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Limit Measures Related to the Conditionally Free Convolution

by

Melanie HINZ and Wojciech MŁOTKOWSKI

Presented by Stanisław KWAPIEŃ

Summary. We describe the limit measures for some class of deformations of the free convolution, introduced by A. D. Krystek and Ł. J. Wojakowski. In particular, we provide a counterexample to a conjecture from their paper.

1. Introduction. The *conditionally free convolution*, defined by Bożejko, Leinert and Speicher [3], is an associative and commutative operation $\boxplus$ on pairs of compactly supported probability measures on $\mathbb{R}$. It is related to the Voiculescu [8, 9] free convolution, namely, if

$$(\mu_1,\nu_1)\boxplus(\mu_2,\nu_2)=(\mu,\nu) \tag{1}$$

then $\nu=\nu_1\boxplus\nu_2$, and if $\mu_1=\nu_1$ and $\mu_1=\nu_2$ then $\mu=\mu_1\boxplus\mu_2$.

Recall that an important tool for studying a probability measure μ on $\mathbb{R}$ is the *Cauchy transform* which is the analytic function $G_\mu:\mathbb{C}_+\to\mathbb{C}$ defined by

$$G_\mu(z):=\int_{\mathbb{R}}\frac{d\mu(t)}{z-t}, \tag{2}$$

where $\mathbb{C}_+:=\{z\in\mathbb{C}:\operatorname{Im} z>0\}$. If μ is compactly supported then $G_\mu(z)$ can be represented as a continued fraction

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$$G_\mu(z) = \cfrac{1}{z-u_0-\cfrac{\alpha_0}{z-u_1-\cfrac{\alpha_1}{z-u_2-\cfrac{\alpha_2}{z-u_3-\cfrac{\alpha_3}{\ddots}}}}}. \tag{3}$$

The *Jacobi parameters* satisfy: $\alpha_k \geq 0$, $u_k \in \mathbb{R}$ and if $\alpha_m = 0$ for some $m \geq 0$ then $\alpha_n = u_n = 0$ for all $n > m$. The coefficient α_0 is called the *variance* of μ and denoted by $V(\mu)$. Let $\mathcal{M}_c$ denote the class of compactly supported probability measures on $\mathbb{R}$. We will need the following two properties, which can be found in Chihara's monograph [4].

PROPOSITION 1.1. *Assume that $\mu \in \mathcal{M}_c$ with the Cauchy transform* (3). *Then*:

(i) *μ is symmetric* (*i.e.* $\mu(A) = \mu(-A)$ *for every Borel subset A of* $\mathbb{R}$) *if and only if $u_k = 0$ for every $k \geq 0$.*
(ii) *The support of μ is contained in the halfline* $[0,\infty)$ *if and only if there exists a sequence $\{\lambda_m\}_{m\geq 0}$ of nonnegative numbers such that for every $m \geq 0$ we have $\alpha_m = \lambda_{2m} \cdot \lambda_{2m+1}$ and $u_m = \lambda_{2m} + \lambda_{2m-1}$, under the convention that $\lambda_{-1} = 0$.* ■

The numbers λ_m are the nonnegative (i.e. *upper*) Jacobi parameters for the symmetric measure μ_{sym} defined by $\int_{\mathbb{R}} f(t^2)\, d\mu_{\mathrm{sym}}(t) = \int_{\mathbb{R}} f(t)\, d\mu(t)$.

To define the conditionally free convolution we define two transforms. For $\mu, \nu \in \mathcal{M}_c$ we define $R_\nu, R_{\mu,\nu}$ as the complex functions which satisfy

$$\frac{1}{G_\nu(z)} = z - R_\nu(G_\nu(z)), \tag{4}$$

$$\frac{1}{G_\mu(z)} = z - R_{\mu,\nu}(G_\nu(z)) \tag{5}$$

(the former is the Voiculescu free transform). For $\mu_1, \nu_1, \mu_2, \nu_2 \in \mathcal{M}_c$ the conditionally free convolution (1) is defined by the equalities

$$R_\nu(z) = R_{\nu_1}(z) + R_{\nu_2}(z), \tag{6}$$

$$R_{\mu,\nu}(z) = R_{\mu_1,\nu_1}(z) + R_{\mu_2,\nu_2}(z). \tag{7}$$

Now, assume that a map $T : \mathcal{M}_c \to \mathcal{M}_c$ satisfies the following condition (*Bożejko property*): if

$$(\mu_1, T\mu_1) \boxplus (\mu_2, T\mu_2) = (\mu, \nu) \tag{8}$$

then $\nu = T\mu$. Defining $\mu_1 \boxplus_T \mu_2 := \mu$ we obtain an associative and commutative operation $\boxplus_T$ on $\mathcal{M}_c$. For example, if $T\mu = \mu$ for every $\mu \in \mathcal{M}_c$ then

$\boxplus_T$ is the Voiculescu free convolution $\boxplus$, and if $T\mu = \delta_0$ for every $\mu \in \mathcal{M}_c$ then $\boxplus_T$ becomes the Boolean convolution $\uplus$.

2. The ϕ-convolution. From now on we fix $\phi \in \mathcal{M}_c$ which is infinitely divisible with respect to $\boxplus$ (examples can be found in [7, 2, 5]). Let

$$G_\phi(z) = \cfrac{1}{z - \beta_0 - \cfrac{\gamma_0}{z - \beta_1 - \cfrac{\gamma_1}{z - \beta_2 - \cfrac{\gamma_2}{z - \beta_3 - \cfrac{\gamma_3}{\ddots}}}}} \tag{9}$$

be its Cauchy transform. Krystek and Wojakowski [6] defined a convolution $\boxplus_\phi$ in the following way. For $\mu \in \mathcal{M}_c$, we put $T\mu := \phi^{\boxplus V(\mu)}$ (the free power of ϕ). Then T has the Bożejko property (Theorem 7 in [6]) and we set $\boxplus_\phi := \boxplus_T$. The authors of [6] found the related limit measures only in the case when ϕ is either the Wigner or the free Poisson measure.

We are going to exhibit the relation between the Jacobi parameters of ϕ and those of the limit measures with respect to $\boxplus_\phi$. In particular we will verify the hypothesis given in [6, Remark 10].

For $\mu \in \mathcal{M}_c$ and $\lambda > 0$ we define *dilation* of μ by $\mathcal{D}_\lambda\mu(A) := \mu(\lambda^{-1}A)$. Denote by $\gamma_m^{(\lambda)}$, $\beta_m^{(\lambda)}$ the Jacobi parameters of the free power $\phi^{\boxplus\lambda}$.

THEOREM 2.1 (The central limit theorem). *Assume that $\mu \in \mathcal{M}_c$ satisfies $\int_{\mathbb{R}} t\,d\mu(t) = 0$ and $\int_{\mathbb{R}} t^2\,d\mu(t) = \lambda$. Then the sequence*

$$\xi_{\lambda,N} := \mathcal{D}_{1/\sqrt{N}}\mu \boxplus_\phi \cdots \boxplus_\phi \mathcal{D}_{1/\sqrt{N}}\mu \tag{10}$$

(N summands) is $*$-weakly convergent to the measure $\xi_\lambda \in \mathcal{M}_c$ such that

$$G_{\xi_\lambda}(z) = \cfrac{1}{z - \cfrac{\lambda}{z - \beta_0^{(\lambda)} - \cfrac{\gamma_0^{(\lambda)}}{z - \beta_1^{(\lambda)} - \cfrac{\gamma_1^{(\lambda)}}{z - \beta_2^{(\lambda)} - \cfrac{\gamma_2^{(\lambda)}}{\ddots}}}}}. \tag{11}$$

In particular, ξ is symmetric if and only if ϕ is symmetric.

Proof. By (a slightly generalized version of) Theorem 8 in [6] and (5) we have $1/G_{\xi_\lambda}(z) = z - \lambda G_{\phi^{\boxplus\lambda}}(z)$, which proves (11). It remains to use Proposition 1.1(i). ■

THEOREM 2.2 (The Poisson limit theorem). *For $\lambda > 0$ the sequence*

$$\varrho_{\lambda,N} := \left(\left(1-\frac{\lambda}{N}\right)\delta_0 + \frac{\lambda}{N}\,\delta_1\right) \boxplus_\phi \cdots \boxplus_\phi \left(\left(1-\frac{\lambda}{N}\right)\delta_0 + \frac{\lambda}{N}\,\delta_1\right) \tag{12}$$

(N summands) is $*$-weakly convergent to the measure ϱ_λ which satisfies

$$G_{\varrho_\lambda}(z) = \cfrac{1}{z-\lambda-\cfrac{\lambda}{z-\beta_0^{(\lambda)}-1-\cfrac{\gamma_0^{(\lambda)}}{z-\beta_1^{(\lambda)}-\cfrac{\gamma_1^{(\lambda)}}{z-\beta_2^{(\lambda)}-\cfrac{\gamma_2^{(\lambda)}}{\ddots}}}}}. \tag{13}$$

Moreover, the support of ϱ_λ is contained in $[0,\infty)$ if and only if the support of ϕ is contained in $[0,\infty)$.

Proof. According to Theorem 9 in [6] and formula (5) we have

$$\frac{1}{G_{\varrho_\lambda}(z)} = z - \frac{\lambda}{1-G_{\phi^{\boxplus\lambda}}(z)} = z-\lambda-\cfrac{\lambda}{\cfrac{1}{G_{\phi^{\boxplus\lambda}}(z)}-1}, \tag{14}$$

which leads to (13).

Assume that $\operatorname{supp}(\phi) \subseteq [0,\infty)$; then also $\operatorname{supp}(\phi^{\boxplus\lambda}) \subseteq [0,\infty)$. Let $\{\lambda_m\}_{m=0}^{\infty}$ be the sequence of upper Jacobi parameters of $(\phi^{\boxplus\lambda})_{\mathrm{sym}}$, according to Proposition 1.1(ii). Then the numbers $\lambda'_0 := \lambda$, $\lambda'_1 := 1$ and $\lambda'_k := \lambda_{k-2}$ for $k \geq 2$ are the upper Jacobi parameters for $(\varrho_\lambda)_{\mathrm{sym}}$. On the other hand, if the support of ϱ_λ is contained in $[0,\infty)$ and if λ'_m are the upper Jacobi parameters of $(\varrho_\lambda)_{\mathrm{sym}}$ then $\lambda'_0 = \lambda$, $\lambda'_1 = 1$ and the numbers λ_{m+2}, $m \geq 0$, are the upper Jacobi parameters of $(\phi^{\boxplus\lambda})_{\mathrm{sym}}$. ■

3. A family of infinitely divisible measures. Here we will show that the limit measures ξ_λ and ϱ_λ are $\boxplus_\phi$-infinitely divisible. More generally, for $\lambda > 0$, $u, v \in \mathbb{R}$, let $\mu = \mu(\lambda,u,v)$ denote a measure such that

$$G_\mu(z) = \cfrac{1}{z-u-\cfrac{\lambda}{z-\beta_0^{(\lambda)}-v-\cfrac{\gamma_0^{(\lambda)}}{z-\beta_1^{(\lambda)}-\cfrac{\gamma_1^{(\lambda)}}{z-\beta_2^{(\lambda)}-\cfrac{\gamma_2^{(\lambda)}}{\ddots}}}}}. \tag{15}$$

Then we have

THEOREM 3.1. *For $\lambda_1, \lambda_2, \lambda > 0$, $u_1, u_2, u, v \in \mathbb{R}$ and $t > 0$ we have*

$$\mu(\lambda_1, u_1, v) \boxplus_\phi \mu(\lambda_2, u_2, v) = \mu(\lambda_1 + \lambda_2, u_1 + u_2, v) \tag{16}$$

and

$$\mu(\lambda, u, v)^{\boxplus_\phi t} = \mu(t\lambda, tu, v). \tag{17}$$

In particular, $\mu(\lambda, u, v)$ is infinitely divisible with respect to $\boxplus_\phi$.

Proof. For $\mu = \mu(\lambda, u, v)$ we have $V(\mu) = \lambda$ and

$$\frac{1}{G_\mu(z)} = z - u - \frac{\lambda}{\dfrac{1}{G_{\phi^{\boxplus\lambda}}(z)} - v} = z - u - \frac{\lambda G_{\phi^{\boxplus\lambda}}(z)}{1 - vG_{\phi^{\boxplus\lambda}}(z)}, \tag{18}$$

hence, by (5),

$$R_{\mu,\phi^{\boxplus\lambda}}(z) = u + \frac{\lambda z}{1 - vz}, \tag{19}$$

which leads to the formulas (16) and (17). ∎

4. An example. In [6] the authors conjecture that for every $\boxplus$-infinitely divisible compactly supported probability measure ϕ the limit measures have eventually constant Jacobi parameters. In view of Theorems 2.1 and 2.2 this is equivalent to the statement that every $\boxplus$-infinitely divisible compactly supported probability measure ϕ has eventually constant Jacobi parameters. The aim of this section is to provide a counterexample to this statement, thus disproving the conjecture. We are indebted to Professor Nobuaki Obata for suggesting the measure that will serve here as the counterexample.

Fix $\lambda > 0$ and let ϱ_λ denote the free Poisson (i.e. the Marchenko–Pastur) distribution with parameter λ. It is know that ϱ_λ is $\boxplus$-infinitely divisible and has compact support contained in $[0, \infty)$. Take the reflection $\widehat{\varrho}_\lambda$, i.e. $\widehat{\varrho}_\lambda(E) := \varrho_\lambda(-E)$. Then we have

$$R_{\varrho_\lambda}(w) = \frac{\lambda}{1 - w} \quad \text{and} \quad R_{\widehat{\varrho}_\lambda}(w) = -R_{\varrho_\lambda}(-w) = \frac{-\lambda}{1 + w}.$$

Now we define $\phi := \varrho_\lambda \boxplus \widehat{\varrho}_\lambda$. Then ϕ is $\boxplus$-infinitely divisible and compactly supported. We also have

$$R_\phi(w) = \frac{2\lambda w}{1 - w^2}. \tag{20}$$

By (4) we have

$$\frac{1}{G_\phi(z)} = z - R_\phi(G_\phi(z)) = z - \frac{2\lambda G_\phi(z)}{1 - G_\phi(z)^2}, \tag{21}$$

which leads to the equation

$$zG_\phi(z)^3 + (2\lambda - 1)G_\phi(z)^2 - zG_\phi(z) + 1 = 0. \tag{22}$$

We are going to prove

THEOREM 4.1. *The Jacobi parameters of ϕ are not eventually constant.*

Proof. Suppose that the Jacobi parameters of ϕ are eventually constant. Then we have

$$G_\phi(z) = \cfrac{1}{z-\beta_0-\cfrac{\gamma_0}{z-\beta_1-\cfrac{\gamma_1}{\ddots\,\cfrac{}{z-\beta_n-\gamma_n G_0(z)}}}} \tag{23}$$

for some $n \geq 0$, $\beta_k \in \mathbb{R}$, $\gamma_k > 0$, where

$$G_0(z) = \cfrac{1}{z-u-\cfrac{a}{z-u-\cfrac{a}{\ddots}}} \tag{24}$$

for some $u \in \mathbb{R}$, $a \geq 0$. Then $G_0(z)$ satisfies the equation

$$aG_0(z)^2 - (z-u)G_0(z) + 1 = 0. \tag{25}$$

By induction on n one can show that

$$G_\phi(z) = \frac{A(z) + B(z)G_0(z)}{C(z) + D(z)G_0(z)}, \tag{26}$$

where $A(z), B(z), C(z), D(z)$ are polynomials of degree $n, n-1, n+1, n$, respectively, with real coefficients. From (26) we have

$$G_0(z) = \frac{C(z)G_\phi(z) - A(z)}{B(z) - D(z)G_\phi(z)}. \tag{27}$$

Combining (25) and (27) we get

$$G_\phi(z)^2 = R(z)G_\phi(z) + S(z), \tag{28}$$

where $R(z)$ and $S(z)$ are rational functions with real coefficients. Substituting this three times to (22) we find that $G_\phi(z)$ is a rational function with real coefficients. That, in turn, implies that ϕ has finite support. In view of Theorem 3.1 in [1], the only $\boxplus$-infinitely divisible measures with finite support are the one-point measures δ_u, $u \in \mathbb{R}$; but then

$$G_{\delta_u}(z) = \frac{1}{z-u} \quad \text{and} \quad R_{\delta_u}(w) = u,$$

contrary to (20). ∎

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Melanie Hinz
Institut für Mathematik und Informatik
Ernst-Moritz-Arndt Universität Greifswald
Jahnstrasse 15a
D-17487 Greifswald, Germany
and
Institute of Mathematics
Wrocław University
Pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: hinz@math.uni.wroc.pl

Wojciech Młotkowski
Institute of Mathematics
Wrocław University
Pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: mlotkow@math.uni.wroc.pl

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