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PARTIAL DIFFERENTIAL EQUATIONS

Global Attractor for the Convective Cahn–Hilliard Equation in H^k

by

Xiaopeng ZHAO and Ning DUAN

Presented by Bogdan BOJARSKI

Summary. We consider the convective Cahn–Hilliard equation with periodic boundary conditions. Based on the iteration technique for regularity estimates and the classical theorem on existence of a global attractor, we prove that the convective Cahn–Hilliard equation has a global attractor in H^k.

1. Introduction. In this paper, we are concerned with the long time behavior of solutions to the convective Cahn–Hilliard equation

$$u_t + D^4 u = D^2(u^3 - u) + uDu, \quad x \in \Omega = (0, L),\ t > 0. \tag{1.1}$$

On the basis of physical considerations, equation (1.1) is supplemented with the periodic boundary value conditions

$$u(x + L, t) = u(x, t), \quad x \in \mathbb{R},\ t > 0, \tag{1.2}$$

and the initial condition

$$u(x, 0) = \varphi(x), \quad x \in \mathbb{R}. \tag{1.3}$$

Equation (1.1) arises naturally as a continuous model for the formation of facets and corners in crystal growth (see [5, 6]). Here $u(x,t)$ denotes the slope of the interface, the convective term uDu (see [6]) stems from the effect of kinetics that provides an independent flux of the order parameter, similar to the effect of an external field in spinodal decomposition of a driven system.

In the last years, many authors have paid much attention to the convective Cahn–Hilliard equation (see [6, 18]). It was K. H. Kwek [7] who

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first studied the convective Cahn–Hilliard equation in a special case with a special convection u. Based on the discontinuous Galerkin finite element method, he proved the existence of a classical solution. Recently, Liu [8] considered the equation

$$u_t + D^4 u = D^2(\gamma_2 u^3 + \gamma_1 u^2 - u) + \beta D\left(-\frac{1}{4}u^4 + \frac{1}{2}u^2\right).$$

He proved the global existence and uniqueness and asymptotic behavior of classical solutions for the initial boundary value problem. Liu [9] also studied the following convective Cahn–Hilliard equation with degenerate mobility:

$$\frac{\partial u}{\partial t} + D[m(u)(kD^3 u - DA(u))] - \gamma DB(u) = 0.$$

A. Eden and V. K. Kalantarov [4] considered (1.1) as an infinite-dimensional dynamical system and showed that solutions enter an absorbing ball in a finite time. Moreover, they showed that the solutions of problem (1.1)–(1.3) fall into the Gevrey class and deduced as a simple corollary that four nodes are determining for solutions. Recently, A. Eden and V. K. Kalantarov [3] also studied (1.1) with periodic boundary conditions in the 3D case. They considered a relevant continuous dynamical system on $\dot{L}^2(\Omega)$, and proved that (1.1) has absorbing balls in $\dot{L}^2(\Omega)$, $\dot{H}^1_{\mathrm{per}}(\Omega)$ and $\dot{H}^2_{\mathrm{per}}(\Omega)$. Combining this with the compactness property of the solution semigroup they deduced the existence of a global attractor for (1.1).

There is much literature concerning the convective Cahn–Hilliard equation; for more recent results we refer the reader to [9, 16, 17] and the references therein.

The dynamic properties of the convective Cahn–Hilliard equation, such as the global asymptotical behavior of solutions and existence of global attractors, are important for the study of convective Cahn–Hilliard systems. In this paper, we are interested in the existence of global attractors for the convective Cahn–Hilliard equation. Based on A. Eden and V. K. Kalantarov's work [4] and T. Ma and S. Wang's recent work [10], we shall prove that the convective Cahn–Hilliard equation (1.1) has a global attractor in H^k ($k > 0$), which attracts any bounded subset of $H^k(0, L)$ in the H^k-norm.

This paper is organized as follows. In the next section, we give some preparations and we state the main theorem about the existence of a global attractor. In Section 3, we prove that problem (1.1)–(1.3) has global attractors in $H^k(0, L)$. Some ideas important for this paper come from [10, 12, 13, 14], etc.

2. Preliminaries. Assume X and X_1 are two Banach spaces, and $X_1 \subset X$ is a compact and dense inclusion. Consider the following equation defined on X:

$$u_t = Lu + Gu, \qquad u(0) = \varphi, \tag{2.1}$$

where u is an unknown function, $L : X_1 \to X$ a linear operator and $G : X_1 \to X$ a nonlinear operator. Then the solution of (2.1) can be expressed as

$$u(t, \varphi) = S(t)\varphi,$$

where $S(t) : X \to X$ $(t \geq 0)$ is the semigroup generated by (2.1).

Next, we recall the classical theorem on existence of a global attractor by R. Temam [15].

LEMMA 2.1. *Assume that $S(t) : X \to X$ is the semigroup generated by problem* (2.1), *and the following conditions hold for some set $B \subset X$:*

- (H1) *For any bounded set $A \subset X$ there exists a time $t_A \geq 0$ such that for all $\varphi \in A$ and $t > t_A$, we have $S(t)\varphi \in B$.*
- (H2) *For any bounded set $u \subset X$ and some $T > 0$ sufficiently large, the set $\overline{\bigcup_{t \geq T} S(t)u}$ is compact in X.*

Then the ω-limit set $\mathcal{A} = \omega(B)$ of B is a global attractor of problem (2.1), *and $\mathcal{A}$ is connected providing B is connected.*

In this paper, we usually assume that the linear operator $L : X_1 \to X$ in (2.1) is a sectorial operator, which generates an analytic semigroup e^{tL}, and L induces the fractional power operators and fractional order spaces as follows:

$$\mathscr{L}^{\alpha} = (-L)^{\alpha} : X_{\alpha} \to X, \qquad \alpha \in \mathbb{R},$$

where $X_\alpha = D(\mathscr{L}^\alpha)$ is the domain of $\mathscr{L}^\alpha$. By semigroup theory, $X_\beta \subset X_\alpha$ is a compact inclusion for any $\beta > \alpha$. For more about the space H_α, we recommend [10].

Thus, Lemma 2.1 can be equivalently expressed as the following lemma, which can be found in [10, 12, 13, 14].

LEMMA 2.2. *Assume that $u(t, \varphi) = S(t)\varphi$ $(\varphi \in X, t \geq 0)$ is a solution of* (2.1) *and $S(t)$ the semigroup generated by* (2.1). *Assume further that X_α is the fractional order space generated by L and:*

- (B1) *For some $\alpha \geq 0$ there is a bounded absorbing set $B \subset X_\alpha$, which means that for any $\varphi \in X_\alpha$ there exists $t_\varphi > 0$ such that*

$$u(t, \varphi) \in B, \qquad \forall t > t_\varphi.$$

- (B2) *There is a $\beta > \alpha$ such that for any bounded set $U \subset X_\beta$ there are $T > 0$ and $C > 0$ such that*

$$\|u(t, \varphi)\|_{X_\beta} \leq C, \qquad \forall t > T, \ \varphi \in U.$$

Then (2.1) *has a global attractor* $\mathscr{A} \subset X_\alpha$ *which attracts any bounded set of* X_α *in the* X_α*-norm.*

We also have the following lemma which can be found in [10, 12, 13, 14].

LEMMA 2.3. *Assume that* $L : X_1 \to X_\alpha$ *is a sectorial operator which generates an analytic semigroup* $T(t) = e^{tL}$. *If all eigenvalues* λ *of* L *satisfy* $\operatorname{Re}\lambda < -\lambda_0$ *for some* $\lambda_0 > 0$, *then for* $\mathscr{L}^\alpha$ ($\mathscr{L} = -L$) *we have:*

(C1) $T(t) : X \to X_\alpha$ *is bounded for all* $\alpha \in \mathbb{R}$ *and* $t > 0$.
(C2) $T(t)\mathscr{L}^\alpha x = \mathscr{L} T(t)x$ *for all* $x \in X_\alpha$.
(C3) *For each* $t > 0$, $\mathscr{L}^\alpha T(t) : X \to X$ *is bounded, and*

$$\|\mathscr{L}^\alpha T(t)\| \le C_\alpha t^{-\alpha} e^{-\delta t}$$

for some $\delta > 0$, *where* $C_\alpha > 0$ *is a constant depending only on* α.
(C4) *The* X_α*-norm can be defined by* $\|x\|_{X_\alpha} = \|\mathscr{L}^\alpha x\|_X$.

For problem (1.1)–(1.3), we assume that the initial function has zero mean, i.e. $\int_0^L \varphi(x)\,dx = 0$. Then it follows that

$$\int_0^L u(x,t)\,dx = \int_0^L \varphi(x)\,dx = 0, \qquad \forall t > 0.$$

Now, we introduce the following spaces:

$$\begin{cases} H = \dot{L}^2(\Omega), \\ H_{1/2} = \dot{H}^2_{\mathrm{per}}(\Omega) = H^2_{\mathrm{per}}(\Omega) \cap H, \\ H_1 = \dot{H}^4_{\mathrm{per}}(\Omega) = H^4_{\mathrm{per}}(\Omega) \cap H, \end{cases} \tag{2.2}$$

where $\Omega = (0, L)$. We define a linear operator $L : H_1 \to H$ and a nonlinear operator $G : H_1 \to H$ by

$$\begin{cases} Lu = -D^4 u, \\ g(u) = D^2(u^3 - u) + uDu, \\ Gu = g(u), \end{cases} \tag{2.3}$$

It is known that L given by (2.3) is a sectorial operator and the fractional power operator $(-L)^{1/2}$ is given by

$$(-L)^{1/2} = -\Delta = -D^2.$$

The space $H_{1/2}$ is the same as in (2.2), and $H_{1/4}$ is given by $H_{1/4}$ = closure of $H_{1/2}$ in $H^1(\Omega)$ and $H_k = H^{4k} \cap H_1$ for $k \ge 1$.

We will give a theorem on the existence of a global attractor in $H^2(\Omega)$ for problem (1.1)–(1.3), which can be deduced easily from the results of A. Eden and V. K. Kalantarov [4].

THEOREM 2.4. *Assume $\Omega = (0, L)$, $\varphi \in \dot{H}^2_{\mathrm{per}}(\Omega)$ and conditions* (1.2)–(1.3) *hold. Then the semigroup $\{S(t)\}_{t\geq 0}$ associated with* (1.1) *has a global attractor $\mathscr{A}$ in $\dot{H}^2_{\mathrm{per}}(\Omega)$ which is compact and connected.*

In order to prove Theorem 2.4, we should verify that equation (1.1) satisfies the two conditions of Lemma 2.1. By A. Eden and V. K. Kalantarov's recent work [4], we have $\|u(t,\varphi)\|_{H^2} \leq C$, where C is a positive constant, so condition (H1) is proved. We have to prove that (1.1) satisfies condition (H2), which suffices to prove that for $t \geq t_0 > 0$, $\|u(t,\varphi)\|_{H^3} \leq C$, where C is a positive constant. Differentiating (1.1) with respect to x, multiplying the result by D^5u, integrating on Ω and using the uniform Gronwall inequality we can deduce (H2). Since the proof is easy, we omit it.

We also have the following corollary which was proved in [4].

COROLLARY 2.5. *Assume $\Omega = (0, L)$ and $\varphi \in \dot{H}^2_{\mathrm{per}}(\Omega)$. Then for problem* (1.1)–(1.3), *we have*

$$\|u(t,\varphi)\|_{L^\infty} \leq C,$$

where C is a positive constant.

The main result is given by the following theorem, which provides the existence of a global attractor of the convective Cahn–Hilliard equation in H^k for any k.

THEOREM 2.6. *Assume $\Omega = (0, L)$, $\varphi \in \dot{H}^2_{\mathrm{per}}(\Omega)$ and conditions* (1.2)–(1.3) *hold. Then for any $\alpha > 0$, equation* (1.1) *has a global attractor $\mathscr{A}$ in H_α and $\mathscr{A}$ attracts any bounded subset of H_α in the H_α-norm.*

In the following, C, C_i $(i = 1, 2, \ldots)$ will represent generic constants that may change from line to line even in the same inequality, and we denote $\Omega = (0, L)$.

3. Proof of Theorem 2.6. It is well known that the solution $u(t,\varphi)$ of problem (1)–(3) can be written as

$$u(t,\varphi) = e^{tL}\varphi + \int_0^t e^{(t-\tau)L} Gu\, d\tau. \tag{3.1}$$

Using (2.3) and (3.1), we obtain

$$u(t,\varphi) = e^{tL}\varphi + \int_0^t e^{(t-\tau)L} g(u)\, d\tau. \tag{3.2}$$

By Lemma 2.2, to prove Theorem 2.6, we first prove the following lemma.

LEMMA 3.1. *Assume $\Omega = (0, L)$ and $\varphi \in \dot{H}^2_{\mathrm{per}}(\Omega)$. Then for any $\alpha \geq 0$, the semigroup $S(t)$ generated by problem* (1.1)–(1.3) *is uniformly compact in H_α.*

Proof. It suffices to prove that for any bounded set $U \subset H_\alpha$, there exists $C > 0$ such that

$$\|u(t,\varphi)\|_{H_\alpha} \le C, \quad \forall t \ge 0,\ \varphi \in U \subset H_\alpha,\ \alpha \ge 0. \tag{3.3}$$

For $\alpha = 1/2$, this follows from Theorem 2.4, i.e. for any bounded set $U \subset H_{1/2}$ there is a constant $C > 0$ such that

$$\|u(t,\varphi)\|_{H_{1/2}} \le C, \quad \forall t \ge 0,\ \varphi \in U \subset H_{1/2},\ \alpha \ge 0. \tag{3.4}$$

We only need to prove (3.3) for any $\alpha > 1/2$.

This is done in a few steps. First, we prove that for any bounded set $U \subset H_\alpha$ $(1/2 \le \alpha < 1)$ there exists a constant $C > 0$ such that

$$\|u(t,\varphi)\|_{H_\alpha} \le C, \quad \forall t \ge 0,\ \varphi \in U,\ \alpha < 1. \tag{3.5}$$

By Corollary 2.5 and the following embedding theorems for fractional order spaces (see Pazy [11]):

$$H_{1/2} \hookrightarrow L^{2p}(\Omega), \quad H_{1/2} \hookrightarrow W^{1,2}(\Omega), \quad H_{1/2} \hookrightarrow W^{1,4}(\Omega),$$

we obtain

$$\begin{aligned} \|g(u)\|_H^2 &= \int_\Omega |g(u)|^2\,dx = \int_\Omega |D^2(u^3-u) + uDu|^2\,dx \\ &= \int_\Omega \big|6u|Du|^2 + 3u^2D^2u - D^2u + uDu\big|^2\,dx \\ &\le C\int_\Omega (u^2|Du|^4 + u^4|D^2u|^2 + |D^2u|^2 + u^2|Du|^2)\,dx \\ &\le C\int_\Omega (|Du|^4 + |Du|^2 + |D^2u|^2)\,dx \\ &\le C(\|u\|_{W^{1,4}}^4 + \|u\|_{W^{1,2}}^2 + \|u\|_{H_{1/2}}^2) \\ &\le C(\|u\|_{H_{1/2}}^4 + \|u\|_{H_{1/2}}^2 + \|u\|_{H_{1/2}}^2), \end{aligned} \tag{3.6}$$

which means that $g : H_{1/2} \to H$ is bounded. Hence, we deduce that

$$\begin{aligned} \|u(t,\varphi)\|_{H_\alpha} &= \Big\| e^{tL}\varphi + \int_0^t e^{(t-\tau)L} g(u)\,d\tau \Big\|_{H_\alpha} \\ &\le C\|\varphi\|_{H_\alpha} + \int_0^t \|(-L)^\alpha e^{(t-\tau)L} g(u)\|_H\,d\tau \\ &\le C\|\varphi\|_{H_\alpha} + \int_0^t \|(-L)^\alpha e^{(t-\tau)L}\| \cdot \|g(u)\|_H\,d\tau \\ &\le C\|\varphi\|_{H_\alpha} + C\int_0^t \tau^{-\alpha} e^{-\delta\tau}\,d\tau \\ &\le C, \quad \forall t \ge 0,\ \varphi \in U \subset H_\alpha, \end{aligned} \tag{3.7}$$

where $0 < \alpha < 1$. Thus, (3.5) is proved.

Second, we prove that for any bounded set $U \subset H_\alpha$ $(1 \le \alpha < 5/4)$, there exists a constant $C > 0$ such that

$$\|u(t,\varphi)\|_{H_\alpha} \le C, \quad \forall t \ge 0,\ \varphi \in U \subset H_\alpha,\ 1 \le \alpha < 5/4. \tag{3.8}$$

By Corollary 1.1 and the following embedding theorems of fractional order spaces (see Pazy [11]):

$$H_\alpha \hookrightarrow W^{3,2}(\Omega), \quad H_\alpha \hookrightarrow W^{2,4}(\Omega), \quad H_\alpha \hookrightarrow W^{2,2}(\Omega),$$
$$H_\alpha \hookrightarrow W^{1,6}(\Omega), \quad H_\alpha \hookrightarrow W^{1,4}(\Omega),$$

where $3/4 \le \alpha < 1$, we obtain

$$\begin{aligned}
&\|g(u)\|^2_{H_{1/4}} \\
&= \int_\Omega |Dg(u)|^2\,dx = \int_\Omega |D(D^2(u^3-u)+uDu))|^2\,dx \\
&= \int_\Omega (6|Du|^3 + 18u|DuD^2u| + (3u^2-1)D^3u + uD^2u + |Du|^2)^2\,dx \\
&\le C\int_\Omega (|Du|^6 + |DuD^2u|^2 + |D^3u|^2 + |D^2u|^2 + |Du|^4)\,dx \\
&\le C\int_\Omega (|Du|^6 + |D^2u|^4 + |D^2u|^2 + |D^3u|^2 + |Du|^4)\,dx \\
&\le C(\|u\|^6_{W^{1,6}} + \|u\|^4_{W^{2,4}} + \|u\|^2_{W^{2,2}} + \|u\|^2_{W^{3,2}} + \|u\|^4_{W^{1,4}}) \\
&\le C(\|u\|^6_{H_\alpha} + \|u\|^2_{H_\alpha} + \|u\|^4_{H_\alpha}),
\end{aligned} \tag{3.9}$$

which means that $g : H_\alpha \to H_{1/4}$ is bounded for $3/4 \le \alpha < 1$. Using (3.5) and (3.9), we obtain

$$\|g(u(t,\varphi))\|_{H_{1/4}} \le C, \quad \forall t \ge 0,\ \varphi \in U,\ 3/4 \le \alpha < 1. \tag{3.10}$$

By using the same method as in the first step, from (3.10) we have

$$\begin{aligned}
\|u(t,\varphi)\|_{H_\alpha} &= \Big\| e^{tL}\varphi + \int_0^t e^{(t-\tau)L} g(u)\,d\tau \Big\|_{H_\alpha} \\
&\le C\|\varphi\|_{H_\alpha} + \int_0^t \|(-L)^\alpha e^{(t-\tau)L} g(u)\|_H\,d\tau \\
&\le C\|\varphi\|_{H_\alpha} + \int_0^t \|(-L)^{\alpha-1/4} e^{(t-\tau)L}\| \cdot \|g(u)\|_{H_{1/4}}\,d\tau \\
&\le C\|\varphi\|_{H_\alpha} + C\int_0^t \tau^{-\beta} e^{-\delta\tau}\,d\tau \\
&\le C, \quad \forall t \ge 0,\ \varphi \in U \subset H_\alpha,
\end{aligned} \tag{3.11}$$

where $\beta = \alpha - 1/4$ $(0 < \beta < 1)$. Thus (3.8) is proved.

Third, we prove that for any bounded set $U \subset H_\alpha$ $(5/4 \le \alpha < 3/2)$ there exists a constant $C > 0$ such that

$$\|u(t,\varphi)\|_{H_\alpha} \le C, \quad \forall t \ge 0, \varphi \in U \subset H_\alpha,\ 5/4 \le \alpha < 3/2. \tag{3.12}$$

By Corollary 2.5 and the following embedding theorems (see Pazy [11]):

$$\begin{aligned} &H_\alpha \hookrightarrow W^{1,4}(\Omega), \quad H_\alpha \hookrightarrow W^{2,4}(\Omega), \quad H_\alpha \hookrightarrow W^{3,4}(\Omega),\\ &H_\alpha \hookrightarrow W^{4,2}(\Omega), \quad H_\alpha \hookrightarrow W^{3,2}(\Omega), \quad H_\alpha \hookrightarrow W^{1,8}(\Omega), \end{aligned}$$

where $1 \le \alpha < 5/4$, we obtain

$$\begin{aligned} \|g(u)\|^2_{H_{1/2}} &= \int_\Omega |D^2 g(u)|^2\,dx = \int_\Omega |D^2(D^2(u^3-u)+uDu)|^2\,dx \\ &= \int_\Omega (36|Du|^2|D^2u| + 18u|D^2u|^2 + 24uDuD^3u + (3u^2-1)D^4u \\ &\quad + 3DuD^2u + uD^3u)^2\,dx \\ &\le C\int_\Omega (|Du|^4|D^2u|^2 + u^2|D^2u|^4 + u^2|Du|^2|D^3u|^2 + u^4|D^4u|^2 \\ &\quad + |D^4u|^2 + |Du|^2|D^2u|^2 + u^2|D^3u|^2)\,dx \\ &\le C\int_\Omega (|Du|^8 + |D^2u|^4 + |D^3u|^4 + |D^4u|^2 + |Du|^4 + |D^3u|^2)\,dx \\ &\le C(\|u\|^8_{W^{1,8}} + \|u\|^4_{W^{2,4}} + \|u\|^4_{W^{1,4}} + \|u\|^4_{W^{3,4}} + \|u\|^2_{W^{4,2}} + \|u\|^2_{W^{3,2}}) \\ &\le C(\|u\|^8_{H_\alpha} + \|u\|^4_{H_\alpha} + \|u\|^2_{H_\alpha}), \end{aligned} \tag{3.13}$$

which means that $g : H_\alpha \to H_{1/2}$ is bounded for $1 \le \alpha < 5/4$. Using (3.8) and (3.13), we obtain

$$\|g(u(t,\varphi))\|_{H_{1/2}} \le C, \quad \forall t \ge 0,\ \varphi \in U,\ 1 \le \alpha < 5/4. \tag{3.14}$$

By using the same method as in the first and second steps, from (3.14) we have

$$\begin{aligned} \|u(t,\varphi)\|_{H_\alpha} &= \Big\| e^{tL} + \int_0^t e^{(t-\tau)L} g(u)\,d\tau \Big\|_{H_\alpha} \\ &\le C\|\varphi\|_{H_\alpha} + \int_0^t \|(-L)^{\alpha-1/2} e^{(t-\tau)L}\| \cdot \|g(u)\|_{H_{1/2}}\,d\tau \\ &\le C\|\varphi\|_{H_\alpha} + C\int_0^t \tau^{-\beta} e^{-\delta\tau}\,d\tau \\ &\le C, \quad \forall t \ge 0,\ \varphi \in U \subset H_\alpha, \end{aligned} \tag{3.15}$$

where $\beta = \alpha - 1/2$ $(0 < \beta < 1)$. Thus (3.12) is proved.

Fourth, we prove that for any bounded set $U \subset H_\alpha$ $(3/2 \le \alpha < 7/4)$ there exists a constant $C > 0$ such that

$$\|u(t,\varphi)\|_{H_\alpha} \le C, \quad \forall t \ge 0,\ \varphi \in U \subset H_\alpha,\ 3/2 \le \alpha < 7/4. \tag{3.16}$$

By Corollary 2.5 and the following embedding theorems (see Pazy [11]):

$$\begin{aligned} &H_\alpha \hookrightarrow W^{1,8}(\Omega), \quad H_\alpha \hookrightarrow W^{3,4}(\Omega), \quad H_\alpha \hookrightarrow W^{4,2}(\Omega),\\ &H_\alpha \hookrightarrow W^{1,4}(\Omega), \quad H_\alpha \hookrightarrow W^{2,8}(\Omega), \quad H_\alpha \hookrightarrow W^{1,6}(\Omega),\\ &H_\alpha \hookrightarrow W^{2,4}(\Omega), \quad H_\alpha \hookrightarrow W^{4,4}(\Omega), \quad H_\alpha \hookrightarrow W^{5,2}(\Omega), \end{aligned}$$

where $5/4 \le \alpha < 3/2$, we obtain

$$\begin{aligned} \|g(u)\|^2_{H_{3/4}} &= \int_\Omega |D^3(D^2(u^3-u)+uDu)|^2\,dx\\ &= \int_\Omega (60|Du|^2D^3u + 90Du|D^2u|^2 + 60uD^2uD^3u + 30uDuD^4u\\ &\quad + (3u^2-1)D^5u + 3|D^2u|^2 + 4DuD^3u + uD^4u)^2\,dx\\ &\le C\int_\Omega (|Du|^4|D^3u|^2 + |Du|^2|D^2u|^4 + |D^2u|^2|D^3u|^2\\ &\quad + |Du|^2|D^4u|^2 + |D^5u|^2 + |D^2u|^4 + |Du|^2|D^3u|^2 + |D^4u|^2)\,dx\\ &\le C\int_\Omega (|Du|^8 + |D^3u|^4 + |Du|^4 + |D^2u|^8 + |D^2u|^4\\ &\quad + |D^4u|^4 + |D^4u|^2 + |D^5u|^2)\,dx\\ &\le C(\|u\|^8_{W^{1,8}} + \|u\|^4_{W^{3,4}} + \|u\|^4_{W^{1,4}} + \|u\|^8_{W^{2,8}} + \|u\|^4_{W^{2,4}}\\ &\quad + \|u\|^2_{W^{4,2}} + \|u\|^4_{W^{4,4}} + \|u\|^2_{W^{5,2}})\,dx\\ &\le C(\|u\|^2_{H_\alpha} + \|u\|^4_{H_\alpha} + \|u\|^6_{H_\alpha} + \|u\|^8_{H_\alpha}), \end{aligned}$$

which means that $g : H_\alpha \to H_{3/4}$ is bounded for $5/4 \le \alpha < 3/2$. Using (3.12) and (3.17), we obtain

$$\|g(u(t,\varphi))\|_{H_{3/4}} \le C, \quad \forall t \ge 0,\ \varphi \in U,\ 5/4 \le \alpha < 3/2. \tag{3.17}$$

By using the same method as in the above steps, and from (3.17), we have

$$\begin{aligned} \|u(t,\varphi)\|_{H_\alpha} &= \Big\| e^{tL}\varphi + \int_0^t e^{(t-\tau)L} g(u)\,d\tau \Big\|_{H_\alpha}\\ &\le C\|\varphi\|_{H_\alpha} + \int_0^t \|(-L)^{\alpha-3/4}e^{(t-\tau)L}\| \cdot \|g(u)\|_{H_{3/4}}\,d\tau\\ &\le C\|\varphi\|_{H_\alpha} + C\int_0^t \tau^{-\beta}e^{-\delta\tau}\,d\tau\\ &\le C, \quad \forall t \ge 0,\ \varphi \in U \subset H_\alpha, \end{aligned} \tag{3.18}$$

where $\beta = \alpha - 3/4$ $(0 < \beta < 1)$. Thus (3.16) is proved.

Using the same method as in the proof of (3.16), by iteration we can prove that for any bounded set $U \subset H_\alpha$ ($\alpha > 0$), there exists a constant $C > 0$ such that (3.3) holds. i.e. for all $\alpha \geq 0$ the semigroup $S(t)$ generated by problem (1.1)–(1.3) is uniformly compact in H_α. ■

We also have the following lemma.

LEMMA 3.2. *Assume* $\Omega = (0, L)$ *and* $\varphi \in \dot{H}^2_{\mathrm{per}}(\Omega)$. *Then for any* $\alpha \geq 0$, *problem* (1.1)–(1.3) *has a bounded absorbing set in* H_α.

Proof. It suffices to prove that for any bounded set $U \subset H_\alpha$ ($\alpha \geq 0$) there exist $T > 0$ and a constant $C > 0$, independent of φ, such that

$$\|u(t,\varphi)\|_{H_\alpha} \leq C, \quad \forall t \geq T, \varphi \in U \subset H_\alpha. \tag{3.19}$$

For $\alpha = 1/2$, this follows from Theorem 2.4. And we only need to prove (3.19) for any $\alpha > 1/2$. We prove the lemma in the following steps.

First, we will prove that for any $1/2 \leq \alpha < 1$, the problem (1.1)–(1.3) has a bounded absorbing set in H_α. Using (3.2) gives

$$u(t,\varphi) = e^{(t-T)L}u(T,\varphi) + \int_T^t e^{(t-T)L} g(u)\, d\tau. \tag{3.20}$$

Assume B is a bounded absorbing set of problem (1.1)–(1.3) and $B \subset H_{1/2}$; we also let $T_0 > 0$ be the time such that

$$u(t,\varphi) \in B, \quad \forall t > T_0,\ \varphi \in U \subset H_\alpha,\ \alpha \geq 1/2.$$

Note that

$$\|e^{tL}\| \leq Ce^{-d\lambda_1 t},$$

where $\lambda_1 > 0$ is the first eigenvalue of the equation

$$-\Delta u = \lambda u, \quad u(L,t) = u(0,t). \tag{3.21}$$

Then for any given $T > 0$ and $\varphi \in U \subset H_\alpha$ ($\alpha \geq 1/2$), we obtain

$$\lim_{t\to\infty} \|e^{(t-T)L}u(T,\varphi)\|_{H_\alpha} = 0. \tag{3.22}$$

Using (3.6), (3.20) and the assertion (C3) of Lemma 2.3 gives

$$\begin{aligned} \|u(t,\varphi)\|_{H_\alpha} &\leq \|e^{(t-T_0)L}u(T_0,\varphi)\|_{H_\alpha} + \int_{T_0}^t \|(-L)^\alpha e^{(t-T)L}\| \cdot \|g(u)\|_H\, d\tau \\ &\leq \|e^{(t-T_0)L}u(T_0,\varphi)\|_{H_\alpha} + C\int_{T_0}^t \|(-L)^\alpha e^{(t-T)L}\| \\ &\leq \|e^{(t-T_0)L}u(T_0,\varphi)\|_{H_\alpha} + C\int_0^{T-T_0} \tau^{-\alpha}e^{-\delta\tau}\, d\tau \\ &\leq \|e^{(t-T_0)L}u(T_0,\varphi)\|_{H_\alpha} + C, \end{aligned} \tag{3.23}$$

where $C > 0$ is a constant independent of φ. Then by (3.22) and (3.23), we see that (3.19) holds for all $1/2 \leq \alpha < 1$.

Second, we can use the same method as in the above step to prove that for any $3/4 < \alpha < 5/4$ and for any $1 < \alpha < 3/2$, problem (1.1)–(1.3) has a bounded absorbing set in H_α. By iteration, we conclude that (3.19) holds for all $\alpha \geq 1/2$. ■

Proof of Theorem 2.6. Apply Lemmas 2.2, 3.1 and 3.2. ■

Hence, we have the following remark.

REMARK. The attractors $\mathscr{A}_\alpha \subset H_\alpha$ in Theorem 2.6 are the same for all $\alpha \geq 0$, i.e. $\mathscr{A}_\alpha = \mathscr{A}$ for all $\alpha \geq 0$. Hence, $\mathscr{A} \subset C^\infty(\Omega)$. Theorem 2.6 implies that for any $\varphi \in H$, the solution $u(t,\varphi)$ of problem (1.1)–(1.3) satisfies

$$\lim_{t\to\infty} \inf_{v\in\mathscr{A}} \|u(t,\varphi) - v\|_{C^k} = 0, \quad \forall k \geq 1.$$

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Xiaopeng Zhao
School of Mathematics
Jilin University
Changchun, 130012, P.R. China
E-mail: zhaoxp10@mails.jlu.edu.cn

Ning Duan
The No. 2 High School of Tangshan
Tangshan, 063000, P.R. China
E-mail: duanning@sina.cn

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