely singular continuous spectrum. By a general result of Carey and Pincus [4] there exists a trace class operator T with $\sigma_{\mathrm{ac}}(J+T) = \sigma_{\mathrm{sc}}(J+T) = \emptyset$. Can one choose T to be a Jacobi operator?

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