MEASURE AND INTEGRATION

The Young Measure Representation for Weak Cluster Points of Sequences in $M$-spaces of Measurable Functions

by

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Summary. Let $\langle X, Y \rangle$ be a duality pair of $M$-spaces $X, Y$ of measurable functions from $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^d$. The paper deals with $Y$-weak cluster points $\phi$ of the sequence $\phi(\cdot, z_j(\cdot))$ in $X$, where $z_j : \Omega \to \mathbb{R}^m$ is measurable for $j \in \mathbb{N}$ and $\phi : \Omega \times \mathbb{R}^m \to \mathbb{R}^d$ is a Carathéodory function. We obtain general sufficient conditions, under which, for some negligible set $A_\phi$, the integral $I(\phi, \nu_x) := \int_{\mathbb{R}^m}^{\nu} \phi(x, \lambda) \, d\nu_x(\lambda)$ exists for $x \in \Omega \setminus A_\phi$ and $\phi(x) = I(\phi, \nu_x)$ on $\Omega \setminus A_\phi$, where $\nu = \{\nu_x\}_{x \in \Omega}$ is a measurable-dependent family of Radon probability measures on $\mathbb{R}^m$.

1. Notations and some basic facts on Young measures. Let $\mu$ denote a complete separable $\sigma$-finite $\sigma$-additive positive measure on a $\sigma$-algebra $\mathfrak{A}$ of subsets of a set $\Omega$. Measurability will always mean $\mathfrak{A}$-measurability. Let $E$ be a separable Banach space. We will denote by $L^\infty(\Omega, E; \mu)$, or briefly $L^\infty(E)$, the Banach space (of all equivalence classes) of essential $E$-norm-bounded measurable functions $u : \Omega \to E$ with norm $\|u\|_{L^\infty} := \text{ess sup}_{x \in \Omega} \|u(x)\|_E$. Let $L^1(\Omega, E; \mu)$, or briefly $L^1(E)$, denote the Bochner–Lebesgue space (of all equivalence classes) of $\mu$-integrable strongly measurable functions from $\Omega$ into $E$.

Let $\mathcal{M}(\mathbb{R}^m)$ be the Banach space of bounded signed Radon measures on $\mathbb{R}^m$ and $C_0(\mathbb{R}^m)$ be the Banach space of all continuous functions $f : \mathbb{R}^m \to \mathbb{R}$ with $\lim_{|\lambda| \to \infty} f(\lambda) = 0$ equipped with the sup-norm, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^m$. It is known that $(C_0(\mathbb{R}^m))^* \cong \mathcal{M}(\mathbb{R}^m)$. Let $L^\infty(\mathcal{M}(\mathbb{R}^m))$ denote the Banach space (of all equivalence classes) of $C_0(\mathbb{R}^m)$-valued measurable functions.

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weakly measurable functions \( \nu : \Omega \to \mathcal{M}(\mathbb{R}^m) \) with norm \( \|\nu\|_{\infty} := \|x \mapsto |\nu_x|(\mathbb{R}^m)|_{L^\infty} < \infty \), where \( |\nu_x|(\mathbb{R}^m) \) is the total variation of \( \nu_x \) on \( \mathbb{R}^m \) and, for abbreviation, we write \( L^\infty \) (resp. \( \nu_x \)) instead of \( L^\infty(\mathbb{R}) \) (resp. \( \nu(x) \)). It is known that \( L^\infty_\infty(\mathcal{M}(\mathbb{R}^m)) \) can be interpreted as dual space \( (L^1(C_0(\mathbb{R}^m)))^* \) via the injection \( \nu \mapsto \langle \cdot, \nu \rangle \mu \), where \( \langle h, \nu \rangle \mu := \int_\Omega \langle v(x), h(x) \rangle d\mu(x) \) for all \( h \in L^1(C_0(\mathbb{R}^m)) \). Given a measurable function \( z : \Omega \to \mathbb{R}^m \), define the \textit{parametrized Dirac measure} \( \delta_z \in L^\infty_\infty(\mathcal{M}(\mathbb{R}^m)) \) by

\[
x \in \Omega \mapsto \delta_z(x) := \delta_{z(x)} \quad \text{(the Dirac measure supported at } z(x) \text{).}
\]

An element \( \nu \in L^\infty_\infty(\mathcal{M}(\mathbb{R}^m)) \) is called a Young parametrized measure if \( \nu_x(\mathbb{R}^m) = 1 \) \( \mu \)-a.e. Define \( (\phi \circ z)(x) := \phi(z(x)) \). A function \( f : \Omega \times \mathbb{R}^m \to E \) is said to be Carathéodory if \( f(\cdot, u) \) is measurable for every \( u \in \mathbb{R}^m \) and \( f(x, \cdot) \) is continuous for almost all \( x \in \Omega \).

The formulations and proofs of the main results of the present paper are based on the following fundamental theorem [2, 3] about the Young measure representation in case of the pair \( (X, Y) = (L^1(\mathbb{R}), L^\infty(\mathbb{R})) \) (see Theorem 1.1; cf. [20, p. 98–100], [8, Section 8.1, pp. 518–525], [5, 21]).

**Theorem 1.1** (The Young measure representation; Ball [3], Balder [2]). Suppose that a sequence of measurable functions \( z_j : \Omega \to \mathbb{R}^m \) satisfies the global tightness condition with respect to \( \mu \):

\[
(\text{GB}) \quad \lim_{L \to \infty} \sup_{j \in \mathbb{N}} \mu\{x \in \Omega : |z_j(x)| \geq L\} = 0.
\]

Then there exist a subsequence \( z_{j_k} \) and a Young measure \( \nu = \{\nu_x\}_{x \in \Omega} \) such that \( \delta_{z_{j_k}} \) is \( L^1(C_0(\mathbb{R}^m)) \)-weakly convergent to \( \nu \) in \( L^\infty_\infty(\mathcal{M}(\mathbb{R}^m)) \). Moreover, given a Carathéodory function \( \psi : \Omega \times \mathbb{R}^m \to \mathbb{R} \), the following statements hold.

1. **(Y1)** If \( \psi \circ z_{j_k} \) is \( L^\infty(\mathbb{R}) \)-weakly convergent to \( \bar{\psi} \) in \( L^1(\mathbb{R}) \), then, for some \( \mu \)-negligible set \( A_\psi \subset \mathcal{A} \), the integral \( \int_{\mathbb{R}^m} \psi(x, \lambda) d\nu_x(\lambda) \in \mathbb{R} \) exists for \( x \in \Omega \setminus A_\psi \) and

\[
(1.1) \quad \bar{\psi}(x) = \int_{\mathbb{R}^m} \psi(x, \lambda) d\nu_x(\lambda) \quad \text{on } \Omega \setminus A_\psi.
\]

2. **(Y2)** If \( \psi \circ z_{j_k} \) is sequentially \( L^\infty(\mathbb{R}) \)-weakly pre-compact in \( L^1(\mathbb{R}) \), then, for some \( \mu \)-negligible set \( A_\psi \subset \mathcal{A} \), the integral \( I(\psi, \nu_x) := \int_{\mathbb{R}^m} \psi(x, \lambda) d\nu_x(\lambda) \in \mathbb{R} \) exists for all \( x \in \Omega \setminus A_\psi \) and \( \psi \circ z_{j_k} \) is \( L^\infty(\mathbb{R}) \)-weakly convergent to \( \tilde{\psi} \in L^1(\mathbb{R}) \), where \( \tilde{\psi}(x) := I(\psi, \nu_x) \) for \( x \in \Omega \setminus A_\psi \) and \( \tilde{\psi}(x) := 0 \) otherwise.

The generalization of Theorem 1.1 for the \( L^{\Psi^*}(\mathbb{R}) \)-weak limit of \( \tau \circ z_{j_k} \) in the Orlicz space \( L^{\Psi}(\mathbb{R}) \) is proved by P. Málek et al. [11, Th. 4.2.1, pp. 171–176] in the case when \( \Psi \) and \( \Psi^* \) are complementary non-power Orlicz functions, \( \Psi \) satisfies the \( \triangle_2 \)-condition [12], and \( \tau : \mathbb{R}^m \to \mathbb{R} \) is continuous.
2. Formulation of results. A linear space $Z \subset L^0(\mathbb{R}^m)$ is called an $M$-space if the inclusions $z \in Z$ and $\alpha \in L^\infty(\mathbb{R})$ imply that $\alpha z \in Z$ [16, 18]. If $m = 1$ then it is easy to check that $M$-spaces $Z$ are just vector lattices. The Köthe associate space $Z'$ of an $M$-space $Z$ is defined e.g. in [7, 9] for $m = 1$, and in [15, 14, 18] for $m \geq 2$. By [16, Theorem 3.1], equivalently in case $m \geq 2$, $Z'$ is defined by

$$Z' = \{ z' \in L^0(\mathbb{R}^m) : z'(x) \in \text{vsupp} Z(x) \text{ } \mu\text{-a.e.}, \langle z, z' \rangle_\mu \in \mathbb{R}, \forall z \in Z \}.$$ 

Here $\langle z, z' \rangle_\mu := \int_{\Omega}(z(x), z'(x))d\mu(x)$, where $(\cdot, \cdot)$ denotes the Euclidean scalar product on $\mathbb{R}^m$, and the so-called vector support vsupp $Z$ can be equivalently defined by

$$\text{vsupp} Z(x) := \{ z_1(x), z_2(x), \ldots \} \text{ } \mu\text{-a.e.}$$

for some sequence $z_n \in Z$ such that $z \in Z \Rightarrow z(x) \in \{ z_1(x), z_2(x), \ldots \} \text{ } \mu\text{-a.e.}$ If $Z, Y \subset L^0(\mathbb{R}^m)$ are $M$-spaces and $Y \subset Z'$, then $\langle Z, Y \rangle$ is a duality pair with respect to $(z, z')_\mu (z \in Z, z' \in Y)$, and we write $\langle Z, Y \rangle_\mu$.

Let $\langle Z, Y \rangle$ be a duality pair of vector spaces. A set $\mathcal{N} \subset Z$ is called sequentially $Y$-weakly pre-compact in $Z$ (or conditionally sequentially $Y$-weakly compact in $Z$) if each sequence $z_j \in \mathcal{N}$ has some $Y$-weak Cauchy subsequence $z_{j(k)}$. The space $Z$ is called sequentially $Y$-weakly complete if each $Y$-weak Cauchy sequence is $Y$-weakly convergent in $Z$.

**Theorem 2.1.** Let $X, Y \subset L^0(\mathbb{R}^d)$ be $M$-spaces, supp $X = \Omega$, vsupp $X(x) = \text{vsupp} Y(x) \text{ } \mu\text{-a.e.}$, and $Y \subset X'$, where $X'$ is the Köthe associate space of $X$ with respect to $\mu$. Suppose that a sequence $z_j \in L^0(\mathbb{R}^m)$ satisfies (GB) with respect to $\mu$, and a Carathéodory function $\phi : \Omega \times \mathbb{R}^m \to \mathbb{R}^d$ satisfies $\phi(x, \mathbb{R}^m) \subset \text{vsupp} X(x) \text{ } \mu\text{-a.e.}$ Moreover, let $z_{j(k)}$ and $\nu$ be as in Theorem 1.1. Then the following statements hold.

(Y3) If $\phi \circ z_{j(k)}$ is $Y$-weakly convergent to $\phi$ in $X$, then, for some $\mu$-negligible set $A_\phi \in \mathcal{A}$, the integral $\int_{\mathbb{R}^m} \phi(x, \lambda) d\nu_x(\lambda)$ exists in vsupp $X(x)$ for $x \in \Omega \setminus A_\phi$ and

\begin{equation}
\overline{\phi}(x) = \int_{\mathbb{R}^m} \phi(x, \lambda) d\nu_x(\lambda) \quad \text{on } \Omega \setminus A_\phi.
\end{equation}

(Y4) If $X = Y'$ and $\phi \circ z_{j(k)}$ is sequentially $Y$-weakly pre-compact in $X$, then, for some $\mu$-negligible set $A_\phi \in \mathcal{A}$, the integral $I(\phi, \nu_x) := \int_{\mathbb{R}^m} \phi(x, \lambda) d\nu_x(\lambda)$ exists in vsupp $X(x)$ for all $x \in \Omega \setminus A_\phi$ and $\phi \circ z_{j(k)}$ is $Y$-weakly convergent to $\phi$ in $X$, where $\phi(x) := I(\phi, \nu_x)$ for $x \in \Omega \setminus A_\phi$ and $\overline{\phi}(x) := 0$.

**Condition 2.2** (Local tightness condition, [11, p. 171], [20]). A sequence $z_j \in L^0(\mathbb{R}^m)$ satisfies
(LB) \[ \limsup_{L \to \infty} \mu \{ x \in C_q : |z_j(x)| \geq L \} = 0 \quad (\forall q \in \mathbb{N}) \]

for a nondecreasing sequence \( C_q \in \mathcal{A} \) with \( \mu(C_q) < \infty \) and \( \bigcup_{q \in \mathbb{N}} C_q = \Omega \).

**Theorem 2.3.** Let \( \mu(\Omega) = \infty \) and let \( X, Y \) and \( \phi \) be as in Theorem 2.1. If a sequence \( z_j \in L^0(\mathbb{R}^m) \) satisfies (LB) with respect to \( \mu \), then the statements (Y3)–(Y4) of Theorem 2.1 remain true.

A normed space \( Z \subseteq L^0(\mathbb{R}^m) \) with norm \( \| \cdot \|_Z \) is called a normed M-space if the inclusions \( z \in Z \) and \( \alpha \in L^\infty(\mathbb{R}) \) imply that \( \alpha z \in Z \) and \( \|\alpha z\|_Z \leq \|\alpha\|_{L^\infty} \|z\|_Z \) [16, 18]. The regular part \( Z^\circ \) of a normed M-space \( Z \) is defined to be the normed M-subspace of all elements \( z \in Z \) satisfying \( \lim_{\mu_* \circ (D) \to 0} \|\chi_D z\|_Z = 0 \), where \( \mu_* := \mu \) if \( \mu(\Omega) < \infty \) and \( \mu_* \) is a fixed finite positive measure equivalent to \( \mu \) if \( \mu(\Omega) = \infty \), and \( \chi_D \) denotes the characteristic function of \( D \in \mathcal{A} \).

**Proposition 2.4.** Let \( X, Y \subseteq L^0(\mathbb{R}^d) \) be M-spaces with \( X \subseteq Y' \), where \( Y' \) is the Köthe associate space of \( Y \) with respect to \( \mu \). Suppose that a sequence \( z_j \in L^0(\mathbb{R}^m) \) and a Carathéodory function \( \phi : \Omega \times \mathbb{R}^m \to \mathbb{R}^d \) satisfy one of the following conditions:

1. (SC1) There exist nondecreasing continuous functions \( g, \gamma : [0, \infty) \to [0, \infty) \) such that
   - (a) \( \lim_{t \to \infty} g(t) = \infty \) and \( \lim_{t \to \infty} \gamma(t)/g(t) = 0 \);
   - (b) \( \{(g \circ z_j)u_0\}_{j \in \mathbb{N}} \) is \( Y \)-weakly bounded in \( X \), where \( u_0 : \Omega \to (0, \infty) \) is measurable, \( u_0 Y \subseteq L^1(\mathbb{R}^d) \), and \( v\text{supp}(X(\cdot)) = v\text{supp}(Y(\cdot)) \) \( \mu \)-a.e.;
   - (c) \( |\phi(x, \lambda)| \leq \gamma(|\lambda|)u_0(x) \) for \( \mu \)-almost all \( x \in \Omega \) and all \( \lambda \in \mathbb{R}^m \);
2. (SC2) There exists a Banach M-space \( \Gamma \) with \( Y \subseteq \Gamma^\circ \), \( (\Gamma^\circ)'^\prime \subseteq X \) and \( \sup_{j \in \mathbb{N}} \|\phi \circ z_j\|_{(\Gamma^\circ)'} < \infty \).

Then the sequence \( \phi \circ z_j \) is sequentially \( Y \)-weakly pre-compact in \( X \).

**Remark 2.5.** Proposition 2.4/(SC1) is a generalization of [20, Proposition 6.5] (where \( Y = L^1(\mathbb{R}) \) with \( \mu(\Omega) < \infty \)).

In the case of \( \phi : \Omega \times \mathbb{R}^m \to E \) with \( \dim E = \infty \), results analogous to Theorems 2.1 and 2.3 can be proved but only for a pair \( (X, Y) \) of Köthe–Bochner spaces \( X, Y \) of \( E/-/E^* \)-valued functions (see Theorem 2.6). Given a separable Banach space \( E \) and a vector lattice \( K \subseteq L^0(\mathbb{R}) \), the Köthe–Bochner space \( K(E) \) is defined as the space (of equivalence classes) of strongly measurable \( E \)-valued functions \( z \) such that \( \|z(\cdot)\|_E \in K \).

**Theorem 2.6.** Let \( K, \tilde{K} \subseteq L^0(\mathbb{R}) \) be vector lattices, \( E \) be a Banach space and \( E^* \) be its dual. Assume that:
(a) \( \text{supp } K = \text{supp } \tilde{K} = \Omega \) and \( \tilde{K} \subset K' \), where \( K' \) is the Köthe associate space of \( K \) with respect to \( \mu \);
(b) \( E \) is separable and reflexive with \( \dim E = \infty \).

If \( \phi : \Omega \times \mathbb{R}^m \to E \) is a Carathéodory function and a sequence \( z_j \in L^0(\mathbb{R}^m) \) satisfies either (GB) or (LB), then the statements (Y3)–(Y4) of Theorem 2.1 remain true for the Köthe–Bochner spaces \( X = K(E) \) and \( Y = \tilde{K}(E^*) \) provided (2.1) (resp. \( \tilde{\phi} \)) is substituted by

\[
(2.2) \quad \tilde{\phi}(x) = (P)- \int_{\mathbb{R}^m} \phi(x, \lambda) \, d\nu_x(\lambda) \quad \text{on } \Omega \setminus A_\phi
\]

(resp. \( \hat{\phi}(x) = (P)- \int_{\mathbb{R}^m} \phi(x, \lambda) \, d\nu_x(\lambda) \) for \( x \in \Omega \setminus A_\phi \)), where, for \( x \in \Omega \setminus A_\phi \), the above integral exists as the Pettis integral of the function \( \phi(x, \cdot) : \mathbb{R}^m \to E \) with respect to the measure \( \nu_x \).

**Proposition 2.7.** Let \( z_j \in L^0(\mathbb{R}^m) \) \( (j \in \mathbb{N}) \). Then (LB) with respect to \( \mu \) follows from the condition:

(LK) For \( q \in \mathbb{N} \) there exist a normed lattice with monotone norm \( K(q) \subset L^0(\mathbb{R}) \) and a continuous nondecreasing function \( g_q : [0, \infty) \to [0, \infty) \)

such that \( \lim_{t \to \infty} g_q(t) = \infty \) and \( \sup_{j \in \mathbb{N}} \| \chi_{C_q} g_q(|z_j(\cdot)|) \|_{K(q)} < \infty \) for a nondecreasing sequence \( C_q \in \mathcal{A} \) with \( \mu(C_q) < \infty \) and \( \bigcup_{q \in \mathbb{N}} C_q = \Omega \).

**Remark 2.8.** Proposition 2.7 is an extension of the statement in [3, Remark 1, p. 209] (where \( K(q) = L^1(\mathbb{R}) \)).

**Remark 2.9.** If \( Z \subset L^0(\mathbb{R}^m) \) is a normed \( M \)-space and \( \sup_{j \in \mathbb{N}} \| z_j \|_Z < \infty \), then (LB) holds. Indeed, by [9, Corollary of Theorem IV.3.1], [23] \( (m = 1) \) and [16, Theorem 2.1/(3)] \( (m \geq 2) \), the sequence \( z_j \) is bounded in \( L^0(\mathbb{R}^m) \) equipped with the quasi-norm \( \| z \|_{L^0(\mathbb{R}^m)} := \int_{\Omega} \frac{|z(x)|}{1 + |z(x)|} \, d\mu(x) \).

Hence, by [9, Section III.1.3–III.1.4], this sequence is bounded in \( \mu \) on any \( C_q \), and so (LB) follows. In particular, \( Z \) can be assumed to be either a Banach lattice of scalar-valued functions (a solid space) or a non-solid generalized Orlicz space (see, e.g., [1, 12, 17]) of \( \mathbb{R}^m \)-valued functions with \( m \geq 2 \).

### 3. Proofs of results of Section 2

**Proof of Theorem 2.1.** We divide this proof into Steps 3.1–3.2.

**Step 3.1 (Proof of (Y3)).** Given \( y \in Y \), define \( \phi_y : \Omega \times \mathbb{R}^m \to \mathbb{R} \) by \( \phi_y(x, \lambda) := (y(x), \phi(x, \lambda)) \). As \( Y \) is an \( M \)-space we have \( \alpha y \in Y \) for every \( \alpha \in L^\infty(\mathbb{R}) \), and from \( Y \subset X' \) we infer that

\[
\langle \phi \circ z_{jk}, \alpha y \rangle_\mu = \langle \phi_y \circ z_{jk}, \alpha \rangle_\mu \in \mathbb{R}.
\]

By Theorem 1.1/(Y2) for \( \phi_y \) together with the assumption for \( \phi \circ z_{jk} \), we
deduce that
\[ \langle \phi \circ z_{jk}, \alpha y \rangle_{\mu} = \langle \phi_y \circ z_{jk}, \alpha \rangle_{\mu} \rightarrow \langle \tilde{\phi}, \alpha y \rangle_{\mu} = \langle \tilde{\phi}_y, \alpha \rangle_{\mu} \in \mathbb{R} \]
for all \( \alpha \in L^\infty(\mathbb{R}) \), where, for some \( \tilde{D}_{\phi_y} \in \mathfrak{A} \) with \( \mu(\Omega \setminus \tilde{D}_{\phi_y}) = 0 \), the integral
\[ \int_{\mathbb{R}^m} \phi_y(x, \lambda) \, d\nu_x(\lambda) \in \mathbb{R} \]
exists for \( x \in \tilde{D}_{\phi_y} \), and \( \phi_y(x) := \int_{\mathbb{R}^m} \phi_y(x, \lambda) \, d\nu_x(\lambda) \)
for \( x \in \tilde{D}_{\phi_y} \) and \( \tilde{\phi}_y(x) := 0 \) otherwise. Hence,
\[ \langle \tilde{\phi}, \chi_{D}y \rangle_{\mu} = \langle \tilde{\phi}_y, \chi_{D} \rangle_{\mu} = \int_{D} \left[ \int_{\mathbb{R}^m} (y(x), \phi(x, \lambda)) \, d\nu_x(\lambda) \right] \, d\mu(x) \in \mathbb{R} \quad (D \in \mathfrak{A}, D \subseteq \tilde{D}_{\phi_y}). \]

On the other hand, \( \langle \tilde{\phi}, \chi_{DY}y \rangle_{\mu} = \int_{D} (y(x), \tilde{\phi}(x)) \, d\mu(x) \) for any \( D \in \mathfrak{A} \) with \( D \subseteq \tilde{D}_{\phi_y} \). By the Radon–Nikodym theorem, we deduce that for \( y \in Y \) there exists \( D_{\phi_y} \in \mathfrak{A} \) such that \( D_{\phi_y} \subseteq \tilde{D}_{\phi_y} \), \( \mu(\tilde{D}_{\phi_y} \setminus D_{\phi_y}) = 0 \), and
\[ (y(x), \tilde{\phi}(x)) = \int_{\mathbb{R}^m} (y(x), \phi(x, \lambda)) \, d\nu_x(\lambda) \in \mathbb{R} \quad (\forall x \in D_{\phi_y}). \]

Now, we consider \( X \subseteq L^0(\Omega, \mathbb{R}^d) \) and \( Y \subseteq X' \) for \( d > 1 \) (the case \( d = 1 \) can be handled analogously upon using [9, Corollary IV.3.2] for \( \text{supp} Y = \text{supp} X = \Omega \)). By [16, Theorem 3.1], there exists a sequence of representative families \( G_q = \{u_{1q}, \ldots, u_{dq}\} \) of the \( M \)-space \( Y \) such that the sets \( \text{supp} G_q \in \mathfrak{A} \) are mutually disjoint, and
\[ \begin{align*}
(1) \quad & \mu(\text{supp} Y \setminus \bigcup_{q=1}^\infty \text{supp} G_q) = 0; \\
(2) \quad & |u_{1q}(x)| = \cdots = |u_{dq}(q)(x)| = 1 \text{ and } |u_{iq}(x)| = 0 \quad (i \notin \{1, \ldots, d(q)\}) \text{ for } x \in \text{supp} G_q \text{ and } d(q) = \dim \text{vsupp} Y(x) \text{ on } \text{supp} G_q.
\end{align*} \]

By the definition [16] of the representative family \( G_q \), we have \( u_{iq} \in Y \) and the linear hull of \( \{u_{1q}(x), \ldots, u_{dq}(x)\} \) coincides with \( \text{vsupp} Y(x) \) for \( x \in \text{supp} G_q \). Hence, by (3.1), for \( \chi_{\text{supp} G_q} u_{pq} \in Y \) (1 \( \leq p \leq d(q) \)) there exists \( D_{pq} \in \mathfrak{A} \) such that \( D_{pq} \subseteq \text{supp} G_q \), \( \mu(\text{supp} G_q \setminus D_{pq}) = 0 \), and
\[ \langle \chi_{\text{supp} G_q} (x) u_{pq}(x), \tilde{\phi}(x) \rangle = \int_{\mathbb{R}^m} \langle \chi_{\text{supp} G_q} (x) u_{pq}(x), \phi(x, \lambda) \rangle \, d\nu_x(\lambda) \in \mathbb{R} \]
for \( x \in D_{pq} \). By the assumption, there exists \( D_0 \in \mathfrak{A} \) with \( \mu(\Omega \setminus D_0) = 0 \) such that \( \tilde{\phi}(x), \phi(x, \lambda) \in \text{vsupp} X(x) = \text{vsupp} Y(x) \) for all \( x \in D_0 \) and for all \( \lambda \in \mathbb{R}^m \). Hence, for \( x \in D_0 \cap \bigcap_{p=1}^{d(q)} D_{pq} \) and 1 \( \leq p \leq d(q) \), the integral \( \int_{\mathbb{R}^m} \phi(x, \lambda) \, d\nu_x(\lambda) \) exists in the finite-dimensional Euclidean space \( \text{vsupp} Y(x) = \text{vsupp} X(x) \) and
\[ \langle u_{pq}(x), \tilde{\phi}(x) \rangle = \left( u_{pq}(x), \int_{\mathbb{R}^m} \phi(x, \lambda) \, d\nu_x(\lambda) \right) \in \mathbb{R}. \]
Therefore,
\[
\overline{\phi}(x) = \int_{\mathbb{R}^m} \phi(x, \lambda) \, d\nu_x(\lambda) \in \text{v supp } X(x)
\]
for \( x \in D\phi := \bigcup_{q=1}^{\infty} [D_0 \cap \bigcap_{p=1}^{d(q)} D_{pq}] \), and \( \mu(\Omega \setminus D\phi) = 0 \). Hence the statement (Y3) follows for \( A\phi := \Omega \setminus D\phi \).

**Step 3.2 (Proof of (Y4)).** Observe that, as \( X = Y' \), there exist a subsequence \( j(k) \) of \( j_k \) and \( \tilde{\phi} \in X \) such that \( \phi \circ z_{j(k)} \) is \( Y \)-weakly convergent to \( \tilde{\phi} \) in \( X \), due to the \( Y \)-weak completeness theorem of J. Diuonné [7] (if \( X \) is a normed lattice with \( Y = X' \)); W. Luxemburg and A. Zaanen [10], P. P. Zabrejko [23, Theorem 32] (if \( X \) is a normed lattice); H. Nakano [13] \((d = 1 \text{ with } Y = X') \); O. Burkinshaw and P. Dodds [4, Corollary 4.2 of Theorem 4.1] \((d = 1 \text{ and } \text{[15, Theorem 2.8/(1)], [18] (d \geq 2)})\).

By Theorem 2.1/(Y3) applied to \( \phi \circ z_{j(k)} \), we can find \( A\phi \in \mathfrak{A} \) such that \( \mu(A\phi) = 0 \) and the integral \( \int_{\mathbb{R}^m} \phi(x, \lambda) \, d\nu_x(\lambda) \) exists in \( \text{v supp } X(x) \) for \( x \in \Omega \setminus A\phi \).

We proceed to show that \( \phi \circ z_{j_k} \) is \( Y \)-weakly convergent to \( \tilde{\phi} \) in \( X \), where \( \tilde{\phi}(x) := \int_{\mathbb{R}^m} \phi(x, \lambda) \, d\nu_x(\lambda) \) for \( x \in \Omega \setminus A\phi \) and \( \tilde{\phi}(x) := 0 \) otherwise.

On the contrary, suppose that \( \phi \circ z_{j_k} \) is not \( Y \)-weakly convergent to \( \tilde{\phi} \) in \( X \). Then there exist \( \varepsilon > 0, h_0 \in Y \) and a subsequence \( q_k \) of \( j_k \) such that \( |\langle \phi \circ z_{q_k}, h_0 \rangle - \langle \tilde{\phi}, h_0 \rangle| > \varepsilon > 0 \). By the above \( Y \)-weak completeness theorem together with Theorem 2.1/(Y3), for the sequence \( \phi \circ z_{q_k} \), we can find a subsequence \( i_k \) of \( q_k \), \( \hat{\phi} \in X \) and \( A\hat{\phi} \in \mathfrak{A} \) such that \( \langle \phi \circ z_{i_k}, h \rangle \to \langle \hat{\phi}, h \rangle (\forall h \in Y) \), \( \mu(A\hat{\phi}) = 0 \), the integral \( \int_{\mathbb{R}^m} \phi(x, \lambda) \, d\nu_x(\lambda) \) exists in \( \text{v supp } X(x) \) for \( x \in \Omega \setminus A\hat{\phi} \), and \( \hat{\phi}(x) = \int_{\mathbb{R}^m} \phi(x, \lambda) \, d\nu_x(\lambda) \) on \( \Omega \setminus A\hat{\phi} \). Therefore, \( \hat{\phi} \) and \( \tilde{\phi} \) define the same element (equivalence class) in \( X \), and \( \langle \hat{\phi}, h_0 \rangle = \langle \tilde{\phi}, h_0 \rangle \). Hence, we get a contradiction.

**Proposition 3.1 ([15, Lemma 4.2.2]).** Let \( \mu(\Omega) = \infty \). Then, for a sequence \( z_j \in L^0(\mathbb{R}^m) \), the condition (LB) holds with respect to \( \mu \) if and only if the condition (GB) holds with respect to \( \mu_* \).

**Proof of Theorem 2.3.** By Proposition 3.1, (LB) for \( \mu \) and \( z_j \) implies (GB) for \( \mu_* \) and \( z_j \). So, we may apply Theorem 2.1 for \( z_j \) with respect to \( \mu_* \). Recall that if \( \mu(\Omega) = \infty \) then the measure \( \mu \) is called separable (see [9, 23]) provided \( \mu_* \) is separable, which is equivalent to separability of \( L^0(\mathbb{R}^m) \). We divide the proof into Steps 3.3–3.4.

**Step 3.3 (Proof of (Y3)).** Denote by \( \alpha_* \in L^1((0, \infty)) \) the Radon–Nikodym derivative \( d\mu_* / d\mu \). Define
\[
\tilde{Y} := \{ z^* : \alpha_* z^* \in Y \},
\]
\[
\tilde{Y}_{\mu_*} := \{ z \in L^0(\mathbb{R}^m) : z(x) \in \text{v supp } \tilde{Y}(x) \text{ \( \mu_* \)-a.e., } \langle z, z^* \rangle_{\mu_*} \in \mathbb{R}, \forall z^* \in \tilde{Y} \},
\]
where \( \langle z, z' \rangle_{\mu_*} := \int_{\Omega} (z(x), z'(x)) d\mu_*(x) \). Then \( \tilde{Y}_{\mu_*}' \) is in fact the Köthe associate space of \( \tilde{Y} \) with respect to \( \mu_* \). Observe that, for \( \alpha_* z' = z' \in Y \),
\[
\langle z, z' \rangle_{\mu_*} = \int_{\Omega} (z(x), z'(x)/\alpha_*(x)) d\mu(x) = \langle z, z' \rangle_{\mu}.
\]

As \( \zeta \in L^1(\Omega, C_0(\mathbb{R}^m); \mu) \) if and only if \( \tilde{\zeta} := \zeta/\alpha_* \in L^1(\Omega, C_0(\mathbb{R}^m); \mu_*), \) we have
\[
\langle \nu, \tilde{\zeta} \rangle_{\mu_*} := \int_{\Omega} \int_{\mathbb{R}^m} \tilde{\zeta}(x, \lambda) d\nu_x(\lambda) d\mu_*(x) = \langle \nu, \zeta \rangle_{\mu}.
\]
Hence, \( \delta_{z_{jk}} \) is \( L^1(\Omega, C_0(\mathbb{R}^m); \mu_*) \)-weakly convergent to \( \nu \) in \( L^\infty_\omega(\Omega, M(\mathbb{R}^m); \mu_*) \) and \( \phi \circ z_{jk} \) is \( \tilde{Y} \)-weakly convergent to \( \tilde{\phi} \) in \( X \) with respect to the duality pair \( \langle X, \tilde{Y} \rangle_{\mu_*} \). By Theorem 2.1/(Y3), there exists \( A_\phi \in \mathfrak{A} \) such that \( \mu_*(A_\phi) = 0 \), the integral \( \int_{\mathbb{R}^m} \phi(x, \lambda) d\nu_x(\lambda) \) exists in \( \text{vsupp} \ X(x) \) for all \( x \in \Omega \setminus A_\phi \), and (2.1) holds for all \( x \in \Omega \setminus A_\phi \). As \( \mu \) is equivalent to \( \mu_* \), we see that \( \mu(A_\phi) = 0 \).

**Step 3.4 (Proof of (Y4)).** Observe that \( X = Y' \) implies \( X = \tilde{Y}_{\mu_*}' \). Since the sequence \( \phi \circ z_{jk} \) is sequentially \( Y \)-weakly pre-compact in \( X \), we conclude that \( \phi \circ z_{jk} \) is sequentially \( \tilde{Y} \)-weakly pre-compact in \( X \) with respect to the duality pair \( \langle X, \tilde{Y} \rangle_{\mu_*} \). By Theorem 2.1/(Y4), there exists \( A_\phi \in \mathfrak{A} \) such that \( \mu_*(A_\phi) = 0 \), the integral \( \int_{\mathbb{R}^m} \phi(x, \lambda) d\nu_x(\lambda) \) exists in \( \text{vsupp} \ X(x) \) for all \( x \in \Omega \setminus A_\phi \), and \( \phi \circ z_{jk} \) is \( \tilde{Y} \)-weakly convergent to \( \tilde{\phi} \) in \( X \) with respect to \( \langle \cdot, \cdot \rangle_{\mu_*} \), where \( \tilde{\phi}(x) := \int_{\mathbb{R}^m} \phi(x, \lambda) d\nu_x(\lambda) \) for \( x \in \Omega \setminus A_\phi \) and \( \tilde{\phi}(x) := 0 \) otherwise. Since \( \mu \) is equivalent to \( \mu_* \), we conclude that \( \mu(A_\phi) = 0 \) and \( \phi \circ z_{jk} \) is \( Y \)-weakly convergent to \( \tilde{\phi} \) in \( X \).

**Proof of Proposition 2.4.** We divide this proof into Steps 3.5–3.6.

**Step 3.5.** Assume that (SC1) holds. We claim that the sequence \( \phi \circ z_j \) is \( Y \)-absolutely bounded in \( X \), i.e.

\[
(3.2) \quad y \in Y \Rightarrow \lim_{\mu_*(D) \to 0} \sup_{j \in \mathbb{N}} \int_D |(y(x), (\phi \circ z_j)(x))| d\mu(x) = 0,
\]

\[
\sup_{j \in \mathbb{N}} \int_{\Omega} |(y(x), (\phi \circ z_j)(x))| d\mu(x) < \infty.
\]

Indeed, we deduce that
\[
\int_D |(y(x), (\phi \circ z_j)(x))| d\mu(x) \leq \int_D |y(x)| \gamma(|z_j(x)|) u_0(x) d\mu(x) \leq \left( \int_{D \cap \{ \gamma(|z_j(\cdot)|) \leq l \}} |y(x)| \gamma(|z_j(x)|) u_0(x) d\mu(x) + \int_{D \cap \{ \gamma(|z_j(\cdot)|) \geq l \}} |y(x)| \gamma(|z_j(x)|) u_0(x) d\mu(x) \right)
\]

\[
\leq l \int_D |y(x)| u_0(x) d\mu(x) + \int_{\{ \gamma(|z_j(\cdot)|) \geq l \}} |y(x)| \gamma(|z_j(x)|) u_0(x) d\mu(x).
\]
Since $\gamma$ is nondecreasing, we can choose $m_l \to \infty$ such that $\{t \geq 0 : \gamma(t) \geq l\} \subset \{t \geq 0 : t \geq m_l\}$. Then
\[
\int_{\{\gamma(|z_j(\cdot)|) \geq l\}} |y(x)|\gamma(|z_j(x)|)u_0(x) \, d\mu(x) \\
\leq \int_{\{\gamma(|z_j(\cdot)|) \geq m_l\}} |y(x)|\gamma(|z_j(x)|)u_0(x) \, d\mu(x) \\
\leq \frac{1}{M_l} \int_{\{\gamma(|z_j(\cdot)|) \geq m_l\}} |y(x)|g(|z_j(x)|)u_0(x) \, d\mu(x) \\
\leq \frac{1}{M_l} \int_\Omega |y(x)|g(|z_j(x)|)u_0(x) \, d\mu(x) \leq \frac{C}{M_l} \to 0
\]
as $l \to \infty$ uniformly in $j$, where $C \in (0, \infty)$, $g(t) \geq M_l \gamma(t)$ for $t \geq m_l$, and $M_l \to \infty$ as $l \to \infty$. Hence, for any $\varepsilon > 0$ there exists $l_0$ such that
\[
\int_{\{\gamma(|z_j(\cdot)|) \geq l_0\}} |y(x)|\gamma(|z_j(x)|)u_0(x) \, d\mu(x) \leq \varepsilon \quad \forall j \in \mathbb{N}.
\]
As $y \in Y$ and $u_0Y \subset L^1(\mathbb{R}^d)$ we have $\lim_{\mu_*(D) \to 0} \int_D |y(x)|u_0(x) \, d\mu(x) = 0$. Therefore, there exists $\delta > 0$ such that $\mu_*(D) < \delta$ implies
\[
\int_D |y(x)u_0(x)| \, d\mu(x) \leq \frac{\varepsilon}{l_0}.
\]
Hence, we infer that
\[
\mu_*(D) < \delta \Rightarrow \int_D |(y(x), (\phi \circ z_j)(x))| \, d\mu(x) \leq l_0 \frac{\varepsilon}{l_0} + \varepsilon = 2\varepsilon.
\]
So, the first part of (3.2) follows. The second part of (3.2) follows by the same arguments.

Since $\text{vsupp} X(x) = \text{vsupp} Y(x)$ $\mu$-a.e. and $X \subset Y'$, (3.2) implies that the sequence $\phi \circ z_{j_k}$ is sequentially $Y$-weakly pre-compact in $X$, due to the $Y$-weak pre-compactness theorem of J. Dieudonné [7] (if $X$ is a normed lattice with $X = X'', Y = X'$); W. Luxemburg and A. Zaanen [10], P. P. Zabrejko [23, Theorem 33] (if $X$ is a normed lattice); H. Nakano [13] ($m = 1$ with $X = X'', Y = X''$); O. Burkinshaw and P. Dodds [4, Theorem 3.4, Proposition 2.4] ($m = 1$), and [15, Theorem 2.8/(2)], [18] ($m \geq 2$).

**Step 3.6.** Assume that (SC2) holds. It is known that $(\Gamma^\infty)'$ can be interpreted as the dual space $(\Gamma^\infty)^*$ by the injection $z' \mapsto \langle \cdot, z' \rangle_{\mu}$ (see, e.g., [1, 23], [9, Theorems VI.1.4 and IV.3.6] ($d = 1$), [15, Corollary 2.2, Proposition 2.2], [18] ($d \geq 2$)). By [9, Theorem IV.3.3] ($m = 1$) and [16, Theorem 2.5], [15, 18] ($m \geq 2$), the separability of $\mu$ implies the separability of $\Gamma^\infty$. Hence, by the Alaoglu–Bourbaki theorem together with [9, Theorem V.7.6], the $\Gamma^\infty$-weak
topology on any closed ball of \((\Gamma^o)^*\) is compact and metrizable. Therefore, for any sequence \(a_i\) in the \((\Gamma^o)'\)-norm-bounded set \(\{\phi \circ z_{jk}\}_{k \in \mathbb{N}}\) there exist a subsequence \(p(i)\) of the sequence \(i\) and \(a \in (\Gamma^o)'\) such that \(a_{p(i)}\) is \((\Gamma^o)'\)-weakly convergent to \(a\) in \((\Gamma^o)'\). Since \(Y \subset \Gamma^o\) and \((\Gamma^o)' \subset X\), \(a_{p(i)}\) is \((\Gamma^o)'\)-weakly convergent to \(a\) in \(X\). Hence, \(a_{p(i)}\) is a \(Y\)-weak Cauchy sequence in \(X\). Thus, the statement of Proposition 2.4/(SC2) follows.

Proof of Theorem 2.6. It suffices to modify Step 3.1 of the proof of Theorem 2.1/(Y3). Since \(\text{supp} \tilde{K} = \Omega\), by [9, Corollary IV.3.2] there exists a sequence of disjoint sets \(\Omega_q \in \mathfrak{A}\) such that \(\chi_{\Omega_q} \in \tilde{K}\) and \(\mu(\Omega \setminus \bigcup_{q=1}^{\infty} \Omega_q) = 0\). Since \(E\) is a separable reflexive space, so is \(E^*\). Hence, there exists \(\{\tilde{u}_p\}_{p \in \mathbb{N}}\) dense in \(E^*\). By (3.1) for \(\chi_{\Omega_q} \tilde{u}_p \in Y\), for some \(\tilde{D}_{pq} \in \mathfrak{A}\), \(\tilde{D}_{pq} \subset \Omega_q\) and \(\mu(\Omega_q \setminus \tilde{D}_{pq}) = 0\) and \(\langle \chi_{\Omega_q}(x)\tilde{u}_p, \phi(x, \lambda) \rangle = \int_{\mathbb{R}^m} \langle \chi_{\Omega_q}(x)\tilde{u}_p, \phi(x, \lambda) \rangle \, d\nu_x(\lambda) \in \mathbb{R}\) for \(x \in \tilde{D}_{pq}\). Therefore, for \(x \in \bigcap_{p \in \mathbb{N}} \tilde{D}_{pq}\),
\[
\langle \tilde{u}_p, \phi(x) \rangle = \int_{\mathbb{R}^m} \langle \tilde{u}_p, \phi(x, \lambda) \rangle \, d\nu_x(\lambda) \in \mathbb{R}.
\]

Put \(\psi(x, \lambda) := \|\phi(x, \lambda)\|_E\). Since \(\text{supp} K = \text{supp} \tilde{K} = \Omega\) and the sequence \(\phi \circ z_{jk}\) is \(\tilde{K}(E^*)\)-weakly pre-compact in \(K(E)\), by M. Talagrand [22, Corollary 9 of Theorem 6] and M. Nowak [19, Theorem 3.3] we deduce that the sequence \(\psi \circ z_{jk}\) is \(\tilde{K}\)-weakly pre-compact in \(K\). By Theorem 2.1/(Y4) for \(\psi \circ z_{jk}\), there exists \(D_\psi \in \mathfrak{A}\) such that \(\mu(\Omega \setminus D_\psi) = 0\) and the integral \(\int_{\mathbb{R}^m} \psi(x, \lambda) \, d\nu_x(\lambda) \in \mathbb{R}\) exists for all \(x \in D_\psi\).

Fix \(u^* \in E^*\). Then we can choose a sequence \(\hat{u}_i := \tilde{u}_{p(i)}\) from the dense set \(\{\tilde{u}_p\}_{p \in \mathbb{N}}\) with \(\|\hat{u}_i - u^*\|_{E^*} \to 0\) as \(i \to \infty\). Hence, \(x \in \bigcap_{p=1}^{\infty} \tilde{D}_{pq} \cap D_\psi\) implies that \(\langle \hat{u}_i, \phi(x, \lambda) \rangle \to \langle u^*, \phi(x, \lambda) \rangle\) for all \(\lambda \in \mathbb{R}^m\), \(\langle \hat{u}_i, \phi(x) \rangle \to \langle u^*, \phi(x) \rangle\), and \(\|\hat{u}_i, \phi(x, \lambda)\| \leq \sup_{i \in \mathbb{N}} \|\hat{u}_i\|_{E^*} \psi(x, \lambda) < \infty\). Hence, by the Lebesgue dominated convergence theorem, we infer that
\[
\int_{\mathbb{R}^m} \langle \hat{u}_i, \phi(x, \lambda) \rangle \, d\nu_x(\lambda) \to \int_{\mathbb{R}^m} \langle u^*, \phi(x, \lambda) \rangle \, d\nu_x(\lambda) \in \mathbb{R}
\]
as \(i \to \infty\) for \(x \in \bigcap_{p=1}^{\infty} \tilde{D}_{pq} \cap D_\psi\). Hence, \(x \in \bigcap_{p=1}^{\infty} \tilde{D}_{pq} \cap D_\psi\) implies that \(\langle u^*, \phi(x) \rangle = \int_{\mathbb{R}^m} \langle u^*, \phi(x, \lambda) \rangle \, d\nu_x(\lambda) \in \mathbb{R}\) for all \(u^* \in E^*\). Therefore, for \(x \in \bigcap_{p=1}^{\infty} \tilde{D}_{pq} \cap D_\psi\), the Pettis integral \((P)\)-\(\int_{\mathbb{R}^m} \phi(x, \lambda) \, d\nu_x(\lambda) \in E\) exists and coincides with \(\overline{\phi}(x)\) [6, p. 53]. So, we obtain
\[
\left( x \in D_\phi := \bigcup_{q=1}^{\infty} \bigcap_{p=1}^{\infty} \tilde{D}_{pq} \cap D_\psi \right) \Rightarrow \overline{\phi}(x) = (P)-\int_{\mathbb{R}^m} \phi(x, \lambda) \, d\nu_x(\lambda) \in E,
\]
and \(\mu(\Omega \setminus D_\phi) = 0\). Hence, the statement (Y3) of Theorem 2.6 follows for \(A_\phi := \Omega \setminus D_\phi\). □
Hence, \( \lim \parallel \chi_{C_q} g_q(z_j(\cdot)) \parallel_{K(q)} \leq \parallel \chi_{C_q \cap D^j_q} g_q(z_j(\cdot)) \parallel_{K(q)} \geq \parallel \chi_{C_q \cap D^j_q} g_q(L) \parallel_{K(q)} = g_q(L) \parallel \chi_{C_q \cap D^j_q} L \parallel_{K(q)}. \)

Hence, \( \lim_{L \to \infty} \sup_{j \in \mathbb{N}} \parallel \chi_{C_q \cap D^j_q} \parallel_{K(q)} = 0. \) By Lemma 3.2, for all \( \varepsilon > 0 \) there exists \( r_q(\varepsilon) > 0 \) such that, given \( j \in \mathbb{N} \), if \( \parallel \chi_{C_q \cap D^j_q} \parallel_{K(q)} \leq r(\varepsilon) \) then \( \parallel \chi_{C_q \cap D^j_q} \parallel_{L^0(\Omega, \mathbb{R})} = \frac{1}{2} \mu_*(C_q \cap D^j_q) \leq \varepsilon. \) Therefore, there exists \( L^q_\varepsilon \) such that \( L \geq L^q_\varepsilon \) implies that \( \parallel \chi_{C_q \cap D^j_q} \parallel_{K(q)} \leq r(\varepsilon) \) for all \( j \in \mathbb{N}. \) It follows that \( \frac{1}{2} \mu_*(C_q \cap D^j_q) \leq \varepsilon \) for all \( j \in \mathbb{N} \) and all \( L \geq L^q_\varepsilon. \) This gives (GB) for \( \mu_* \) and \( z_j \) on \( C_q \subset \Omega. \) By Proposition 3.1, (LB) follows for \( \mu \) and \( z. \)

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120 H. T. Nguyêñ and D. Pączka


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