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GROUP THEORY AND GENERALIZATIONS

Finite Groups with Weakly s-Permutably Embedded and Weakly s-Supplemented Subgroups

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Summary. Suppose G is a finite group and H is a subgroup of G. H is called weakly s-permutably embedded in G if there are a subnormal subgroup T of G and an s-permutably embedded subgroup H_{se} of G contained in H such that $G = HT$ and $H \cap T \leq H_{se}$; H is called weakly s-supplemented in G if there is a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-permutable in G. We investigate the influence of the existence of s-permutably embedded and weakly s-supplemented subgroups on the structure of finite groups. Some recent results are generalized.

1. Introduction. All groups considered in this paper are finite. A subgroup H of a group G is said to be *s-permutable* in G if H permutes with all Sylow subgroups of G, i.e., $HS = SH$ for any Sylow subgroup S of G. This concept was introduced by Kegel in [K]. Following Ballester-Bolinches and Pedraza-Aguilera [BP], we call H *s-permutably embedded* in G if for each prime p dividing $|H|$, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-permutable subgroup of G. As a generalization of the above notions, Y. Li, S. Qiao and Y. Wang [LQW] introduce a new subgroup embedding property: A subgroup H of a group G is called *weakly s-permutably embedded* in G if there are a subnormal subgroup T of G and an s-permutably embedded subgroup H_{se} of G contained in H such that $G = HT$ and $H \cap T \leq H_{se}$. As another generalization of s-permutable subgroups, Skiba [S] introduced the following concept: A subgroup H of a group G is called *weakly s-supplemented* in G if there is a subgroup T of G such

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that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-permutable in G. In fact, this concept is also a generalization of c-supplemented subgroups given in [W]. Skiba proposed in [S] two open questions related to weakly s-supplemented subgroups. In this paper, we prove some theorems which show that in most cases (for maximal and minimal subgroups) Question 6.4 in [S] has a positive solution. There are examples where weakly s-supplemented subgroups are not weakly s-permutably embedded, and in general the converse is also false. The aim of this article is to unify and improve some earlier results using weakly s-permutably embedded and weakly s-supplemented subgroups.

2. Preliminaries

LEMMA 2.1 ([LQW, Lemma 2.5]). *Let H be a weakly s-permutably embedded subgroup of a group G.*

(i) *If $H \leq L \leq G$, then H is weakly s-permutably embedded in L.*
(ii) *If $N \lhd G$ and $N \leq H \leq G$, then H/N is weakly s-permutably embedded in G/N.*
(iii) *If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is weakly s-permutably embedded in G/N.*

LEMMA 2.2 ([S, Lemma 2.10]). *Let H be a weakly s-supplemented subgroup of a group G.*

(i) *If $H \leq L \leq G$, then H is weakly s-supplemented in L.*
(ii) *If $N \lhd G$ and $N \leq H \leq G$, then H/N is weakly s-supplemented in G/N.*
(iii) *If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is weakly s-supplemented in G/N.*

LEMMA 2.3 ([LWW, Lemma 2.3]). *Suppose that H is s-permutable in G, and P is a Sylow p-subgroup of H, where p is a prime. If $H_G = 1$, then P is s-permutable in G.*

LEMMA 2.4 ([Sc, Lemma A]). *If P is a s-permutable p-subgroup of G for some prime p, then $N_G(P) \geq O^p(G)$.*

LEMMA 2.5 ([LWW, Lemma 2.4]). *Suppose P is a p-subgroup of G contained in $O_p(G)$. If P is s-permutably embedded in G, then P is s-permutable in G.*

LEMMA 2.6 ([DH, A, 1.2]). *Let U, V, and W be subgroups of a group G. Then the following statements are equivalent:*

(i) $U \cap VW = (U \cap V)(U \cap W)$;
(ii) $UV \cap UW = U(V \cap W)$.

Lemma 2.7 ([LG, Lemma 2.6]). *Let H be a solvable normal subgroup of a group G ($H \neq 1$). If no minimal normal subgroup of G which is contained in H is contained in $\Phi(G)$, then the Fitting subgroup $F(H)$ of H is the direct product of minimal normal subgroups of G which are contained in H.*

Lemma 2.8 ([S, Lemma 2.16]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups. Suppose that G is a group with a normal subgroup N such that $G/N \in \mathcal{F}$. If N is cyclic, then $G \in \mathcal{F}$.*

Lemma 2.9 ([LL, Lemma 2.3]). *Let G be a group and p a prime dividing $|G|$ with $(|G|, p-1) = 1$. If G has cyclic Sylow p-subgroups, then G is p-nilpotent.*

3. Main results

Theorem 3.1. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is either weakly s-permutably embedded or weakly s-supplemented in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

(1) *G is not a non-abelian simple group.*

Suppose G is simple. Let P_1 be a maximal subgroup of P. If P_1 is weakly s-permutably embedded in G, then there are a subnormal subgroup T of G and an s-permutably embedded subgroup $(P_1)_{se}$ of G contained in P_1 such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{se}$. Since G is simple, we have $T = G$ and $P_1 = (P_1)_{se}$ is s-permutably embedded in G. Thus there is an s-permutable subgroup K of G such that P_1 is a Sylow p-subgroup of K. Obviously, $K_G = 1$. By Lemma 2.3, P_1 is s-permutable in G. Therefore $N_G(P_1) \geq O^p(G) = G$ by Lemma 2.4. It follows that $P_1 \lhd G$, a contradiction. Now we may suppose that every maximal subgroup P_1 of P is weakly s-supplemented in G. Then there is a subgroup T of G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{sG} \leq O_p(G) = 1$. By [GS2, Theorem 2.2], we have the same contradiction.

(2) *G has a unique minimal normal subgroup N such that G/N is p-nilpotent. Moreover $\Phi(G) = 1$.*

Let N be a minimal normal subgroup of G. We will show G/N satisfies the hypothesis of the theorem. Since P is a Sylow p-subgroup of G, PN/N is a Sylow p-subgroup of G/N. Let M_1/N be a maximal subgroup of PN/N. Then $M_1 = N(M_1 \cap P)$. Let $P_1 = M_1 \cap P$. It follows that $P_1 \cap N = M_1 \cap P \cap N = P \cap N$ is a Sylow p-subgroup of N. Since

$$p = |PN/N : M_1/N| = |PN : (M_1 \cap P)N| = |P : M_1 \cap P| = |P : P_1|,$$

P_1 is a maximal subgroup of P. If P_1 is weakly s-supplemented in G, then there is a subgroup T of G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{sG}$. Thus $G/N = M_1/N \cdot TN/N = P_1N/N \cdot TN/N$. Since $(|N : P_1 \cap N|, |N : T \cap N|) = 1$, we have $(P_1 \cap N)(T \cap N) = N = N \cap G = N \cap P_1T$. By Lemma 2.6, $(P_1N) \cap (TN) = (P_1 \cap T)N$. It follows that

$$(P_1N/N) \cap (TN/N) = (P_1 \cap T)N/N \leq (P_1)_{sG}N/N \leq (P_1N/N)_{sG}.$$

Hence M_1/N is weakly s-supplemented in G/N. If P_1 is weakly s-permutably embedded in G, then we can prove M_1/N is weakly s-permutably embedded in G/N too. Therefore, G/N satisfies the hypothesis of the theorem. The choice of G implies that G/N is p-nilpotent. The uniqueness of N and $\Phi(G) = 1$ are obvious.

(3) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $N \leq O_{p'}(G)$ by Step (2). Since

$$G/O_{p'}(G) \cong (G/N)/(O_{p'}(G)/N)$$

is p-nilpotent, it follows that G is p-nilpotent, a contradiction.

(4) $O_p(G) = 1$.

If $O_p(G) \neq 1$, Step (2) yields $N \leq O_p(G)$ and $\Phi(O_p(G)) \leq \Phi(G) = 1$. Therefore, G has a maximal subgroup M such that $G = MN$ and $G/N \cong M$ is p-nilpotent. Since $O_p(G) \cap M$ is normalized by N and M, hence by G, the uniqueness of N yields $N = O_p(G)$. Clearly, $P = N(P \cap M)$. Since $P \cap M < P$, there exists a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. Then $P = NP_1$. By the hypothesis, P_1 is either weakly s-permutably embedded or weakly s-supplemented in G. If P_1 is weakly s-permutably embedded in G, there are a subnormal subgroup T of G and an s-permutably embedded subgroup $(P_1)_{se}$ of G contained in P_1 such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{se}$. Thus there is an s-permutable subgroup K of G such that $(P_1)_{se}$ is a Sylow p-subgroup of K. If $K_G \neq 1$, then $N \leq K_G \leq K$. It follows that $N \leq (P_1)_{se} \leq P_1$, and so $P = N(P \cap M) = NP_1 = P_1$, a contradiction. If $K_G = 1$, by Lemma 2.3, $(P_1)_{se}$ is s-permutable in G. Then $(P_1)_{se} \leq O_p(G) = N \leq O^p(G)$ since N is the unique minimal normal subgroup of G. Since $|G : T|$ is a power of p, $O^p(G) \leq T$. Hence $P_1 \cap T \leq (P_1)_{se} \leq O^p(G) \cap P_1 \leq T \cap P_1$, and so $P_1 \cap T = (P_1)_{se} = O^p(G) \cap P_1$. Consequently, $G = PO^p(G)$ implies that $(P_1)_{se} \lhd G$. By the minimality of N, we have $(P_1)_{se} = N$ or $(P_1)_{se} = 1$. If $(P_1)_{se} = N$, then $N \leq P_1$ and $P = NP_1 = P_1$, a contradiction. Thus $P_1 \cap T = (P_1)_{se} = 1$, and so $|T|_p = p$. Hence T is p-nilpotent by Lemma 2.9. Let $T_{p'}$ be the normal p-complement of T. Then $T_{p'}$ is a normal Hall p'-subgroup of G since $T \lhd\lhd G$, a contradiction. Thus we may assume P_1 is weakly s-supplemented in G. Then there is a subgroup T of G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{sG}$. From

Lemma 2.4 we have $O^p(G) \leq N_G((P_1)_{sG})$. Since $(P_1)_{sG}$ is subnormal in G, we have $P_1 \cap T \leq (P_1)_{sG} \leq O_p(G) = N$. Hence

$$((P_1)_{sG})^G = ((P_1)_{sG})^{O^p(G)P} = ((P_1)_{sG})^P \leq P_1.$$

Therefore $(P_1)_{sG} = 1$, which implies $P_1 \cap T = 1$ and so $|T|_p = p$. Hence T is p-nilpotent. Let $T_{p'}$ be the normal p-complement of T. Since M is p-nilpotent, we may suppose M has a normal Hall p'-subgroup $M_{p'}$ and $M \leq N_G(M_{p'}) \leq G$. The maximality of M implies that $M = N_G(M_{p'})$ or $N_G(M_{p'}) = G$. If the latter holds, then $M_{p'} \lhd G$, $M_{p'}$ is actually the normal p-complement of G, which contradicts the choice of G. Hence we must have $M = N_G(M_{p'})$. By applying a deep result of Gross [Gr, Main Theorem] and Feit–Thompson's theorem, there exists $g \in G$ such that $T_{p'}^g = M_{p'}$. Hence $T^g \leq N_G(T_{p'}^g) = N_G(M_{p'}) = M$. However, $T_{p'}$ is normalized by T, so g can be considered as an element of P_1. Thus $G = P_1T^g = P_1M$ and $P = P_1(P \cap M) = P_1$, a contradiction.

(5) *G has Hall p'-subgroups and any two Hall p'-subgroups of G are conjugate in G.*

If every maximal subgroup of P is weakly s-permutably embedded in G, then G is p-nilpotent by [LQW, Theorem 4.7], a contradiction. Thus there is a maximal subgroup P_1 of P such that P_1 is weakly s-supplemented in G. Then there exists a subgroup T of G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{sG} \leq O_p(G) = 1$. Then T is p-nilpotent and so T has a normal p-complement $T_{p'}$. Obviously, $T_{p'}$ is also a Hall p'-subgroup of G. A new application of the result of Gross [Gr, Main Theorem] and Feit–Thompson's theorem shows that any two Hall p'-subgroups of G are conjugate in G.

(6) *The final contradiction.*

If $NP < G$, then NP satisfies the hypothesis of the theorem. The choice of G implies that NP is p-nilpotent. Let $N_{p'}$ be the normal p-complement of N. It is clear that $N_{p'} \lhd G$, so that $N_{p'} = 1$ by Step (3) and N is a non-trivial p-group, contrary to Step (4). Therefore we must have $G = NP$. By Step (5), G has Hall p'-subgroups. Then we may suppose that N has a Hall p'-subgroup $N_{p'}$. By Frattini's argument, $G = NN_G(N_{p'}) = (P \cap N)N_{p'}N_G(N_{p'}) = (P \cap N)N_G(N_{p'})$ and so $P = P \cap G = P \cap (P \cap N)N_G(N_{p'}) = (P \cap N)(P \cap N_G(N_{p'}))$. Since $N_G(N_{p'}) < G$, $P \cap N_G(N_{p'}) < P$. We take a maximal subgroup P_1 of P such that $P \cap N_G(N_{p'}) \leq P_1$. Then $P = (P \cap N)P_1$. By the hypothesis, P_1 is either weakly s-permutably embedded or weakly s-supplemented in G. If P_1 is weakly s-permutably embedded in G, there are a subnormal subgroup T of G and an s-permutably embedded subgroup $(P_1)_{se}$ of G contained in P_1 such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{se}$. So there is an s-permutable subgroup K of G such that

$(P_1)_{se}$ is a Sylow p-subgroup of K. If $K_G \neq 1$, then $N \leq K_G \leq K$ and so $(P_1)_{se} \cap N$ is a Sylow p-subgroup of N. We know $(P_1)_{se} \cap N \leq P_1 \cap N \leq P \cap N$ and $P \cap N$ is a Sylow p-subgroup of N, so $(P_1)_{se} \cap N = P_1 \cap N = P \cap N$. Consequently, $P = (N \cap P)P_1 = (P_1 \cap N)P_1 = P_1$, a contradiction. Therefore $K_G = 1$. By Lemma 2.3, $(P_1)_{se}$ is s-permutable in G and so $(P_1)_{se} \lhd G$. Hence $P_1 \cap T \leq (P_1)_{se} \leq O_p(G) = 1$. Since $|T|_p = p$, T is p-nilpotent and so G is p-nilpotent, a contradiction. Therefore we may suppose P_1 is weakly s-supplemented in G. Then there is a subgroup T of G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{sG} \leq O_p(G) = 1$. Since $|T|_p = p$, T is p-nilpotent. Let $T_{p'}$ be the normal p-complement of T. Then $T_{p'}$ is a Hall p'-subgroup of G. By Step (5), $T_{p'}$ and $N_{p'}$ are conjugate in G. Since $T_{p'}$ is normalized by T, there exists $g \in P_1$ such that $T_{p'}^g = N_{p'}$. Hence $G = (P_1T)^g = P_1T^g = P_1N_G(T_{p'}^g) = P_1N_G(N_{p'})$ and $P = P \cap G = P \cap P_1N_G(N_{p'}) = P_1(P \cap N_G(N_{p'})) \leq P_1$, a contradiction. ∎

Corollary 3.2. *Let p be the smallest prime dividing the order of a group G, and H a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is either weakly s-permutably embedded or weakly s-supplemented in G, then G is p-nilpotent.*

Proof. By Lemmas 2.1 and 2.2, every maximal subgroup of P is either weakly s-permutably embedded or weakly s-supplemented in H. By Theorem 3.1, H is p-nilpotent. Now, let $H_{p'}$ be the normal p-complement of H. Then $H_{p'} \lhd G$. If $H_{p'} \neq 1$, then it is easy to see that $G/H_{p'}$ satisfies all the hypotheses of our corollary for the normal subgroup $H/H_{p'}$ of $G/H_{p'}$ by Lemmas 2.1 and 2.2. Now by induction, we see that $G/H_{p'}$ is p-nilpotent and so G is p-nilpotent. Hence we assume $H_{p'} = 1$ and therefore $H = P$ is a p-group. Since G/H is p-nilpotent, let K/H be the normal p-complement of G/H. By Schur–Zassenhaus's theorem, there exists a Hall p'-subgroup $K_{p'}$ of K such that $K = HK_{p'}$. By Theorem 3.1, K is p-nilpotent and so $K = H \times K_{p'}$. Hence $K_{p'}$ is a normal p-complement of G. This completes the proof. ∎

Corollary 3.3. *If every maximal subgroup of any Sylow subgroup of a group G is either weakly s-permutably embedded or weakly s-supplemented in G, then G is a Sylow tower group of supersolvable type.*

Proof. Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. Then every maximal subgroup of P is either weakly s-permutably embedded or weakly s-supplemented in G. By Theorem 3.1, G is p-nilpotent. Let U be the normal p-complement of G. By Lemmas 2.1 and 2.2, U satisfies the hypothesis of the corollary. It follows by induction that U, and hence G, is a Sylow tower group of supersolvable type. ∎

THEOREM 3.4. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups. A group G is in $\mathcal{F}$ if and only if there is a normal subgroup H of G such that $G/H \in \mathcal{F}$ and every maximal subgroup of any Sylow subgroup of H is either weakly s-permutably embedded or weakly s-supplemented in G.*

Proof. The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let G be a counterexample of minimal order. By Lemmas 2.1 and 2.2, every maximal subgroup of any Sylow subgroup of H is either weakly s-permutably embedded or weakly s-supplemented in H. By Corollary 3.3, H is a Sylow tower group of supersolvable type. Let p be the largest prime divisor of $|H|$ and let P be a Sylow p-subgroup of H. Then P is normal in G. Consider G/P. It is easy to prove G/P satisfies the hypothesis of the theorem. By the choice of G we have $G/P \in \mathcal{F}$. Let N be a minimal normal subgroup of G contained in P.

(1) $P = N$.

If $N < P$, then $(G/N)/(P/N) \cong G/P \in \mathcal{F}$. We will show that $G/N \in \mathcal{F}$. By Lemmas 2.1 and 2.2, every maximal subgroup of P/N is either weakly s-permutably embedded or weakly s-supplemented in G/N. By the minimality of G, we have $G/N \in \mathcal{F}$. Since $\mathcal{F}$ is a saturated formation, N is the unique minimal normal subgroup of G contained in P and $N \nleq \Phi(G)$. By Lemma 2.7, it follows that $P = F(P) = N$, a contradiction.

(2) *The final contradiction.*

Since $N \lhd G$, we may take a maximal subgroup N_1 of N such that $N_1 \lhd G_p$, where G_p is a Sylow p-subgroup of G. Then N_1 is either weakly s-permutably embedded or weakly s-supplemented in G. Let T be a supplement of N_1 in G. Then $G = N_1T = NT$ and $N = N \cap N_1T = N_1(N \cap T)$. This implies that $N \cap T \neq 1$. But since $N \cap T$ is normal in G, and N is minimal normal in G, we have $N \cap T = N$ and so $T = G$. If N_1 is weakly s-permutably embedded in G, then N_1 is s-permutably embedded in G. By Lemma 2.5, N_1 is s-permutable in G. By Lemma 2.4, $O^p(G) \leq N_G(N_1)$. Thus $N_1 \lhd G_pO^p(G) = G$. It follows that $|N| = p$ and so $G \in \mathcal{F}$ by Lemma 2.8, a contradiction. If N_1 is weakly s-supplemented in G, then $N_1 = (N_1)_{sG}$. We get the same contradiction. ∎

THEOREM 3.5. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. A group G is in $\mathcal{F}$ if and only if there is a normal subgroup E of G such that $G/E \in \mathcal{F}$ and every cyclic subgroup $\langle x \rangle$ of any Sylow subgroup of E with prime order or order 4 (if the Sylow 2-subgroups are non-abelian) is either weakly s-permutably embedded or weakly s-supplemented in G.*

Proof. We need only prove the sufficiency part since the necessity part is evident. Suppose that the assertion is false and let G be a counterexample of minimal order.

(1) *E is solvable.*

Let K be any proper subgroup of E. Then $|K| < |G|$ and $K/K \in \mathcal{U}$. Let $\langle x\rangle$ be any cyclic subgroup of any Sylow subgroup of K with prime order or order 4 (if the Sylow 2-subgroups are non-abelian). It is clear that $\langle x\rangle$ is also a cyclic subgroup of a Sylow subgroup of E with prime order or order 4. By the hypothesis, $\langle x\rangle$ is either weakly s-permutably embedded or weakly s-supplemented in G. By Lemmas 2.1 and 2.2, $\langle x\rangle$ is either weakly s-permutably embedded or weakly s-supplemented in K. This shows that the hypothesis still holds for $(\mathcal{U}, K)$. By the choice of G, K is supersolvable. By [W, Theorem 3.11.9], E is solvable.

(2) *$G^{\mathcal{F}}$ is a p-group, where $G^{\mathcal{F}}$ is the $\mathcal{F}$-residual of G. Moreover $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G and* $\exp(G^{\mathcal{F}}) = p$ *or* $\exp(G^{\mathcal{F}}) = 4$ *(if $p = 2$ and $G^{\mathcal{F}}$ is non-abelian).*

Since $G/E \in \mathcal{F}$, we have $G^{\mathcal{F}} \le E$. Let M be a maximal subgroup of G such that $G^{\mathcal{F}} \nleq M$ (that is, M is an $\mathcal{F}$-abnormal maximal subgroup of G). Then $G = ME$. We claim that the hypothesis holds for $(\mathcal{F}, M)$. In fact, $M/M \cap E \cong ME/E = G/E \in \mathcal{F}$ and by an argument as above, we can prove that the hypothesis holds for $(\mathcal{F}, M)$. By the choice of G, $M \in \mathcal{F}$. Thus (2) holds by [W, Theorem 3.4.2].

(3) *$\langle x\rangle$ is s-permutable in G for any $x \in G^{\mathcal{F}}$.*

Let $x \in G^{\mathcal{F}}$. Then the order of x is p or 4 by Step (2). By the hypothesis, $\langle x\rangle$ is either weakly s-permutably embedded or weakly s-supplemented in G. If $\langle x\rangle$ is weakly s-supplemented in G, then there is a subgroup T of G such that $G = \langle x\rangle T$ and $\langle x\rangle \cap T \le \langle x\rangle_{sG}$. Hence,

$$G^{\mathcal{F}} = G^{\mathcal{F}} \cap G = G^{\mathcal{F}} \cap \langle x\rangle T = \langle x\rangle(G^{\mathcal{F}} \cap T).$$

Since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is abelian, we have

$$(G^{\mathcal{F}} \cap T)\Phi(G^{\mathcal{F}})/\Phi(G^{\mathcal{F}}) \lhd G/\Phi(G^{\mathcal{F}}).$$

Since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G, we have $G^{\mathcal{F}} \cap T \le \Phi(G^{\mathcal{F}})$ or $G^{\mathcal{F}} = (G^{\mathcal{F}} \cap T)\Phi(G^{\mathcal{F}}) = G^{\mathcal{F}} \cap T$. If $G^{\mathcal{F}} \cap T \le \Phi(G^{\mathcal{F}})$, then $\langle x\rangle = G^{\mathcal{F}} \lhd G$. In this case, $\langle x\rangle$ is s-permutable in G. If $G^{\mathcal{F}} = G^{\mathcal{F}} \cap T$, then $T = G$ and so $\langle x\rangle = \langle x\rangle_{sG}$ is s-permutable in G. If $\langle x\rangle$ is weakly s-permutably embedded in G, we can get the same result.

(4) $|G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})| = p$.

Assume that $|G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})| \ne p$ and let $L/\Phi(G^{\mathcal{F}})$ be any cyclic subgroup of $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$. Let $x \in L \backslash \Phi(G^{\mathcal{F}})$. Then $L = \langle x\rangle\Phi(G^{\mathcal{F}})$. Since $\langle x\rangle$ is s-

permutable in G by Step (3), $L/\Phi(G^{\mathcal{F}})$ is s-permutable in $G/\Phi(G^{\mathcal{F}})$. It follows from [S, Lemma 2.11] that $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ has a maximal subgroup which is normal in $G/\Phi(G^{\mathcal{F}})$. However, this is impossible since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G. Thus $|G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})| = p$.

(5) *The final contradiction.*

Since

$$(G/\Phi(G^{\mathcal{F}}))/(G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})) \cong G/G^{\mathcal{F}} \in \mathcal{F},$$

we see that $G/\Phi(G^{\mathcal{F}}) \in \mathcal{F}$ by Lemma 2.8. However, $\Phi(G^{\mathcal{F}}) \leq \Phi(G)$ and $\mathcal{F}$ is a saturated formation, therefore $G \in \mathcal{F}$, a contradiction. ∎

4. Some applications

Corollary 4.1 ([GS3, Theorem 3.4]). *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If all maximal subgroups of P are c-normal in G, then G is p-nilpotent.*

Corollary 4.2 ([GS1, Theorem 3.4]). *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If all maximal subgroups of P are c-supplemented in G, then G is p-nilpotent.*

Corollary 4.3 ([W, Theorem 3.1]). *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$. Suppose that every maximal subgroup of P is c-supplemented in G and $G \in C_{p'}$. Then $G/O_p(G)$ is p-nilpotent and $G \in D_{p'}$.*

Corollary 4.4 ([LL, Theorem 3.1]). *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is c-normal or s-permutably embedded in G, then G is p-nilpotent.*

Corollary 4.5 ([LW2, Theorem 7]). *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is c-supplemented or π-quasinormal in G, then G is p-nilpotent.*

Corollary 4.6 ([LP, Theorem 3.1]). *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is c-supplemented or π-quasinormally embedded in G, then G is p-nilpotent.*

Corollary 4.7 ([WW, Theorem 3.1]). *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$ and H a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is c^*-normal in G, then G is p-nilpotent.*

Corollary 4.8 ([LG, Theorem 3.3]). *Let H be a normal subgroup of a group G such that G/H is supersolvable. If every maximal subgroup of any Sylow subgroup of H is c-normal in G, then G is supersolvable.*

Corollary 4.9 ([W, Theorem 3.3]). *Let H be a normal subgroup of a group G such that G/H is supersolvable. If every maximal subgroup of any Sylow subgroup of H is c-supplemented in G, then G is supersolvable.*

Corollary 4.10 ([BP, Theorem 1]). *If every maximal subgroup of any Sylow subgroup of a group G is s-permutably embedded in G, then G is supersolvable.*

Corollary 4.11 ([AH, Theorem 3.3]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups. If there is a normal subgroup H of a group G such that $G/H \in \mathcal{F}$ and every maximal subgroup of any Sylow subgroup of H is s-permutably embedded in G, then $G \in \mathcal{F}$.*

Corollary 4.12 ([L, Theorem 3.2]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. A group G is in $\mathcal{F}$ if and only if there is a normal subgroup H of G such that $G/H \in \mathcal{F}$ and every maximal subgroup of any Sylow subgroup of H is either s-permutably embedded or c-normal in G.*

Corollary 4.13 ([GS1, Theorem 4.2]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. If there is a normal subgroup H of G such that $G/H \in \mathcal{F}$ and every maximal subgroup of any Sylow subgroup of H is c-supplemented in G, then $G \in \mathcal{F}$.*

Corollary 4.14 ([LW2, Theorem 5]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. If there is a normal subgroup H of G such that $G/H \in \mathcal{F}$ and every maximal subgroup of any Sylow subgroup of H is either π-quasinormal or c-supplemented in G, then $G \in \mathcal{F}$.*

Corollary 4.15 ([LP, Theorem 3.4]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. If there is a normal subgroup H of G such that $G/H \in \mathcal{F}$ and every maximal subgroup of any Sylow subgroup of H is either π-quasinormally embedded or c-supplemented in G, then $G \in \mathcal{F}$.*

Corollary 4.16 ([WW, Theorem 4.1]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. If there is a normal subgroup H of a group G such that $G/H \in \mathcal{F}$ and every maximal subgroup of any Sylow subgroup of H is c^*-normal in G, then $G \in \mathcal{F}$.*

Corollary 4.17 ([BW, Theorem 4.2]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. If every cyclic subgroup of $G^{\mathcal{F}}$ of prime order or order 4 is c-normal in G, then $G \in \mathcal{F}$.*

Corollary 4.18 ([BWG, Theorem 4.1]). *If every cyclic subgroup of $G^{\mathcal{U}}$ of prime order or order 4 is c-supplemented in G, then G is supersolvable.*

Corollary 4.19 ([LW1, Theorem 3.3]). *If every subgroup of a group G of prime order or of order 4 is s-quasinormally embedded in G, then G is supersolvable.*

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