BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 56, No. 1, 2008

OPERATOR THEORY

Two Results on Jachymski–Schröder–Stein Contractions

by

Simeon REICH and Alexander J. ZASLAVSKI

Presented by Bogdan BOJARSKI

Summary. We establish two fixed point theorems for certain mappings of contractive type.

1. Introduction. Throughout this paper, (X, d) is a complete metric space, N_0 a natural number, and $\phi : [0, \infty) \to [0, \infty)$ a function which is upper semicontinuous from the right and satisfies $\phi(t) < t$ for all $t > 0$. We call a mapping $T : X \to X$ for which

$$\min\{d(T^i x, T^i y) : i \in \{1, \dots, N_0\}\} \le \phi(d(x, y)) \quad \text{for all } x, y \in X \tag{1.1}$$

a *Jachymski–Schröder–Stein contraction* (with respect to ϕ).

Such mappings with $\phi(t) = \gamma t$ for some $\gamma \in (0, 1)$ have recently been of considerable interest [1, 7–11]. In the present paper we study general Jachymski–Schröder–Stein contractions and prove two fixed point theorems for them (Theorems 2.1 and 3.1 below). In our first result we establish convergence of iterates to a fixed point, and in the second this conclusion is strengthened to obtain uniform convergence on bounded subsets of X. This last type of convergence is useful in the study of inexact orbits [4]. Our theorems contain the (by now classical) results of [2, 3] as well as Theorem 2 of [8], where condition (1.1) was first introduced. In contrast with our Theorem 2.1, it was assumed in [8, Theorem 2] that the function ϕ was upper semicontinuous and that $\liminf_{t\to\infty}(t - \phi(t)) > 0$. Moreover, our argument is completely different from the one presented in [8], where the Cantor Intersection Theorem was employed. We remark in passing that Cantor's theorem was also used for a linear ϕ in [5, p. 22] (cf. also [6, p. 2]).

2000 *Mathematics Subject Classification*: 47H10, 54E50, 54H25.

Key words and phrases: complete metric space, contractive mapping, fixed point, iteration.

2. Convergence

THEOREM 2.1. *Let (X, d) be a complete metric space and let $T : X \to X$ be a Jachymski–Schröder–Stein contraction. Assume there is $x_0 \in X$ such that T is uniformly continuous on the orbit $\{T^i x_0 : i = 1, 2, \dots\}$. Then there exists $\overline{x} = \lim_{i\to\infty} T^i x_0$ in (X, d). Moreover, if T is continuous at $\overline{x}$, then $\overline{x}$ is the unique fixed point of T.*

Proof. Set

$$T^0 x = x, \qquad x \in X. \tag{2.1}$$

We are going to define a sequence of nonnegative integers $\{k_i\}_{i=0}^{\infty}$ by induction. Set $k_0 = 0$. Assume that $i \geq 0$ is an integer, and that the integer $k_i \geq 0$ has already been defined. Clearly, there exists an integer k_{i+1} such that

$$1 \leq k_{i+1} - k_i \leq N_0 \tag{2.2}$$

and

$$d(T^{k_{i+1}} x_0, T^{k_{i+1}+1} x_0) = \min\{d(T^{j+k_i} x_0, T^{j+k_i+1} x_0) : j = 1, \dots, N_0\}. \tag{2.3}$$

By (1.1), (2.2) and (2.3), the sequence $\{d(T^{k_j} x_0, T^{k_j+1} x_0\}_{j=0}^{\infty}$ is decreasing. Set

$$r = \lim_{j\to\infty} d(T^{k_j} x_0, T^{k_j+1} x_0). \tag{2.4}$$

Assume that $r > 0$. Then by (1.1), (2.2) and (2.3), for each integer $j \geq 0$,

$$d(T^{k_{j+1}} x_0, T^{k_{j+1}+1} x_0) \leq \phi(d(T^{k_j} x_0, T^{k_j+1} x_0)).$$

When combined with (2.4), the monotonicity of the sequence

$$\{d(T^{k_j} x_0, T^{k_j+1} x_0\}_{j=0}^{\infty},$$

and the upper semicontinuity from the right of ϕ, this inequality implies that

$$r \leq \limsup_{j\to\infty} \phi(d(T^{k_j} x_0, T^{k_j+1} x_0)) \leq \phi(r),$$

a contradiction. Thus $r = 0$ and

$$\lim_{j\to\infty} d(T^{k_j} x_0, T^{k_j+1} x_0) = 0. \tag{2.5}$$

We claim that, in fact,

$$\lim_{i\to\infty} d(T^i x_0, T^{i+1} x_0) = 0.$$

Indeed, let $\varepsilon > 0$. Since T is uniformly continuous on the set

$$\Omega := \{T^i x_0 : i = 1, 2, \dots\}, \tag{2.6}$$

there is

$$\varepsilon_0 \in (0, \varepsilon) \tag{2.7}$$

such that

$$\text{if } x, y \in \Omega,\ i \in \{1, \dots, N_0\},\ d(x,y) \le \varepsilon_0, \text{ then } d(T^i x, T^i y) \le \varepsilon. \tag{2.8}$$

By (2.5), there is a natural number j_0 such that

$$d(T^{k_j} x_0, T^{k_j+1} x_0) \le \varepsilon_0 \quad \text{for all integers } j \ge j_0. \tag{2.9}$$

Let p be an integer such that

$$p \ge k_{j_0} + N_0.$$

Then by (2.2) there is an integer $j \ge j_0$ such that

$$k_j < p \le k_j + N_0. \tag{2.10}$$

By (2.9) and the inequality $j \ge j_0$,

$$d(T^{k_j} x_0, T^{k_j+1} x_0) \le \varepsilon_0.$$

Together with (2.10) and (2.9), this implies that

$$d(T^p x_0, T^{p+1} x_0) \le \varepsilon.$$

Thus this inequality holds for any integer $p \ge k_{j_0} + N_0$ and we conclude that

$$\lim_{p \to \infty} d(T^p x_0, T^{p+1} x_0) = 0, \tag{2.11}$$

as claimed.

Now we show that $\{T^i x_0\}_{i=1}^{\infty}$ is a Cauchy sequence. Let us assume the contrary. Then there exists $\varepsilon > 0$ such that for each natural number p, there exist integers $m_p > n_p \ge p$ such that

$$d(T^{m_p} x_0, T^{n_p} x_0) \ge \varepsilon. \tag{2.12}$$

We may assume without loss of generality that for each natural number p,

$$d(T^i x_0, T^{n_p} x_0) < \varepsilon \quad \text{for all integers } i \text{ satisfying } n_p < i < m_p. \tag{2.13}$$

By (2.12) and (2.13), for any integer $p \ge 1$,

$$\begin{aligned} \varepsilon \le d(T^{m_p} x_0, T^{n_p} x_0) &\le d(T^{m_p} x_0, T^{m_p-1} x_0) + d(T^{m_p-1} x_0, T^{n_p} x_0) \\ &\le d(T^{m_p} x_0, T^{m_p-1} x_0) + \varepsilon. \end{aligned}$$

When combined with (2.11), this implies that

$$\lim_{p \to \infty} d(T^{m_p} x_0, , T^{n_p} x_0) = \varepsilon. \tag{2.14}$$

Let $\delta > 0$. By (2.11), there is an integer $p_0 \ge 1$ such that

$$d(T^{i+1} x_0, T^i x_0) \le \delta (4N_0)^{-1} \quad \text{for all integers } i \ge p_0. \tag{2.15}$$

Let $p \ge p_0$ be an integer. By (2.11), there is $j \in \{1, \dots, N_0\}$ such that

$$d(T^{m_p+j} x_0, T^{n_p+j} x_0) \le \phi(d(T^{m_p} x_0, T^{n_p} x_0)). \tag{2.16}$$

By the inequalities $m_p > n_p \ge p$, (2.15) and (2.16),

$$(2.17)\quad \begin{aligned} &d(T^{m_p}x_0, T^{n_p}x_0) \\ &\le \sum_{i=0}^{j-1} d(T^{m_p+i}x_0, T^{m_p+i+1}x_0) + d(T^{m_p+j}x_0, T^{n_p+j}x_0) \\ &\quad + \sum_{i=0}^{j-1} d(T^{n_p+i}x_0, T^{n_p+i+1}x_0) \\ &\le 2j\delta(4N_0)^{-1} + \phi(d(T^{m_p}x_0, T^{n_p}x_0)) < \delta + \phi(d(T^{m_p}x_0, T^{n_p}x_0)). \end{aligned}$$

By (2.14), (2.17), (2.12), and the upper semicontinuity from the right of ϕ,

$$\varepsilon = \lim_{p\to\infty} d(T^{m_p}x_0, T^{n_p}x_0) \le \delta + \limsup_{p\to\infty} \phi(d(T^{m_p}x_0, T^{n_p}x_0)) \le \delta + \phi(\varepsilon).$$

Since δ is an arbitrary positive number, we conclude that $\varepsilon \le \phi(\varepsilon)$. The contradiction we have reached proves that $\{T^i x_0\}_{i=1}^{\infty}$ is indeed a Cauchy sequence. Set

$$\overline{x} = \lim_{i\to\infty} T^i x_0.$$

Clearly, if T is continuous, then $T\overline{x} = \overline{x}$ and $\overline{x}$ is a unique fixed point of T. Theorem 2.1 is proved. ■

3. Uniform convergence. For each $x \in X$ and $r > 0$, set

$$B(x, r) = \{z \in X : d(x, z) \le r\}.$$

THEOREM 3.1. *Let (X, d) be a complete metric space and let $T : X \to X$ be a Jachymski–Schröder–Stein contraction with respect to the function $\phi : [0, \infty) \to [0, \infty)$. Assume that ϕ is upper semicontinuous, T is uniformly continuous on the set $\{T^i x : i = 1, 2, \dots\}$ for each $x \in X$, and that T is continuous on X. Then there exists a unique fixed point $\overline{x}$ of T such that $T^n x \to \overline{x}$ as $n \to \infty$, uniformly on bounded subsets of X.*

Proof. By Theorem 2.1, T has a unique fixed point $\overline{x}$ and

$$(3.1)\qquad T^n x \to \overline{x} \quad \text{as } n \to \infty \text{ for all } x \in X.$$

Let $r > 0$. We claim that $T^n x \to \overline{x}$ as $n \to \infty$, uniformly on $B(\overline{x}, r)$.

Indeed, let

$$(3.2)\qquad \varepsilon \in (0, r).$$

Since T is continuous, there is

$$(3.3)\qquad \varepsilon_0 \in (0, \varepsilon)$$

such that

$$(3.4)\qquad \text{if } x \in X,\ d(x, \overline{x}) \le \varepsilon_0,\ i \in \{1, \dots, N_0\},\ \text{then } d(T^i x, \overline{x}) \le \varepsilon.$$

Since ϕ is upper semicontinuous, there is

$$\delta \in (0, \varepsilon_0) \tag{3.5}$$

such that

$$\text{if } t \in [\varepsilon_0, r], \text{ then } t - \phi(t) \geq \delta. \tag{3.6}$$

Choose a natural number N_1 such that

$$N_1 \delta > 2r. \tag{3.7}$$

Assume that

$$x \in X, \ d(\overline{x}, x) \leq r. \tag{3.8}$$

We will show that

$$d(\overline{x}, T^i x) \leq \varepsilon \text{ for all integers } i \geq N_0 + N_0 N_1. \tag{3.9}$$

To this end, set $k_0 = 0$. Define by induction an increasing sequence of integers $\{k_i\}_{i=1}^{\infty}$ such that

(3.10)

$$k_{i+1} - k_i \in [1, N_0], \ d(T^{k_i+1}x, \overline{x}) = \min\{d(T^{j+k_i}x, \overline{x}) : j \in \{1, \dots, N_0\}\}.$$

By (1.1) and (3.10), the sequence $\{d(T^{k_i}x, \overline{x})\}_{i=0}^{\infty}$ is decreasing. We claim that $d(T^{k_{N_1}}x, \overline{x}) \leq \varepsilon_0$.

Assume the contrary. Then by (3.8) and (1.1),

$$r \geq d(T^{k_j}x, \overline{x}) > \varepsilon_0, \ j = 0, \dots, N_1. \tag{3.11}$$

By (3.10), (1.1), (3.11) and (3.6), we have for $j = 0, \dots, N_1$,

$$d(T^{k_j}x, \overline{x}) - d(T^{k_j+1}x, \overline{x}) \geq d(T^{k_j}x, \overline{x}) - \phi(d(T^{k_j}x, \overline{x})) \geq \delta. \tag{3.12}$$

Together with (3.8), this implies that

$$r \geq d(T^{k_0}x, \overline{x}) - d(T^{k_{N_1+1}}x, \overline{x}) \geq \delta(N_1 + 1),$$

which contradicts (3.7). The contradiction we have reached and the monotonicity of the sequence $\{d(T^{k_j}x, \overline{x})\}_{j=0}^{\infty}$ show that there is $p \in \{0, 1, \dots, N_1\}$ such that

$$d(T^{k_j}x, \overline{x}) \leq \varepsilon_0 \quad \text{for all integers } j \geq p. \tag{3.13}$$

Assume that i is an integer with $i \geq N_0 + N_0 N_1$. By (3.10), there is an integer $j \geq 0$ such that

$$k_j \leq i < k_{j+1}. \tag{3.14}$$

By (3.10), (3.14) and the choice of p,

$$(j + 1)N_0 > i,$$

so $j + 1 > i/N_0 \geq N_1 + 1$, and hence

$$j > N_1 \geq p. \tag{3.15}$$

By (3.15) and (3.13), $d(T^{k_j}x, \overline{x}) \leq \varepsilon_0$. Together with (3.14), (3.10), (3.3) and (3.4), this inequality implies that

$$d(\overline{x}, T^i x) \leq \varepsilon,$$

as claimed. Theorem 3.1 is proved. ■

Acknowledgments. The first author was partially supported by the Fund for the Promotion of Research at the Technion and by the Technion President's Research Fund. Both authors are grateful to the referee for several insightful comments and suggestions.

References

[1] A. D. Arvanitakis, *A proof of the generalized Banach contraction conjecture*, Proc. Amer. Math. Soc. 131 (2003), 3647–3656.

[2] D. W. Boyd and J. S. W. Wong, *On nonlinear contractions*, ibid. 20 (1969), 458–464.

[3] F. E. Browder, *On the convergence of successive approximations for nonlinear functional equations*, Indag. Math. 30 (1968), 27–35.

[4] D. Butnariu, S. Reich and A. J. Zaslavski, *Asymptotic behavior of inexact orbits for a class of operators in complete metric spaces*, J. Appl. Anal. 13 (2007), 1–11.

[5] K. Goebel, *Concise Course on Fixed Point Theorems*, Yokohama Publ., Yokohama, 2002.

[6] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Dekker, New York and Basel, 1984.

[7] J. R. Jachymski, B. Schröder and J. D. Stein, Jr., *A connection between fixed point theorems and tiling problems*, J. Combin. Theory Ser. A 87 (1999), 273–286.

[8] J. R. Jachymski and J. D. Stein, Jr., *A minimum condition and some related fixed-point theorems*, J. Austral. Math. Soc. Ser. A 66 (1999), 224–243.

[9] J. Merryfield, B. Rothschild and J. D. Stein, Jr., *An application of Ramsey's theorem to the Banach contraction principle*, Proc. Amer. Math. Soc. 130 (2001), 927–933.

[10] J. Merryfield and J. D. Stein, Jr., *A generalization of the Banach contraction principle*, J. Math. Anal. Appl. 273 (2002), 112–120.

[11] J. D. Stein, Jr., *A systematic generalization procedure for fixed-point theorems*, Rocky Mountain J. Math. 30 (2000), 735–754.

Simeon Reich and Alexander J. Zaslavski
Department of Mathematics
The Technion–Israel Institute of Technology
32000 Haifa, Israel
E-mail: sreich@tx.technion.ac.il

Received January 24, 2008 (7643)