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Tame Automorphisms of $\mathbb{C}^3$ with Multidegree of the Form (p_1, p_2, d_3)

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Summary. Let $d_3 \geq p_2 > p_1 \geq 3$ be integers such that p_1, p_2 are prime numbers. We show that the sequence (p_1, p_2, d_3) is the multidegree of some tame automorphism of $\mathbb{C}^3$ if and only if $d_3 \in p_1\mathbb{N} + p_2\mathbb{N}$, i.e. if and only if d_3 is a linear combination of p_1 and p_2 with coefficients in $\mathbb{N}$.

1. Introduction. Let $F = (F_1, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be any polynomial mapping. Its *multidegree*, denoted $\operatorname{mdeg} F$, is the sequence of positive integers $(\deg F_1, \ldots, \deg F_n)$. In dimension 2 there is a complete characterization of the sequences (d_1, d_2) such that there is a polynomial automorphism $F : \mathbb{C}^2 \to \mathbb{C}^2$ with $\operatorname{mdeg} F = (d_1, d_2)$. This characterization is a consequence of the Jung [2] and van der Kulk [4] theorem. Moreover in [3] it was proven, among other things, that there is no tame automorphism of $\mathbb{C}^3$ with multidegree $(3, 4, 5), (3, 5, 7), (4, 5, 7)$ or $(4, 5, 11)$.

Recall that a *tame automorphism* is, by definition, a composition of linear automorphisms and triangular automorphisms, where a *triangular automorphism* is a mapping of the form

$$T : \mathbb{C}^n \ni \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \mapsto \begin{Bmatrix} x_1 \\ x_2 + f_2(x_1) \\ \vdots \\ x_n + f_n(x_1, \ldots, x_{n-1}) \end{Bmatrix} \in \mathbb{C}^n.$$

We will denote by $\operatorname{Tame}(\mathbb{C}^n)$ the group of all tame automorphisms of $\mathbb{C}^n$, and by mdeg the mapping from the set of all polynomial endomorphisms

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of $\mathbb{C}^n$ into $\mathbb{N}^n$. Using this notation, the above mentioned facts can be written as follows: $(3,4,5),(3,5,7),(4,5,7),(4,5,11) \notin \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$. In [7] it was proven that for all d_1, d_2 there are only finitely many d_3 such that $(d_1,d_2,d_3) \notin \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$.

In this paper we make a further progress in the investigation of the set $\operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$. Namely we show the following theorem.

THEOREM 1.1. *Let $d_3 \geq p_2 > p_1 \geq 3$ be positive integers. If p_1 and p_2 are prime numbers, then $(p_1,p_2,d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ if and only if $d_3 \in p_1\mathbb{N} + p_2\mathbb{N}$, i.e. if and only if d_3 is a linear combination of p_1 and p_2 with coefficients in $\mathbb{N}$.*

Notice that for all permutations σ of the set $\{1,2,3\}$, $(d_1,d_2,d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ if and only if $(d_{\sigma(1)},d_{\sigma(2)},d_{\sigma(3)}) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$. Since also, $(d_1,d_2,d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ if $d_1 = d_2$ (by Proposition 2.2 below), and $(2,d_2,d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ for all $d_3 \geq d_2 \geq 2$ ([3, Corollary 2.3]), the assumption $d_3 \geq p_2 > p_1 \geq 3$ is not restrictive.

2. Proof of the theorem. First, we recall one classical result (due to Sylvester) from number theory, concerning the so-called coin problem or Frobenius problem [1].

THEOREM 2.1. *If a, b are positive integers such that $\gcd(a,b) = 1$, then for every integer $k \geq (a-1)(b-1)$ there are $k_1, k_2 \in \mathbb{N}$ such that*

$$k = k_1 a + k_2 b.$$

Moreover $(a-1)(b-1) - 1 \notin a\mathbb{N} + b\mathbb{N}$.

In the proof we will also use the following proposition.

PRPOSITION 2.2 ([3, Proposition 2.2]). *If for a sequence of integers $1 \leq d_1 \leq \cdots \leq d_n$ there is $i \in \{1,\ldots,n\}$ such that*

$$d_i = \sum_{j=1}^{i-1} k_j d_j \quad \text{with } k_j \in \mathbb{N},$$

then there exists a tame automorphism F of $\mathbb{C}^n$ with $\operatorname{mdeg} F = (d_1,\ldots,d_n)$.

By the above proposition, in order to prove Theorem 1.1, it is enough to show that if $d_3 \notin p_1\mathbb{N} + p_2\mathbb{N}$, then $(p_1,p_2,d_3) \notin \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$.

In the proof of the above implication we will use some results and notions from the papers of Shestakov and Umirbaev [5, 6].

The first one is the following

DEFINITION 2.1 ([5, Definition 1]). A pair $f, g \in k[X_1,\ldots,X_n]$ is called **-reduced* if

(i) f, g are algebraically independent;
(ii) $\overline{f}, \overline{g}$ are algebraically dependent, where $\overline{h}$ denotes the highest homogeneous part of h;
(iii) $\overline{f} \notin k[\overline{g}]$ and $\overline{g} \notin k[\overline{f}]$.

Definition 2.2 ([5, Definition 1]). Let $f, g \in k[X_1, \ldots, X_n]$ be a *-reduced pair with $\deg f < \deg g$. Put $p = \deg f / \gcd(\deg f, \deg g)$. Then the pair f, g is called *p-reduced*.

Theorem 2.3 ([5, Theorem 2]). *Let $f, g \in k[X_1, \ldots, X_n]$ be a $p-$reduced pair, and let $G(x, y) \in k[x, y]$ with $\deg_y G(x, y) = pq + r$, $0 \leq r < p$. Then*

$$\deg G(f, g) \geq q(p \deg g - \deg g - \deg f + \deg\,[f, g]) + r \deg g.$$

In the above theorem $[f, g]$ means the Poisson bracket of f and g; for us it is only important that

$$\deg\,[f, g] = 2 + \max_{1 \leq i < j \leq n} \deg\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i}\right)$$

if f, g are algebraically independent, and $[f, g] = 0$ if f, g are algebraically dependent.

Notice also that the estimate from Theorem 2.3 is true even if the condition (ii) of Definition 2.1 is not satisfied. Indeed, if $G(x, y) = \sum_{i,j} a_{i,j} x^i y^j$, then by the algebraic independence of $\overline{f}$ and $\overline{g}$ we have

$$\begin{aligned}\deg G(f, g) &= \max_{i,j} \deg(a_{i,j} f^i g^j) \geq \deg_y G(x, y) \cdot \deg g = (qp + r) \deg g \\ &\geq q(p \deg g - \deg f - \deg g + \deg\,[f, g]) + r \deg g.\end{aligned}$$

The last inequality is a consequence of the fact that $\deg\,[f, g] \leq \deg f + \deg g$.

We will also use the following theorem.

Theorem 2.4 ([5, Theorem 3]). *Let $F = (F_1, F_2, F_3)$ be a tame automorphism of $\mathbb{C}^3$. If $\deg F_1 + \deg F_2 + \deg F_3 > 3$ (in other words, if F is not a linear automorphism), then F admits either an elementary reduction or a reduction of types I–IV* (see [5, Definitions 2–4]).

Let us also recall that an automorphism $F = (F_1, F_2, F_3)$ admits an *elementary reduction* if there exists a polynomial $g \in \mathbb{C}[x, y]$ and a permutation σ of $\{1, 2, 3\}$ such that $\deg(F_{\sigma(1)} - g(F_{\sigma(2)}, F_{\sigma(3)})) < \deg F_{\sigma(1)}$.

Proof of Theorem 1.1. Assume that $F = (F_1, F_2, F_3)$ is an automorphism of $\mathbb{C}^3$ such that $\operatorname{mdeg} F = (p_1, p_2, d_3)$. Assume also that $d_3 \notin p_1\mathbb{N} + p_2\mathbb{N}$. By Theorem 2.1 we have

$$d_3 < (p_1 - 1)(p_2 - 1). \tag{2.1}$$

First of all we show that this hypothetical automorphism F does not admit reductions of types I–IV.

By the definitions of those reductions (see [5, Definitions 2–4]), if $F = (F_1, F_2, F_3)$ admits such a reduction, then $2 \mid \deg F_i$ for some $i \in \{1,2,3\}$. Thus if d_3 is odd, then F does not admit a reduction of types I–IV. Assume that $d_3 = 2n$ for some positive integer n.

If F admits a reduction of type I or II, then by the definition (see [5, Definitions 2 and 3]) we have $p_1 = sn$ or $p_2 = sn$ for some odd $s \geq 3$. Since $p_1, p_2 \leq d_3 = 2n < sn$, we obtain a contradiction.

If F admits a reduction of type III or IV, then by the definition (see [5, Definition 4]) we have either

$$n < p_1 \leq \tfrac{3}{2}n, \qquad p_2 = 3n,$$

or

$$p_1 = \tfrac{3}{2}n, \qquad \tfrac{5}{2}n < p_2 \leq 3n.$$

Since $p_1, p_2 \leq d_3 = 2n < \frac{5}{2}n, 3n$, we obtain a contradiction. Thus we have proved that our hypothetical automorphism F does not admit a reduction of types I–IV.

Now we will show that it also does not admit an elementary reduction.

Assume, to the contrary, that

$$(F_1, F_2, F_3 - g(F_1, F_2)),$$

where $g \in \mathbb{C}[x,y]$, is an elementary reduction of (F_1, F_2, F_3). Then we have $\deg g(F_1, F_2) = \deg F_3 = d_3$. But, by Theorem 2.3,

$$\deg g(F_1, F_2) \geq q(p_1 p_2 - p_1 - p_2 + \deg\,[F_1, F_2]) + r p_2,$$

where $\deg_y g(x,y) = q p_1 + r$ with $0 \leq r < p_1$. Since F_1, F_2 are algebraically independent, $\deg\,[F_1, F_2] \geq 2$ and so

$$p_1 p_2 - p_1 - p_2 + \deg\,[F_1, F_2] \geq p_1 p_2 - p_1 - p_2 + 2 > (p_1 - 1)(p_2 - 1).$$

This and (2.1) imply that $q = 0$, and that

$$g(x,y) = \sum_{i=0}^{p_1 - 1} g_i(x) y^i.$$

Since $\operatorname{lcm}(p_1, p_2) = p_1 p_2$, the sets

$$p_1\mathbb{N},\ p_2 + p_1\mathbb{N},\ \ldots,\ (p_1 - 1)p_2 + p_1\mathbb{N}$$

are pairwise disjoint. This yields

$$\deg\Big(\sum_{i=0}^{p_1 - 1} g_i(F_1) F_2^i\Big) = \max_{i = 0, \ldots, p_1 - 1} (\deg F_1 \deg g_i + i \deg F_2).$$

Since also

$$d_3 \notin \bigcup_{r=0}^{p_1 - 1} (r p_2 + p_1\mathbb{N})$$

(because $d_3 \notin p_1\mathbb{N} + p_2\mathbb{N}$), it is easy to see that

$$\deg\Big(\sum_{i=0}^{p_1-1} g_i(F_1)F_2^i\Big) = d_3$$

is impossible.

Now, assume that

$$(F_1, F_2 - g(F_1, F_3), F_3),$$

where $g \in \mathbb{C}[x, y]$, is an elementary reduction of (F_1, F_2, F_3). Since $d_3 \notin p_1\mathbb{N} + p_2\mathbb{N}$, we have $p_1 \nmid d_3$ and $\gcd(p_1, d_3) = 1$. This means, by Theorem 2.3, that

$$\deg g(F_1, F_3) \geq q(p_1 d_3 - d_3 - p_1 + \deg\,[F_1, F_3]) + r d_3,$$

where $\deg_y g(x, y) = qp_1 + r$ with $0 \leq r < p_1$. Since $p_1 d_3 - d_3 - p_1 + \deg\,[F_1, F_3] \geq p_1 d_3 - 2d_3 \geq d_3 > p_2$ and since we want to have $\deg g(F_1, F_3) = p_2$, it follows that $q = r = 0$. This means that $g(x, y) = g(x)$. But since $p_2 \notin p_1\mathbb{N}$, the equality $\deg g(F_1, F_3) = \deg g(F_1) = p_2$ is impossible.

Finally, if we assume that $(F_1 - g(F_2, F_3), F_2, F_3)$ is an elementary reduction of (F_1, F_2, F_3), then in the same way as in the previous case we obtain a contradiction. ■

3. Some consequences

THEOREM 3.1. *Let $p_2 > 3$ be a prime number and $d_3 \geq p_2$ be an integer. Then $(3, p_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ if and only if $d_3 \notin \{2p_2 - 3k \mid k = 1, \ldots, [p_2/3]\}$.*

Proof. Since $p_2 > 3$ is a prime number, we have $p_2 \equiv r \pmod 3$ for some $r \in \{1, 2\}$. It is easy to see that if $d_3 \geq p_2$ and either $d_3 \equiv 0 \pmod 3$ or $d_3 \equiv r \pmod 3$, then $d_3 \in 3\mathbb{N} + p_2\mathbb{N}$. Thus, by Theorem 2.1,

$$2(p_2 - 1) - 1 \neq 0, r \pmod 3.$$

Take any d_3 such that $p_2 \leq d_3 \leq 2p_2 - 3$ and $d_3 \neq 0, r \pmod 3$. Since $d_3 \leq 2p_2 - 3$ and $d_3 \equiv 2p_2 - 3 \pmod 3$, one can see that $d_3 \notin 3\mathbb{N} + p_2\mathbb{N}$, because otherwise we would have $2p_2 - 3 \in 3\mathbb{N} + p_2\mathbb{N}$, contrary to Theorem 2.1. Thus

$$\begin{aligned}
&\{d_3 \in \mathbb{N} \mid d_3 \geq p_2, d_3 \notin 3\mathbb{N} + p_2\mathbb{N}\}\\
&\quad = \{d_3 \in \mathbb{N} \mid p_2 \leq d_3 \leq 2p_2 - 3, d_3 \equiv 2p_2 - 3 \pmod 3\}\\
&\quad = \{2p_2 - 3k \mid k = 1, \ldots, [p_2/3]\}. \ ■
\end{aligned}$$

One can also notice the following easy but probably amusing results (of course one can easily write down more statements like these).

Theorem 3.2. (a) *If* $d_3 \geq 7$, *then* $(5, 7, d_3) \in \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$ *if and only if*

$$d_3 \neq 8, 9, 11, 13, 16, 18, 23.$$

(b) *If* $d_3 \geq 11$, *then* $(5, 11, d_3) \in \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$ *if and only if*

$$d_3 \neq 12, 13, 14, 17, 18, 19, 23, 24, 28, 29, 34, 39.$$

(c) *If* $d_3 \geq 13$, *then* $(5, 13, d_3) \in \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$ *if and only if*

$$d_3 \neq 14, 16, 17, 19, 21, 22, 24, 27, 29, 32, 34, 37, 42, 47.$$

(d) *If* $d_3 \geq 11$, *then* $(7, 11, d_3) \in \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$ *if and only if*

$$d_3 \neq 12, 13, 15, 16, 17, 19, 20, 23, 24, 26, 27, 30, 31, 34, 37, 38, 41, 45, 48, 52, 59.$$

Proof. This is a consequence of Theorems 2.1 and 1.1. For example to prove (a), by Theorems 2.1 and 1.1 we only have to check which of the numbers $7, 8, \ldots, 23 = (5-1)(7-1)-1$ are elements of the set $5\mathbb{N} + 7\mathbb{N}$. ∎

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