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FUNCTIONAL ANALYSIS

Remarks on the Bourgain–Brezis–Mironescu Approach to Sobolev Spaces

by

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Summary. For a function $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ the notion of p-mean variation of order 1, $\mathsf{V}_1^p(f,\mathbb{R}^n)$ is defined. It generalizes the concept of F. Riesz variation of functions on the real line $\mathbb{R}^1$ to $\mathbb{R}^n$, $n > 1$. The characterisation of the Sobolev space $W^{1,p}(\mathbb{R}^n)$ in terms of $\mathsf{V}_1^p(f,\mathbb{R}^n)$ is directly related to the characterisation of $W^{1,p}(\mathbb{R}^n)$ by Lipschitz type pointwise inequalities of Bojarski, Hajłasz and Strzelecki and to the Bourgain–Brezis–Mironescu approach.

1. Introduction. In the paper *Another look at Sobolev spaces* [6] a new approach to Sobolev space theory on (smooth) domains Ω of euclidean space $\mathbb{R}^n$ was proposed. The authors start with the observation that if $f \in W^{1,p}(\Omega)$, $1 \le p < \infty$, and $\rho \in L^1(\mathbb{R}^n)$, $\rho \ge 0$, then

$$\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho(x-y)\,dx\,dy \le C\|f\|^p_{W^{1,p}}\|\rho\|_{L_1} \tag{1.1}$$

with a constant C depending on p and Ω only. Here the seminorm $\|f\|_{W^{1,p}}$ is defined as

$$\|f\|^p_{W^{1,p}} = \int_\Omega |\nabla f|^p\,dx \tag{1.2}$$

where $|\ |$ denotes the euclidean norm. For a sequence of radial mollifiers $\rho_k \in L^1(\mathbb{R}^n)$ with $\rho_k \ge 0$, $\int \rho_k(x)\,dx = 1$ and

$$\rho_k(r) \to 0 \quad \text{uniformly in } r \ge r_0 \text{ for all } r_0 > 0, \tag{1.3}$$

the following theorem is proved (Theorem 2 in [6]).

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THEOREM A. *Let $f \in L^p(\Omega)$, $1 < p < \infty$. If the condition*

$$(1.4)\qquad A_p^p = \liminf_{k\to\infty} \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_k(x-y)\,dx\,dy < \infty$$

holds for a sequence of mollifiers ρ_k, then $f \in W^{1,p}(\Omega)$. Moreover, the limiting value A_p recovers the norm $\|f\|_{W^{1,p}}$: $A_p^p = K_{p,n}\|f\|_{W^{1,p}}^p$ with a constant $K_{p,n}$ depending on p and n only.

Conversely, if $f \in W^{1,p}(\Omega)$ then the condition (1.4) *is satisfied for any sequence of mollifiers ρ_k.*

The formulation of Theorem A implies the global finiteness condition

$$(1.5)\qquad \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\,dx\,dy < \infty.$$

In this paper (1.5) is assumed to hold throughout.

Theorem A shows that the additional asymptotic condition (1.4) characterizes the Sobolev class $W^{1,p}(\Omega)$.

The condition (1.4) expresses the fact that the total mass of the integrand kernel in (1.5) "has a finite p-trace" on the diagonal Δ in the cartesian product $\Omega\times\Omega$.

The main purpose of this note is to discuss the relations of the BBM theory to the description of Sobolev spaces in terms of pointwise Sobolev inequalities, as developed in a series of papers, mainly in collaboration with P. Hajłasz and P. Strzelecki, [3], [4], [2], [11], [12] ([1]).

For the case of (open) subsets of $\mathbb{R}^n$ the BHS theory characterizes the Sobolev spaces $W^{m,p}(\Omega)$ of integer order m, $m \ge 1$, and real $p > 1$ by the pointwise inequalities

$$(1.6)\qquad |R^{m-1}f(x;y)| \le |x-y|^m(a_f(x)+a_f(y)),\quad x,y\in\Omega,$$

where $R^m f(x;y)$ is the mth order Taylor–Whitney remainder term, centered at the point y in Ω, of the function $f(x)$, and $a_f \in L^p(\Omega)$ (see [2], [5]).

Any function a_f in the right hand side of (1.6) will be called a *mean maximal m-gradient* of f.

Here the mean maximal m-gradient appears as a component of the pair (R_f^{m-1}, a_f) representing an element of the Sobolev space $W^{m,p}(\Omega)$. For the case $m=1$ it reduces to the simple form

$$(1.7)\qquad |R^0 f(x;y)| = |f(x)-f(y)| \le |x-y|(a_f(x)+a_f(y)),\qquad a_f \in L^p(\Omega)\subset L^p(\mathbb{R}^n),$$

which does not involve derivatives ([11], [14], [15]).

([1]) For convenience we use the shorthand BHS, BBM for the cases considered.

For a detailed definition of p-mean 1-variation $\mathsf{V}_1^p(f,\Omega)$ of a function $f \in L^p_{\mathrm{loc}}(\Omega)$ see Section 4 below. Here we only write the approximate formula (2)

$$\mathsf{V}_1^p(f,\Omega) \sim \sup_{\mathcal{P}} \sum_{i=1}^{N} |Q_i| \fint_{Q_i}\fint_{Q_i} \frac{|f(x)-f(y)|^p}{|x-y|^p}\,dx\,dy \tag{1.8}$$

showing how $\mathsf{V}_1^p(f,\Omega)$ is described in terms of the least upper bounds of finite sums of averaged powers of difference quotients for cubical partitions (i.e., unions of disjoint coordinate cubes) in Ω.

Our main result—Theorem 4.1 and its converse—characterizes the space $W^{1,p}(\Omega)$ in terms of $\mathsf{V}_1^p(f,\Omega)$.

THEOREM 4.1. *If for a (real valued) function $f \in L^p_{\mathrm{loc}}(\Omega)$ the p-mean 1-variation $\mathsf{V}_1^p(f,\Omega')$ is finite for each open Ω' with compact closure in Ω (i.e.* $\mathrm{dist}(\Omega',\partial\Omega)>0$) *then $f \in W^{1,p}_{\mathrm{loc}}(\Omega)$ $(p>1)$. In other words, $\mathsf{V}_1^p(\Omega) \subset W^{1,p}(\Omega)$.*

The notion of p-mean 1-variation has its predecessors in a broad variety of "variations" in mathematical analysis and geometry. Two of these, briefly recalled in Sections 2 and 3, have their sources in the seminal paper of F. Riesz [23] in *Math. Annalen*, 1910, and are used in our basic Section 4 and final comments in Section 5.

For simplicity in the following we mainly restrict our attention to the model case $\Omega = \mathbb{R}^n$, [12].

The tools which we propose to use in the discussion of equivalence of BBM theory (Theorem A) and BHS theory (inequalities (1.6), (1.7)) are the concepts of mean variations of real valued functions. The concept of p-mean m-variation of positive integer order, $m \geq 1$, seems to be new.

2. F. Riesz variation of functions and measures on the real line $\mathbb{R}$. For a real valued function $F(x)$, $a<x<b$, $I=[a,b]$, and a finite collection $\mathcal{P}=\{I_k\}$, $k=1,\dots,n$, of nonoverlapping subintervals $I_k=(a_k,b_k)$, $I_k \subset I$, called a *partition* in I, the F. Riesz p-variation $(p \geq 1)$ of F on $\mathcal{P}$ is the quantity

$$RV^p(F,\mathcal{P}) = \sum_{k=1}^{n} \frac{|F(b_k)-F(a_k)|^p}{(b_k-a_k)^{p-1}}. \tag{2.1}$$

If these quantities are uniformly bounded for all partitions $\mathcal{P}$ in I then the least upper bound

$$RV^p(F,I) = \sup_{\mathcal{P}} RV^p(F,\mathcal{P}) < \infty \tag{2.2}$$

(2) We use the notation $\fint_Q g\,dx = |Q|^{-1}\int_Q g\,dx$.

defines the Riesz *p-variation* of F on I. For $p > 1$, (2.2) implies that F is absolutely continuous on $[a, b]$ and, when normalized by $F(a) = 0$, has the representation

$$F(x) = \int_a^x f(t)\,dt, \quad f \in L^p(I). \tag{2.3}$$

For $p = 1$ one only concludes that F has bounded variation.

THEOREM (F. Riesz [20]). *The function F is in $W^{1,p}(I)$* (3)*, $p > 1$, iff it has finite p-variation* (2.2). *Moreover in that case*

$$RV^p(F, I) = \int_a^b |f|^p\,dx. \tag{2.4}$$

The proof of F. Riesz' theorem, though rather subtle, is very natural and transparent: it approximates the derivative $f(t)$ by piecewise constant functions

$$\varphi_n(t) = \frac{F(b_{n,k}) - F(a_{n,k})}{b_{n,k} - a_{n,k}}, \quad t \in I_{n,k} \equiv [a_{n,k}, b_{n,k}],$$

for a sequence of partitions $\mathcal{P}_n$ such that $\lim_{n\to\infty} \max_k |I_{n,k}| = 0$.

The sequence $\varphi_n(t)$ converges pointwise to $f(t)$ on a set of full Lebesgue measure in $[a, b]$. Fatou's Lemma and the estimates (2.2) give all what is needed to finish the proof. For details see [21], [24].

Formulas (2.1) and (2.2), as is well known, have a measure-theoretic interpretation [13], [21]. If dF denotes the measure $dF(I_k) = |F(b_k) - F(a_k)|$ on the real line $\mathbb{R}$, and $d\mu$ denotes the Lebesgue measure, $d\mu([a_k, b_k]) = b_k - a_k$, then (2.1) takes the form

$$RV^p(F, \mathcal{P}) = RV^p(dF, \mathcal{P}) = \sum_k \mu(I_k) \left(\frac{dF(I_k)}{d\mu(I_k)} \right)^p. \tag{2.5}$$

While the concept of Riesz p-variation (2.1)–(2.2), as it stands, does not apply in $\mathbb{R}^n$, its measure-theoretic version (2.5) may be immediately generalized to much more general situations, as we recall next.

3. p-mean variations of abstract measures and L^1-functions. The main reference for this section is the short note of L. D. Kudryavtsev [17] (4). It is convenient here to use the term *partition* $\mathcal{P} = \{E_i\}$, $i = 1, \ldots, N_{\mathcal{P}}$,

(3) Of course, for F. Riesz (1910!) the condition $F \in W^{1,p}(I)$ meant precisely that the formula (2.3) holds with $f \in L^p(a, b)$.

(4) In somewhat more general context of Banach space valued measures the corresponding theory has been initiated by Bochner & Taylor [1] and it has an extensive presentation in the monograph [9]. In [1] there is no reference to the paper of F. Riesz [23].

in a measure space (X, Σ, μ) for an arbitrary (finite) collection of disjoint μ-measurable subsets of X $(E_i \in \Sigma)$.

Let ν be a σ-finite countably additive nonnegative set function on Σ. For $p \geq 1$ and a partition $\mathcal{P}$ of E in Σ, $E = \bigcup E_i$, $\mu(E_i) > 0$, $i \leq N_{\mathcal{P}}$, the *p-mean variation* of ν on $\mathcal{P}$, denoted by $\mathsf{V}^p(\nu, E; \mathcal{P})$, is defined as

$$\mathsf{V}^p(\nu, E; \mathcal{P}) = \sum_i \mu(E_i) \left(\frac{\nu(E_i)}{\mu(E_i)} \right)^p. \tag{3.1}$$

Then the formula

$$\mathsf{V}^p(\nu, E) = \sup_{\mathcal{P}} \mathsf{V}^p(\nu, E; \mathcal{P}) \tag{3.2}$$

where the supremum is taken over all partitions $\mathcal{P} = \{E_i\}$ of E with $\mu(E_i) > 0$, defines the p-mean variation $\mathsf{V}^p(\nu, E)$ of ν on E. It is a measure on Σ, denoted by $\mathsf{V}^p(\nu)$.

Proposition 3.1 ([17]). *If $\mathsf{V}^p(\nu, E) < \infty$ and $p > 1$ then the measure ν is absolutely continuous with respect to μ on E: $\nu \ll \mu$. If $p \geq 1$, $\mathsf{V}^p(\nu, E) < \infty$ and $\nu(E') = \int_{E'} g(x)\, d\mu$ for all measurable $E' \subset E$ then $\mathsf{V}^p(\nu) \ll \mu$, $g \in L^p(E)$ and*

$$\mathsf{V}^p(\nu, E) = \int_E g^p(x)\, d\mu. \tag{3.3}$$

Proposition 3.1 has a nice formulation in terms of Radon–Nikodým derivatives ([5]):

$$\text{If } \frac{\mathcal{D}\nu}{\mathcal{D}\mu} = g(x) \text{ a.e.,} \quad \text{then } \frac{\mathcal{D}\mathsf{V}^p(\nu)}{\mathcal{D}\mu}(x) = g^p(x) \text{ a.e.} \tag{3.4}$$

This will be used in Sections 4 and 5 below.

p-mean variations of nonnegative L^1_{loc}-functions ρ are defined as p-mean variations of the associated measures $\nu(E) = \int_E \rho\, d\mu$: $\mathsf{V}^p(\rho, E) = \mathsf{V}^p(\nu, E)$. For subdomains Ω of $\mathbb{R}^n$ p-mean variations $\mathsf{V}^p(\rho, \Omega)$ can be calculated by averaging ρ over families of some standard sets—coordinate cubes, balls, dyadic cubes etc.—covering the domain Ω finely in the sense of Vitali [22, 25].

Corollary 3.1. *$\mathsf{V}^p(\rho, \Omega)$ can be calculated by the formula*

$$\mathsf{V}^p(\rho, \Omega) = \int_\Omega \rho^p\, dx = \text{l.u.b.} \sum |Q_i| \Big(\fint_{Q_i} \rho\, dx \Big)^p \tag{3.5}$$

with the supremum taken over all finite cubical partitions $\{Q_i\}$.

Corollary 3.1 illustrates the general important fact that p-mean variations of measures and functions may be controlled by restricting calculations of

([5]) We use the notation of P. Mattila [19].

averages (3.1) and (3.5) to a subclass of geometrically better organized subsets Q—cubes, balls—than arbitrary measurable subsets E_i. For these the natural dilations $Q \to \lambda Q$, λ real, make sense. Then the important coverings theorems of Vitali or Besicovitch type apply (see [10]).

4. Higher order mean variations. For a cubical partition $\mathcal{P} = \{Q_i\}$ in Ω we define the *p-mean gradient* of f on $\mathcal{P}$, $\mathsf{V}_1^p(f,\mathcal{P})$ (or p-mean 1-variation of f on $\mathcal{P}$) as

$$\mathsf{V}_1^p(f,\mathcal{P}) = \sum_{Q_i\in\mathcal{P}} \mu(Q_i)\Big(\fint_{Q_i}\fint_{Q_i} \frac{|f(x)-f(y)|^p}{|x-y|^p}\,dx\,dy\Big). \tag{4.1}$$

DEFINITION 4.1. The *p-mean* 1-*variation* (variation of order 1) of the function $f \in L^p_{\mathrm{loc}}(\Omega)$ on Ω is the least upper bound

$$\mathsf{V}_1^p(f,\Omega) = \sup \mathsf{V}_1^p(f,\mathcal{P}) \tag{4.2}$$

taken over all cubical partitions $\mathcal{P}$ in Ω.

The variation $\mathsf{V}_1^p(f,\Omega)$ and the p-mean gradient $\mathsf{V}_1^p(f,\mathcal{P})$ have analogous properties of monotonicity under refinements of partitions and lower semicontinuity for pointwise convergence $f_n \to f$ as the p-mean variations $\mathsf{V}^p(f,\Omega)$ and $\mathsf{V}^p(f,\mathcal{P})$.

The linear subspace of functions in $L^p(\Omega)$ with finite p-mean 1-variation will be denoted by $\mathsf{V}_1^p(\Omega)$. It can be made a seminormed Banach space with the seminorm

$$\|f\|_{\mathsf{V}_1^p} = [\mathsf{V}_1^p(f,\Omega)]^{1/p}.$$

As recalled in the introduction, in BHS theory the Sobolev spaces $W^{1,p}(\Omega)$ are characterized by the pointwise inequality

$$|f(x)-f(y)| \le |x-y|(a(x)+a(y)) \tag{4.3}$$

with $a = a_f \in L^p(\Omega)$ (see [4, 5, 11, 12, 2]).

The following is a direct consequence of the definition and the inequality (4.3).

PROPOSITION 4.1. *The p-mean* 1-*variation of $f \in W^{1,p}(\Omega)$ satisfying* (4.3) *is controlled by the p-mean variation of any of the mean maximal gradients of f,*

$$\mathsf{V}_1^p(f,E) \le 2^p \mathsf{V}^p(a,E), \quad E \subset \Omega. \tag{4.4}$$

In particular, for $E = \Omega$ this implies $W^{1,p}(\Omega) \subset \mathsf{V}_1^p(\Omega)$.

Proof. The proof follows from the obvious inequalities

$$(4.5)\quad \fint_E \fint_E \frac{|f(x)-f(y)|^p}{|x-y|^p}\,dx\,dy \le \fint_E \fint_E (a(x)+a(y))^p\,dx\,dy \le 2^p \fint_E a^p(x)\,dx,$$

$$(4.6)\qquad \mathsf{V}_1^1(f,\mathcal{P}) \le 2\mathsf{V}^1(a,\mathcal{P}) \quad\text{and}\quad \mathsf{V}_1^p(f,\mathcal{P}) \le 2^p\mathsf{V}^p(a,\mathcal{P}),\quad p\ge 1,$$

valid even for measurable partitions $\mathcal{P}=\{E\}$ in Ω. ∎

Other pointwise properties of Sobolev functions are recalled in

PROPOSITION 4.2. *Any real valued function f in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $p\ge 1$, satisfying* (4.3) *is approximately differentiable and has a generalized Sobolev gradient $\nabla f(x)$ at almost every point $x\in\mathbb{R}^n$. Moreover, the Sobolev gradient ∇f and the approximate gradient $\nabla_{\mathrm{app}} f(x)$ satisfy the inequality*

$$(4.7)\qquad |\nabla f|(x) = |\nabla_{\mathrm{app}} f(x))| \le C(n)a(x)\quad a.e.$$

with a universal constant C, depending on n only (*see* [12]).

The converse to Proposition 4.1 is our main result.

THEOREM 4.1. *If for a* (*real valued*) *function $f\in L^p_{\mathrm{loc}}(\Omega)$* ($p>1$) *the p-mean* 1*-variation $\mathsf{V}_1^p(f,\Omega')$ is finite for each open Ω' with compact closure in Ω* (*i.e.* $\operatorname{dist}(\Omega',\partial\Omega)>0$) *then $f\in W^{1,p}_{\mathrm{loc}}(\Omega)$. In other words, $\mathsf{V}_1^p(\Omega)\subset W^{1,p}(\Omega)$.*

Theorem 4.1 and Proposition 4.1 give the identification $\mathsf{V}_1^p(\Omega)=W^{1,p}(\Omega)$ for (smooth) subdomains Ω of $\mathbb{R}^n$, $p>1$.

Sketch of the proof of Theorem 4.1. The first ingredient of the proof is the following classical fact of integral geometry.

PROPOSITION 4.3. *Let $\varphi\in C_0^\infty(\mathbb{R}^n)$ and e be a unit vector in $\mathbb{R}^n$. Then, for every $x\in\mathbb{R}^n$,*

$$(4.8)\qquad \int_{(y-x)\cdot e\ge 0} \frac{\varphi(y)-\varphi(x)}{|y-x|}\,\rho_k(y,x)\,dy \to K\nabla_e\varphi(x),\quad K=\frac{\Gamma(\frac{n}{2})}{\pi^{1/2}\Gamma(\frac{n+1}{2})}$$

and $\rho_k(y,x)$ is any sequence of functions in $L^1(\mathbb{R}^n\times\mathbb{R}^n)$ approximating the Dirac $\delta(x-y)$. $\nabla_e\varphi(x)$ is the gradient of φ in direction e at the point x.

For the case when $\rho_k(y,x)=\rho_k(y-x)$ as in (1.3) above, a proof of Proposition 4.3 is sketched in [6].

We use Proposition 4.3 for a discrete approximation

$$(4.9)\qquad \rho_k(x,y)=\sum_{i=1}^{N_k}\frac{\chi_{Q_{i,k}}(x)\chi_{Q_{i,k}}(y)}{|Q_{i,k}|}$$

of $\delta(x-y)$ at x. For x in a model unit cube Q° of $\mathbb{R}^n$ it is defined by a sequence of cubical partitions $\mathcal{P}_k=\{Q_{i,k}\}$, shrinking to x for $k\to\infty$ (i.e. $x\in Q_{i_k(x),k}$ for all k, $\operatorname{diam} Q_{i_k(x),k}\to 0$ as $k\to\infty$).

If $\max_i \operatorname{diam}|Q_{i,k}| \le \delta$ then the support of the kernel ρ_k in (4.9) has the diagonal form

$$\bigcup_{i=1}^{N_k} Q_{i,k} \times Q_{i,k}$$

and is contained in the tubular neighborhood

$$\Delta_\delta = \{(x,y) \in Q^\circ \times Q^\circ : |x-y| < \delta'\} \quad \text{for some positive } \delta' < 2\delta \tag{4.10}$$

of the diagonal Δ of $Q^\circ \times Q^\circ$ ([6]).

Another important ingredient of the proof of Theorem 4.1 is the "discrete" integration by parts procedure which, essentially, is contained in the following formula for double integrals:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)[\varphi(x)-\varphi(y)]}{|x-y|}\,\beta(x,y)\,dy\,dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{[f(y)-f(x)]\varphi(y)\beta(x,y)}{|x-y|}\,dx\,dy. \tag{4.11}$$

As above, $f \in L^1(\mathbb{R}^n)$, $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\beta(x,y)$ is a compactly supported symmetric nonnegative kernel,

$$\beta(x,y) = \beta(y,x), \quad \beta \ge 0. \tag{4.12}$$

(4.11) is obtained by interchanging the variables x and y and using the symmetry (4.12).

Let us denote the left hand side of (4.11) by $I_L \equiv I_L(f,\varphi)$ and the right hand side by I_R. We use (4.11) by restricting interior integration (with respect to y) in I_L to the half-space $\pi_e(x)$ given by $(y-x)\cdot e \ge 0$ in $\mathbb{R}^n$ as in (4.8). Then from (4.11) we obtain the inequality

$$I_{L,\pi_e} = \left| \int_{\mathbb{R}^n}\int_{\pi_e(x)} \frac{f(x)[\varphi(x)-\varphi(y)]}{|x-y|}\,\beta(x,y)\,dy\,dx \right| \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|}{|x-y|}|\varphi(y)|\beta(x,y)\,dx\,dy. \tag{4.13}$$

For $p > 1$ and $1/p + 1/q = 1$, by the Hölder inequality we have

$$I_{L,\pi_e} \le \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^p}\,\beta(x,y)\,dx\,dy \Big)^{1/p} \times \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\varphi(y)|^q \beta(x,y)\,dx\,dy \Big)^{1/q}. \tag{4.14}$$

([6]) The converse does not hold and to correct this technical defect we should introduce the multiple partitions, $2\mathcal{P} = \{2Q_i\}$, or more generally, $\lambda\mathcal{P}$, $\lambda > 1$.

We apply (4.14) to a sequence $\beta_k(x,y)$ of "discrete" kernels $\beta_k(x,y) = \rho_{k,\mathcal{P}}(x,y)$ (4.9) for a sequence of partitions $\mathcal{P}_k$ shrinking to x or approximating the Dirac δ at x as described above.

We have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^p}\,\beta_k(x,y)\,dx\,dy = \sum_{i=1}^{N} \mu(Q_i) \fint_{Q_i}\fint_{Q_i} \frac{|f(x)-f(y)|^p}{|x-y|^p}\,dx\,dy \tag{4.15}$$

and

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\varphi(y)|^q \beta_k(x,y)\,dx\,dy = \sum_{i=1}^{N} \mu(Q_i) \fint_{Q_i}\Big(\fint_{Q_i} |\varphi(y)|^q\,dy\Big)\,dx. \tag{4.16}$$

In view of (3.1), (4.2), the inequality (4.14) takes the form

$$I_{L,\pi_e}(f,\varphi,\beta_k) \le (\mathsf{V}_1^p(f,\mathcal{P}))^{1/p}(\mathsf{V}^q(\varphi,\mathcal{P}))^{1/q}. \tag{4.17}$$

Finally, by definition of the mean variations V_1^p and $\mathsf{V}^q(\varphi)$ we get from (4.17) and Proposition 4.3 for $k\to\infty$ the fundamental inequality for $f\in L^p(\mathbb{R}^n)$ and $\varphi\in C_0^\infty(\mathbb{R}^n)$,

$$\Big|\int_{\mathbb{R}^n} f(x)\nabla_e\varphi(x)\,dx\Big| \le (\mathsf{V}_1^p(f))^{1/p}\|\varphi\|_{L^q}. \tag{4.18}$$

By definition of weak Sobolev derivatives (see [25]) this ends our sketch of the proof of Theorem 4.1. ■

Finally, let us observe the pointwise inequality

$$\frac{\mathcal{D}\mathsf{V}_1^p(f,E)}{\mathcal{D}\mu}(x) \le 2^p a^p(x) \quad \text{a.e.} \tag{4.19}$$

obtained by "differentiating " (4.4) and using Proposition 3.1 (formula (3.4)).

A challenging problem is to recover the "optimal" maximal mean gradient of the function f from its 1-variation $\mathsf{V}_1^p(f,E)$ as the Radon–Nikodým derivative, i.e. "inversion" of the inequality (4.19).

5. Final remarks and acknowledgements. Analogous ideas and concepts are applicable for Sobolev spaces $W^{m,p}(\Omega)$ of higher order $m>1$. In the formulas (4.1)–(4.4) above we then have to use the Taylor remainder terms $R^{m-1}f(x;y)$ instead of $R^0 f(x;y) = f(x)-f(y)$.

The condition (1.5) takes the form

$$\fint_{\Omega}\fint_{\Omega} \frac{|R^{m-1}f(x;y)|^p}{|x-y|^{mp}}\,dx\,dy < \infty.$$

The p-mean variations of order $m > 1$, $\mathsf{V}_m^p(f, \mathcal{P})$ and $\mathsf{V}_m^p(f, \Omega)$, are defined in the same way as for $m = 1$.

In BBM theory and above, the classical duality type argument of S. L. Sobolev plays a crucial role. As is well known ([2, 4, 5, 12]), in BHS theory in $\mathbb{R}^n$ the use of convolution approximations $f_\varepsilon(x) = f * \phi_\varepsilon$ is very convenient. It can also be applied in BBH theory and in the estimates of p-variations $\mathsf{V}_m^p(f, \mathbb{R}^n)$, $m > 1$. However, the precise calculations do not seem to have been elaborated in detail, as yet.

The concept of 1-mean variations for Sobolev functions also reveals some probabilistic aspects of the Sobolev theory. In particular, random selections of partitions $\mathcal{P}$ in Ω during the evaluation procedures for the p-mean variations $\mathsf{V}_1^p(f, \mathcal{P})$ resemble the use of Monte-Carlo methods for numerical estimation of Sobolev norms.

Let me also point out the interesting paper of Y. T. Medvedev [20] generalizing the F. Riesz theorem to Orlicz type spaces L^φ, φ increasing, convex. Extension of this idea is an interesting research topic.

It would also be useful to understand better the role of fractional maximal functions in the above theories [16].

Apparently somehow related to the variations $\mathsf{V}_m^p(f, \Omega)$, $m \geq 1$, is the interesting theory of p-variations of Yu. A. Brudnyi [7, 8] ([7]). In this connection I want to thank S. K. Vodopyanov for pointing out to me the recent papers of Yu. A. Brudnyi.

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