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GENERAL TOPOLOGY

Infinite-Dimensionality modulo Absolute Borel Classes

by

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Dedicated to Professor Tetsuo Furumochi on his sixtieth birthday

Summary. For each ordinal $1 \leq \alpha < \omega_1$ we present separable metrizable spaces X_α, Y_α and Z_α such that

(i) $\mathrm{f}\, X_\alpha$, $\mathrm{f}\, Y_\alpha$, $\mathrm{f}\, Z_\alpha = \omega_0$, where f is either trdef or $\mathcal{K}_0$-trsur,
(ii) $A(\alpha)$-trind $X_\alpha = \infty$ and $M(\alpha)$-trind $X_\alpha = -1$,
(iii) $A(\alpha)$-trind $Y_\alpha = -1$ and $M(\alpha)$-trind $Y_\alpha = \infty$, and
(iv) $A(\alpha)$-trind $Z_\alpha = M(\alpha)$-trind $Z_\alpha = \infty$ and $A(\alpha+1) \cap M(\alpha+1)$-trind $Z_\alpha = -1$.

We also show that there exists no separable metrizable space W_α with $A(\alpha)$-trind $W_\alpha \neq \infty$, $M(\alpha)$-trind $W_\alpha \neq \infty$ and $A(\alpha) \cap M(\alpha)$-trind $W_\alpha = \infty$, where $A(\alpha)$ (resp. $M(\alpha)$) is the absolutely additive (resp. multiplicative) Borel class.

1. Introduction. All topological spaces in this paper are assumed to be separable metrizable, and all classes of topological spaces are assumed to be non-empty (the empty space $\emptyset$ is a member of each class), and to contain every space homeomorphic to a closed subspace of each of their members (one says that the class is *monotone with respect to closed subspaces*). The letter $\mathcal{P}$ is used to denote such a class. Our terminology mostly follows [1] and [3].

In [4] Lelek introduced the *small inductive dimension modulo a class* $\mathcal{P}$, $\mathcal{P}$-ind, a natural generalization of the small inductive dimension ind and the small inductive compactness degree cmp. Namely, for a space X one defines

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(i) $\mathcal{P}\text{-ind}\, X = -1$ iff $X \in \mathcal{P}$,

(ii) $\mathcal{P}\text{-ind}\, X \leq n$, where n is an integer ≥ 0, if for each point $x \in X$ and each closed subset A of X with $x \notin A$ there exists a partition C in X between x and A such that $\mathcal{P}\text{-ind}\, C < n$.

Recall that a subset C of a space X is said to be a *partition* between disjoint sets A and B in X if there are disjoint open subsets U and V of X such that $A \subset U$, $B \subset V$ and $C = X \setminus (U \cup V)$.

It is evident that if $\mathcal{P} = \{\emptyset\}$ (resp. $\mathcal{P}$ is the class of compact spaces) then $\mathcal{P}\text{-ind}\, X = \operatorname{ind} X$ (resp. $\mathcal{P}\text{-ind}\, X = \operatorname{cmp} X$). Moreover, if $\mathcal{P}_2 \subset \mathcal{P}_1$ then $\mathcal{P}_1\text{-ind}\, X \leq \mathcal{P}_2\text{-ind}\, X$. In particular, $\operatorname{cmp} X \leq \operatorname{ind} X$.

Recall (cf. [1]) that the *absolutely additive* (resp. *multiplicative* and *ambiguous*) Borel classes $A(0), \ldots, A(\alpha), \ldots$ (resp. $M(0), \ldots, M(\alpha), \ldots$ and $A(0) \cap M(0), \ldots, A(\alpha) \cap M(\alpha), \ldots$), where $0 \leq \alpha < \omega_1$, satisfy the conditions above (see Section 3 for details). In the universe of separable metrizable spaces, $A(0) = \{\emptyset\}$, $M(0)$ is the class $\mathcal{K}_0$ of compact metrizable spaces, $A(1)$ is the class $\mathcal{S}_0$ of σ-compact separable metrizable spaces, and $M(1)$ is the class $\mathcal{C}_0$ of separable completely metrizable spaces (cf. [1]).

Moreover, the following *hierarchy of absolute Borel classes* holds (the arrows indicate inclusions of classes):

$$\begin{tikzcd}
 & & & A(1) = \mathcal{S}_0 \arrow[dr] & & A(2)\ \cdots \arrow[dr] & \\
A(0) = \{\emptyset\} \arrow[r] & M(0) = \mathcal{K}_0 \arrow[r] & A(1) \cap M(1) \arrow[ur] \arrow[dr] & & A(2) \cap M(2) \arrow[ur] \arrow[dr] & & \mathcal{AB} \\
 & & & M(1) = \mathcal{C}_0 \arrow[ur] & & M(2)\cdots \arrow[ur] &
\end{tikzcd}$$

where $\mathcal{AB} = \bigcup\{A(\alpha) : \alpha < \omega_1\}$.

It is well known that the Hilbert cube $\mathbb{I}^\infty$ has $\operatorname{trind} \mathbb{I}^\infty = \infty$, where trind is the small transfinite inductive dimension, a natural transfinite extension of ind. Evidently, $\operatorname{trcmp} \mathbb{I}^\infty = -1$. Hence from the hierarchy it follows that all other small transfinite dimensions of $\mathbb{I}^\infty$ modulo absolute Borel classes $\mathcal{P} \neq \{\emptyset\}$ are equal to -1.

In [5] E. Pol defined the *small transfinite inductive compactness degree*, trcmp, a natural transfinite extension of cmp, and constructed a separable completely metrizable σ-compact space E such that $\operatorname{trcmp} E = \infty$. Note that $A(1) \cap M(1)\text{-trind}\, E = -1$. Hence by the hierarchy we have $\operatorname{trind} E = \infty$, and all other small transfinite dimensions of E modulo absolute Borel classes $\mathcal{P} \supset \mathcal{K}_0$ are equal to -1.

In [2] Charalambous suggested considering a natural transfinite extension of $\mathcal{P}$-ind, $\mathcal{P}$-trind, a generalization of both trind and trcmp (see Section 2 for the definition). Now the problem naturally arises about analogs of the spaces $\mathbb{I}^\infty$ and E for all small transfinite dimensions modulo absolute Borel classes different from trind and trcmp.

Recall that a space Y is a *$\mathcal{P}$-hull* (resp. *a $\mathcal{P}$-kernel*) of a space X if $X \subset Y$ (resp. $Y \subset X$) and $Y \in \mathcal{P}$. As in [2] the *small transfinite $\mathcal{P}$-deficiency* and the *small transfinite $\mathcal{P}$-surplus* of a space X are defined by

$$\mathcal{P}\text{-trdef}\, X = \min\{\text{trind}(Y \setminus X) : Y \text{ is a } \mathcal{P}\text{-hull of } X\},$$
$$\mathcal{P}\text{-trsur}\, X = \min\{\text{trind}(X \setminus Y) : Y \text{ is a } \mathcal{P}\text{-kernel of } X\},$$

respectively. Evidently, the functions $\mathcal{P}$-trdef and $\mathcal{P}$-trsur are transfinite extensions of the functions $\mathcal{P}$-def and $\mathcal{P}$-sur from [1]. Observe (cf. [1]) that for $\mathcal{P} = \mathcal{K}_0$ the function $\mathcal{P}$-def is the *compact deficiency*, def. We will denote the transfinite extension $\mathcal{K}_0$-trdef of def by trdef. Note that if $\mathcal{P}_2 \subset \mathcal{P}_1$ then $\mathcal{P}_1$-trdef $X \le \mathcal{P}_2$-trdef X and $\mathcal{P}_1$-trsur $X \le \mathcal{P}_2$-trsur X.

Using an idea of E. Pol, Charalambous [2] presented a space C such that $\mathcal{C}_0$-trdef $C = \omega_0$ and $\mathcal{C}_0$-trind $C = \infty$. This example showed that the Aarts equality $\mathcal{C}_0$-def $X = \mathcal{C}_0$-ind X (valid for each space X) cannot be extended to the transfinite case. Recall (cf. [1]) the equalities $M(\alpha)$-def $X = M(\alpha)$-ind X and $A(\alpha)$-sur $X = A(\alpha)$-ind X, which hold for each ordinal $1 \le \alpha < \omega_1$ and each space X. So the problem arises about extending Charalambous' result to all absolute Borel classes.

The main result of this paper answers the above problems as follows.

THEOREM 1.1. *For each ordinal $1 \le \alpha < \omega_1$ there exist spaces X_α, Y_α, Z_α such that*

(i) $\mathrm{f}\, X_\alpha$, $\mathrm{f}\, Y_\alpha$, $\mathrm{f}\, Z_\alpha = \omega_0$, *where* f *is either* trdef *or* $\mathcal{K}_0$-trsur,
(ii) $A(\alpha)$-trind $X_\alpha = \infty$ *and* $M(\alpha)$-trind $X_\alpha = -1$,
(iii) $A(\alpha)$-trind $Y_\alpha = -1$ *and* $M(\alpha)$-trind $Y_\alpha = \infty$,
(iv) $A(\alpha)$-trind $Z_\alpha = M(\alpha)$-trind $Z_\alpha = \infty$ *and* $A(\alpha+1) \cap M(\alpha+1)$-trind $Z_\alpha = -1$,

but there exists no space W_α such that

$$A(\alpha)\text{-trind}\, W_\alpha \ne \infty,$$
$$M(\alpha)\text{-trind}\, W_\alpha \ne \infty,$$
$$A(\alpha) \cap M(\alpha)\text{-trind}\, W_\alpha = \infty.$$

REMARK 1.1. For each space X and each $1 \le \alpha < \omega_1$ we have:

(i) if $M(\alpha)$-trdef $X \le \omega_0$ and $M(\alpha)$-trind $X = \infty$ (resp. -1), then $M(\alpha)$-trdef $X = \omega_0$ (resp. -1),
(ii) if $A(\alpha)$-trsur $X \le \omega_0$ and $A(\alpha)$-trind $X = \infty$ (resp. -1), then $A(\alpha)$-trsur $X = \omega_0$ (resp. -1).

Hence we know additionally that $M(\alpha)$-trdef $X_\alpha = A(\alpha)$-trsur $Y_\alpha = -1$ and $A(\alpha)$-trsur $X_\alpha = M(\alpha)$-trdef $Y_\alpha = M(\alpha)$-trdef $Z_\alpha = A(\alpha)$-trsur $Z_\alpha = \omega_0$.

THEOREM 1.2. *There exists a space X with* trdef $X = \mathcal{K}_0$-trsur $X = \omega_0$ *such that* $\mathcal{AB}$-trind $X = \infty$.

REMARK 1.2. By the hierarchy we have B-trind $X = \infty$ for each absolute Borel class B and D-trdef $X =$ D-trsur $X = \omega_0$ for each absolute Borel class D except $A(0)$.

2. Small transfinite inductive dimension modulo a class $\mathcal{P}$. Let X be a space and α be either an ordinal ≥ 0 or the integer -1.

Recall ([2]) that the *small transfinite inductive dimension modulo a class* $\mathcal{P}$, $\mathcal{P}$-trind, of X is defined as follows:

(i) $\mathcal{P}$-trind $X = -1$ iff $X \in \mathcal{P}$.
(ii) $\mathcal{P}$-trind $X \leq \alpha$ (≥ 0) if for every point $x \in X$ and every closed subset A of X such that $x \notin A$ there exists a partition C in X between x and A with $\mathcal{P}$-trind $C < \alpha$.
(iii) $\mathcal{P}$-trind $X = \alpha$ if $\mathcal{P}$-trind $X \leq \alpha$ and $\mathcal{P}$-trind $X > \beta$ for every ordinal $\beta < \alpha$.
(iv) $\mathcal{P}$-trind $X = \infty$ if $\mathcal{P}$-trind $X > \alpha$ for every ordinal α.

Note that $\{\emptyset\}$-trind $=$ trind and $\mathcal{K}_0$-trind $=$ trcmp. Some other known functions are $\mathcal{S}_0$-trind $= \mathcal{S}$-trind, the *small transfinite inductive σ-compactness degree*, and $\mathcal{C}_0$-trind $=$ tricd, the *small transfinite inductive completeness degree* ([2]). The following relationships between particular cases of $\mathcal{P}$-trind are evident.

PROPOSITION 2.1.

(i) $\mathcal{P}_1$-trind $= \mathcal{P}_2$-trind *iff* $\mathcal{P}_1 = \mathcal{P}_2$.
(ii) *If* $\mathcal{P}_2 \subset \mathcal{P}_1$ *then* $\mathcal{P}_1$-trind $\leq \mathcal{P}_2$-trind.
(iii) *If* $X \in \mathcal{P}$ *then* $\mathcal{P}_1$-trind $X = \mathcal{P} \cap \mathcal{P}_1$-trind X *for every class* $\mathcal{P}_1$.

Note that trind $\geq \mathcal{P}$-trind for every class $\mathcal{P}$, and trcmp $\geq \max\{\mathcal{S}$-trind, tricd$\}$. Observe also that the function $\mathcal{P}$-trind is monotone with respect to closed subsets.

3. Absolute Borel classes. Recall that every ordinal α can be represented as $\alpha = \lambda(\alpha) + n(\alpha)$, where $\lambda(\alpha)$ is a limit ordinal or 0, and $n(\alpha)$ is an integer ≥ 0. An ordinal α is called *even* (resp. *odd*) if $n(\alpha)$ is even (resp. odd). As in [1] let us denote by $\mathcal{A}_\sigma$ (resp. $\mathcal{A}_\delta$) the family of all countable unions (resp. intersections) of elements from a family $\mathcal{A}$ of sets.

Let $\mathcal{B}(X)$ be the family of Borel subsets of a space X. This family can be generated by an inductive transfinite process (cf. [1]). For each countable ordinal $\alpha \geq 0$ the *Borel class* $F_\alpha(X)$ (resp. $G_\alpha(X)$) is defined transfinitely as follows.

(i) $F_0(X)$ (resp. $G_0(X)$) is the family of all closed (resp. open) subsets of X;
(ii) if α is odd then $F_\alpha(X) = (\bigcup\{F_\beta(X) : \beta < \alpha\})_\sigma$ (resp. $G_\alpha(X) = (\bigcup\{G_\beta(X) : \beta < \alpha\})_\delta$);
(iii) if α is even then $F_\alpha(X) = (\bigcup\{F_\beta(X) : \beta < \alpha\})_\delta$ (resp. $G_\alpha(X) = (\bigcup\{G_\beta(X) : \beta < \alpha\})_\sigma$).

Notice that $\mathcal{B}(X) = \bigcup\{F_\alpha(X) : \alpha < \omega_1\} = \bigcup\{G_\alpha(X) : \alpha < \omega_1\}$. It is also clear that $A \in F_\alpha(X)$ iff $X \setminus A \in G_\alpha(X)$ (briefly, $F_\alpha(X) = \neg G_\alpha(X)$), and if $Z \subset Y \subset X$ then $Z \in F_\alpha(Y)$ (resp. $G_\alpha(Y)$) iff there exists a subset $Z' \subset X$ such that $Z = Z' \cap Y$ and $Z' \in F_\alpha(X)$ (resp. $G_\alpha(X)$).

Recall that for each ordinal $0 \leq \alpha < \omega_1$ the *multiplicative* (resp. *additive*) *Borel class* α, $\Pi^0_\alpha(X)$ (resp. $\Sigma^0_\alpha(X)$) of a space X is the family F_α for α even (resp. odd) and G_α for α odd (resp. even); the *ambiguous class* α, $\Delta^0_\alpha(X)$, of X is $\Pi^0_\alpha(X) \cap \Sigma^0_\alpha(X)$. Some properties of multiplicative, additive and ambiguous classes of a space X can be found in the next statement.

Proposition 3.1 ([8, Proposition 3.6.1]).

(i) *The additive* (*resp. multiplicative*) *classes are closed under countable unions* (*resp. intersections*).
(ii) *The additive, multiplicative and ambiguous classes are closed under finite intersections and finite unions.*
(iii) *The following hierarchy of Borel sets holds* (*the arrows indicate inclusions of families*):

$$\begin{tikzcd}
 & \Sigma^0_0(X) \arrow[dr] & & \Pi^0_1(X) \arrow[dr] & & \Pi^0_2(X)\cdots \\
\Delta^0_0(X) \arrow[ur] \arrow[dr] & & \Delta^0_1(X) \arrow[ur] \arrow[dr] & & \Delta^0_2(X) \arrow[ur] \arrow[dr] & \\
 & \Pi^0_0(X) \arrow[ur] & & \Sigma^0_1(X) \arrow[ur] & & \Sigma^0_2(X)\cdots
\end{tikzcd}$$

(iv) *For each* $0 \leq \alpha < \omega_1$, $\Pi^0_\alpha(X) = \neg\Sigma^0_\alpha(X)$ *and* $\Delta^0_\alpha(X)$ *is an algebra.*
(v) *For each* $0 < \alpha < \omega_1$, $\Sigma^0_\alpha(X) = (\Delta^0_\alpha(X))_\sigma$ *and* $\Pi^0_\alpha(X) = (\Delta^0_\alpha(X))_\delta$.

We need two more facts about the Borel sets.

Proposition 3.2 ([8, Corollary 3.6.8]). *Let X be an uncountable Polish space and $0 \leq \alpha < \omega_1$. Then there exists an element E of $\Sigma^0_\alpha(X)$ which is not in $\Pi^0_\alpha(X)$.*

Note that then $X \setminus E \in \Pi^0_\alpha(X) \setminus \Sigma^0_\alpha(X)$. Observe that both E and $X \setminus E$ are in $\Delta^0_{\alpha+1}(X)$.

Proposition 3.3 ([8, Theorem 5.2.11]). *Let X, Y be compact metric spaces and $f : X \to Y$ a continuous onto mapping. Suppose $A \subset Y$ and $0 \le \alpha < \omega_1$. Then $A \in \Pi^0_\alpha(Y)$ iff $f^{-1}(A) \in \Pi^0_\alpha(X)$.*

We also notice that if $B \subset Y$ and $0 \le \alpha < \omega_1$ then $B \in \Sigma^0_\alpha(Y)$ (resp. $\Delta^0_\alpha(Y)$) iff $f^{-1}(B) \in \Sigma^0_\alpha(X)$ (resp. $\Delta^0_\alpha(X)$).

Following [2] we call a subset A of a space X a *Bernstein set* if $|A \cap B| = |(X \setminus A) \cap B| = c$ (continuum) for every uncountable $B \in \mathcal{B}(X)$. Let $\mathrm{Brn}(X)$ denote the family of all Bernstein sets of X. Notice that $\mathrm{Brn}(X) \neq \emptyset$ for every uncountable Polish space X. Indeed, recall (cf. [8, Theorem 3.2.7]) that every uncountable Borel subset B of X contains a copy of the Cantor set C. Note that C is homeomorphic to C^2. So B contains c disjoint copies of C. One can show as in [8, Example 3.2.8] that X contains a subset A such that $A \cap F$ and $(X \setminus A) \cap F$ are uncountable for each uncountable closed set F in X. So $|A \cap B| = |(X \setminus A) \cap B| = c$. Hence $A \in \mathrm{Brn}(X)$.

Note that if $M \in \mathrm{Brn}(X)$ then $|M| = c$, $X \setminus M \in \mathrm{Brn}(X)$ and $M \notin \mathcal{B}(X)$.

Recall that a space X is said to be *absolutely of multiplicative* (resp. *additive*) *class* α, where $0 \le \alpha < \omega_1$, if X is of multiplicative (resp. additive) Borel class α in Y whenever X is a subspace of a space Y (that is, for any homeomorphic embedding $h : X \to Y$ the image $h(X)$ is of multiplicative (resp. additive) class α in Y). As in [1] let us denote the absolutely multiplicative (resp. additive) Borel class α by $M(\alpha)$ (resp. $A(\alpha)$). For each $0 \le \alpha < \omega_1$ the intersection $M(\alpha) \cap A(\alpha)$ is called the *ambiguous absolutely Borel class* α.

Proposition 3.4 ([1, Theorem II.9.6 and Corollary II.9.7]).

(i) $A(0) = \{\emptyset\}$, $M(0) = \mathcal{K}_0$, $A(1) = \mathcal{S}_0$ *and* $M(1) = \mathcal{C}_0$.
(ii) *For every α with $2 \le \alpha < \omega_1$ a space X is in $M(\alpha)$ (resp. $A(\alpha)$) iff there is a homeomorphic embedding $h : X \to Y$ with $Y \in \mathcal{C}_0$ such that $h(X)$ is of multiplicative (resp. additive) class α in Y.*

It is evident that the classes $M(\alpha), A(\alpha)$ and $A(\alpha) \cap M(\alpha)$, $0 \le \alpha < \omega_1$, are monotone with respect to closed subsets. A class $\mathcal{P}$ of topological spaces is said to be *finitely additive* if each space X, covered by a finite family of elements of $\mathcal{P}$, is also an element of $\mathcal{P}$.

Proposition 3.5 ([1, Theorem II.9.9]). *For each ordinal $\alpha < \omega_1$ the classes $A(\alpha)$, $M(\alpha)$ and $A(\alpha) \cap M(\alpha)$ are finitely additive.*

We will call a space X *absolute Borel* if X is in $A(\alpha)$ (or $M(\alpha)$) for some $\alpha < \omega_1$. We denote by $\mathcal{AB}$ the class of all absolute Borel spaces.

A simple corollary of Propositions 3.1 and 3.4 is the hierarchy of absolute Borel classes in the universe of separable metrizable spaces from the introduction. Let $\mathbb{I}$ denote the closed interval $[0,1]$, Q_1 the space of rational numbers in $\mathbb{I}$, and P_1 the space of irrational numbers in $\mathbb{I}$. Observe that $Q_1 \in \mathcal{S}_0 \setminus \mathcal{C}_0$ and $P_1 \in \mathcal{C}_0 \setminus \mathcal{S}_0$.

PROPOSITION 3.6. *For each ordinal* $1 \leq \alpha < \omega_1$ *there are subsets* Q_α, P_α *and* D_α *of* $\mathbb{I}$ *such that*

(i) $Q_\alpha \in A(\alpha) \setminus M(\alpha)$ *and* $P_\alpha \in M(\alpha) \setminus A(\alpha)$;
(ii) $D_\alpha \in A(\alpha+1) \cap M(\alpha+1)$, *but* $D_\alpha \notin A(\alpha) \cup M(\alpha)$.

Proof. (i) For $\alpha > 1$ the desired subsets Q_α and P_α exist by Propositions 3.2 and 3.4.

(ii) For $\alpha \geq 1$ put $D_\alpha = E_\alpha \cup F_\alpha$, where E_α is in $\Sigma^0_\alpha([0,1/3]) \setminus \Pi^0_\alpha([0,1/3])$ and F_α is in $\Pi^0_\alpha([2/3,1]) \setminus \Sigma^0_\alpha([2/3,1])$. The statement is proved.

4. Infinite-dimensionality modulo a class $\mathcal{P}$. We will follow some idea of E. Pol from [5].

In this section all classes $\mathcal{P}$ of topological spaces are additionally assumed to be finitely additive. A space X is said to have *property* $(*)_\mathcal{P}$ if for every sequence $\{(A_i, B_i)\}_{i=1}^\infty$ of pairs of disjoint compact subsets of X there exist partitions L_i between A_i and B_i such that $\bigcap_{i=1}^N L_i \in \mathcal{P}$ for some integer N. It is evident that if a space X has property $(*)_\mathcal{P}$ then so does each closed subset of X.

REMARK 4.1. Let M be a subspace of a space X, (A,B) a pair of disjoint closed subsets of X, and L a partition in M between $M \cap A$ and $M \cap B$. If M is closed in X or $A, B \subset M$ then there exists a partition L' in X between A and B such that $M \cap L' = L$ (see [3, Lemma 1.2.9] and [1, Lemma I. 4.5]).

PROPOSITION 4.1. *If a space X is covered by a finite family of closed sets, each having property $(*)_\mathcal{P}$, then X also has this property.*

Proof. It is sufficient to consider the case when X is the union of two closed subsets X_1 and X_2 which have property $(*)_\mathcal{P}$. Consider a sequence $\{(A_i, B_i)\}_{i=1}^\infty$ of pairs of disjoint compact subsets of X. Since X_1 has property $(*)_\mathcal{P}$, for each integer $i \geq 0$ there is a partition L'_{2i+1} between $A_{2i+1} \cap X_1$ and $B_{2i+1} \cap X_1$ in X_1 such that $\bigcap_{i=1}^{N_1} L'_{2i+1} \in \mathcal{P}$ for some N_1. Let L_{2i+1} be a partition between A_{2i+1} and B_{2i+1} in X such that $L_{2i+1} \cap X_1 = L'_{2i+1}$. So $\bigcap_{i=1}^{N_1}(L_{2i+1} \cap X_1) \in \mathcal{P}$. Similarly, for each $i \geq 1$ we have a partition L_{2i} between A_{2i} and B_{2i} in X such that $\bigcap_{i=1}^{N_2}(L_{2i} \cap X_2) \in \mathcal{P}$ for some N_2. Put $N = \max\{2N_1+1, 2N_2\}$. Then

$$\begin{aligned}\bigcap_{i=1}^{N} L_i &= \Big(\Big(\bigcap_{i=1}^{N} L_i\Big) \cap X_1\Big) \cup \Big(\Big(\bigcap_{i=1}^{N} L_i\Big) \cap X_2\Big)\\ &= \Big(\bigcap_{i=1}^{N}(L_i \cap X_1)\Big) \cup \Big(\bigcap_{i=1}^{N}(L_i \cap X_2)\Big)\\ &\subset \Big(\bigcap_{i=1}^{N_1}(L_{2i+1} \cap X_1)\Big) \cup \Big(\bigcap_{i=1}^{N_2}(L_{2i} \cap X_2)\Big).\end{aligned}$$

Since $\mathcal{P}$ is finitely additive and monotone with respect to closed subsets, we have $\bigcap_{i=1}^{N} L_i \in \mathcal{P}$. This completes the proof.

PROPOSITION 4.2. *If* $\mathcal{P}\text{-trind}\, X \neq \infty$ *then* X *has property* $(*)_{\mathcal{P}}$.

Proof. Let us apply induction on $\alpha = \mathcal{P}\text{-trind}\, X$. If $\alpha = -1$, then $X \in \mathcal{P}$ and the statement is evidently valid.

Assume that the conclusion holds for $\mathcal{P}\text{-trind}\, X < \alpha \geq 0$. Let now X have $\mathcal{P}\text{-trind}\, X = \alpha$. Consider a sequence $\{(A_i, B_i)\}_{i=1}^{\infty}$ of pairs of disjoint compact subsets of X. Since A_1 is compact and $\mathcal{P}\text{-trind}\, X = \alpha$, there exist open subsets $U_1, \ldots, U_k$ of X such that $A_1 \subset \bigcup_{i=1}^{k} U_i$ and for each $i = 1, \ldots, k$, we have $\mathrm{Cl}\, U_i \cap B_1 = \emptyset$ and $\mathcal{P}\text{-trind}\, \mathrm{Bd}\, U_i \leq \beta_i$ for some $\beta_i < \alpha$. By the inductive assumption and Proposition 4.1, the set $\bigcup_{i=1}^{k} \mathrm{Bd}\, U_i$ has property $(*)_{\mathcal{P}}$. We put $U = \bigcup_{i=1}^{k} U_i$. It is easy to see that $L_1 = \mathrm{Bd}\, U \subset \bigcup_{i=1}^{k} \mathrm{Bd}\, U_i$ is a partition between A_1 and B_1, and L_1 has property $(*)_{\mathcal{P}}$. Note that for each $i \geq 2$ there exists a partition L_i' between $A_i \cap L_1$ and $B_i \cap L_1$ in L_1 such that $\bigcap_{i=2}^{N} L_i' \in \mathcal{P}$ for some N. Now if we take, for each $i = 2, \ldots, N$, a partition L_i between A_i and B_i in X such that $L_i \cap L_1 = L_i'$ then $\bigcap_{i=2}^{N} L_i' = \bigcap_{i=1}^{N} L_i$. So $\bigcap_{i=1}^{N} L_i \in \mathcal{P}$. The proposition is proved.

Let $\mathbb{I}^{\infty} = \{(x_j) : 0 \leq x_j \leq 1,\, j = 1, 2, \ldots\}$ be the product of countably many intervals $\mathbb{I}$. For each $n \geq 2$ denote the subset $\{(x_j) \in \mathbb{I}^{\infty} : x_k = 0$ for $k \geq n+1\}$ by $\mathbb{I}^n$. For each $n \geq 2$ and each $i = 1, \ldots, n$, set $A_i^n = \{(x_j) \in \mathbb{I}^n : x_i = 0\}$ and $B_i^n = \{(x_j) \in \mathbb{I}^n : x_i = 1\}$. Choose for each $n \geq 2$ a subset E_n of $\mathbb{I}^n$ and put

$$X = (\{0\} \times \mathbb{I}^{\infty}) \cup \bigcup_{n=2}^{\infty} (\{1/n\} \times E_n). \tag{4.1}$$

Let $Y = (\{0\} \times \mathbb{I}^{\infty}) \cup \bigcup_{n=2}^{\infty} (\{1/n\} \times \mathbb{I}^n)$ and $Z = \{0, 1/2, 1/3, \ldots\}$. It is obvious that $X \subset Y \subset Z \times \mathbb{I}^{\infty}$. Moreover, Y is compact, and its subspace $Y \setminus X$ is a topological sum of countably many finite-dimensional spaces. Hence, $\mathrm{trind}(Y \setminus X) \leq \omega_0$. Moreover, $\mathrm{trind}(X \setminus (\{0\} \times \mathbb{I}^{\infty})) \leq \omega_0$. It follows that

$$\mathrm{trdef}\, X \leq \omega_0 \quad \text{and} \quad \mathcal{K}_0\text{-trsur}\, X \leq \omega_0. \tag{4.2}$$

Proposition 4.3. *If for each integer $m \geq 1$ there exist an integer $k(m) \geq m+1$ such that for every $n \geq k(m)$ and for arbitrary partitions L_i^n between A_i^n and B_i^n in $\mathbb{I}^n$, $i \leq n$, we have $E_n \cap \bigcap_{i=1}^m L_i^n \notin \mathcal{P}$, then $\mathcal{P}$-trind $X = \infty$.*

Proof. We will apply Proposition 4.2. For each $i \geq 1$ let L_i be an arbitrary partition between the compact sets $A_i = \{(0,(x_j)) \in \{0\} \times \mathbb{I}^\infty : x_i = 0\}$ and $B_i = \{(0,(x_j)) \in \{0\} \times \mathbb{I}^\infty : x_i = 1\}$ in X. It suffices to show that $\bigcap_{i=1}^N L_i \notin \mathcal{P}$ for every $N \geq 1$. Consider, for each $i \geq 1$, a partition L_i' between A_i and B_i in Y such that $L_i' \cap X = L_i$. Note that for every i there exists an integer $n_i \geq 2$ such that for each $n \geq n_i$ the set $L_i^n = L_i' \cap (\{1/n\} \times \mathbb{I}^n)$ is a partition between $\{1/n\} \times A_i^n$ and $\{1/n\} \times B_i^n$ in $\{1/n\} \times \mathbb{I}^n$. Let N be an arbitrary integer and $n = \max\{n_1, \ldots, n_N, k(N)\}$. Note that

$$\begin{aligned} C = \Big(\bigcap_{i=1}^N L_i^n\Big) \cap (\{1/n\} \times E_n) &= \Big(\bigcap_{i=1}^N L_i'\Big) \cap (\{1/n\} \times E_n) \\ &= \Big(\bigcap_{i=1}^N L_i\Big) \cap (\{1/n\} \times E_n) \end{aligned}$$

is a closed subset of $\bigcap_{i=1}^N L_i$. Moreover, $C \notin \mathcal{P}$ by the assumption. Hence $\bigcap_{i=1}^N L_i \notin \mathcal{P}$. The proposition is proved.

Let us recall the following.

Proposition 4.4 ([7, Lemma 5.2]). *Let L_{i_j}, $j = 1, \ldots, p$, be partitions between the opposite faces $A_{i_j}^n$ and $B_{i_j}^n$ in $\mathbb{I}^n$, where $1 \leq i_1 < \cdots < i_p \leq n$ and $1 \leq p < n$. Then for any $k \neq i_j$, $j = 1, \ldots, p$, there is a continuum $C \subset \bigcap_{j=1}^p L_{i_j}$ meeting the faces A_k^n and B_k^n.*

Now we are ready to prove

Proposition 4.5. *Let L_i be a partition between the opposite faces A_i^n and B_i^n in the cube $\mathbb{I}^n$, $i \leq p$, for some $p < n$. Let also $L = \bigcap_{i=1}^p L_i$, $E = \{(x_i) \in \mathbb{I}^n : x_n \in F\} \subset \mathbb{I}^n$, where F is a subset of* $[0,1]$, *and let Q_α, P_α, D_α, $1 \leq \alpha < \omega_1$, be the subsets of* $[0,1]$ *from Proposition* 3.6. *Then*

(i) $L \cap E \notin \mathcal{K}_0$ *if* $F \notin \mathcal{K}_0$,
(ii) $L \cap E \notin M(\alpha)$ *if* $F = Q_\alpha$,
(iii) $L \cap E \notin A(\alpha)$ *if* $F = P_\alpha$,
(iv) $L \cap E \notin M(\alpha) \cup A(\alpha)$ *if* $F = D_\alpha$,
(v) $L \cap E$ *is not a Borel set of* $\mathbb{I}^n$ *if* $F \in \mathrm{Brn}([0,1])$.

Proof. By Proposition 4.4 there is a continuum $C \subset L$ meeting the faces A_n^n and B_n^n. Let π_n be the projection of $\mathbb{I}^n$ onto the nth coordinate $\mathbb{I}$, i.e., $\pi_n(x_1, \ldots, x_n) = x_n$, and let π_n^C be the restriction of π_n to C. Observe that

π_n^C is a continuous mapping of C onto $[0,1]$. Notice also that $C \cap E = (\pi_n^C)^{-1}(F)$ and $C \cap E$ is a closed subset of $L \cap E$.

(i) Since $F \notin \mathcal{K}_0$ we have $C \cap E = (\pi_n^C)^{-1}(F) \notin \mathcal{K}_0$, and so $L \cap E \notin \mathcal{K}_0$.

(ii) Since $Q_\alpha \notin M(\alpha)$, by Proposition 3.3 we have $C \cap E = (\pi_n^C)^{-1}(Q_\alpha) \notin M(\alpha)$, and hence $L \cap E \notin M(\alpha)$.

(iii) Since $P_\alpha \notin A(\alpha)$, by Proposition 3.3 we have $C \cap E = (\pi_n^C)^{-1}(P_\alpha) \notin A(\alpha)$, and hence $L \cap E \notin A(\alpha)$.

(iv) Since D_α is neither in $M(\alpha)$ nor in $A(\alpha)$, Proposition 3.3 shows that $C \cap E = (\pi_n^C)^{-1}(D_\alpha) \notin M(\alpha) \cup A(\alpha)$, and hence $L \cap E$ has the same property.

(v) Since F is not a Borel set in $[0,1]$, $C \cap E = (\pi_n^C)^{-1}F$ is not a Borel set by Proposition 3.3. Hence $L \cap E$ is not a Borel set in $\mathbb{I}^n$.

LEMMA 4.1. *For every space* Y, *if* $\operatorname{trdef} Y \le \omega_0$ *and* $\operatorname{trcmp} Y = \infty$, *then* $\operatorname{trdef} Y = \omega_0$.

Proof. Observe that if $\operatorname{trdef} Y$ is finite so $\operatorname{trdef} Y = \operatorname{def} Y \ge \operatorname{cmp} Y$ (the last inequality can be found in [1]). This contradiction proves the lemma.

PROPOSITION 4.6. *Let* F *be a non-compact subset of* $\mathbb{I}$ *and* X *from* (4.1), *where* $E_n = \{(x_i) \in \mathbb{I}^n : x_n \in F\}$ *for each* $n \ge 2$. *Then* $\operatorname{trcmp} X = \infty$ *and* $\operatorname{trdef} X = \mathcal{K}_0\text{-}\operatorname{trsur} X = \omega_0$. *Moreover*, *for any* $1 \le \alpha < \omega_1$ *we have*:

(i) *if* $F = Q_\alpha$, *then* $M(\alpha)\text{-}\operatorname{trind} X = \infty$, $A(\alpha)\text{-}\operatorname{trind} X = A(\alpha)\text{-}\operatorname{trsur} X = -1$ *and* $M(\alpha)\text{-}\operatorname{trdef} X = \omega_0$,

(ii) *if* $F = P_\alpha$, *then* $A(\alpha)\text{-}\operatorname{trind} X = \infty$, $M(\alpha)\text{-}\operatorname{trind} X = M(\alpha)\text{-}\operatorname{trdef} X = -1$ *and* $A(\alpha)\text{-}\operatorname{trsur} X = \omega_0$,

(iii) *if* $F = D_\alpha$, *then* $M(\alpha)\text{-}\operatorname{trind} X = A(\alpha)\text{-}\operatorname{trind} X = \infty$, $A(\alpha+1) \cap M(\alpha+1)\text{-}\operatorname{trind} X = -1$ *and* $M(\alpha)\text{-}\operatorname{trdef} X = M(\alpha)\text{-}\operatorname{trsur} X = \omega_0$,

(iv) *if* $F \in \operatorname{Brn}([0,1])$, *then* $\mathcal{AB}\text{-}\operatorname{trind} X = \infty$.

Proof. For each integer $m \ge 1$ put $k(m) = m+1$. Consider $m \ge 1$ and $n \ge k(m)$. Let L_i^n be an arbitrary partition between A_i^n and B_i^n in $\mathbb{I}^n$ for each $i = 1, \dots, m$. By Proposition 4.5(i) we have $E_n \cap \bigcap_{i=1}^m L_i^n \notin \mathcal{K}_0$. Hence, by Proposition 4.3, it follows that $\operatorname{trcmp} X = \infty$. Then, by Lemma 4.1 and (4.2), we have $\operatorname{trdef} X = \omega_0$. Observe that for any compact subspace Y of X and each $n \ge 2$ there is a subset of $(\{1/n\} \times E_n) \setminus Y$ homeomorphic to $\mathbb{I}^{n-1}$. Thus $\mathcal{K}_0\text{-}\operatorname{trsur} X \ge \omega_0$. Then, by (4.2), it follows that $\mathcal{K}_0\text{-}\operatorname{trsur} X = \omega_0$.

(i) By Propositions 4.5(ii) and 4.3 we have $M(\alpha)\text{-}\operatorname{trind} X = \infty$. It is clear (see the hierarchy of absolute Borel classes) that $M(\alpha)\text{-}\operatorname{trdef} X \le \operatorname{trdef} X = \omega_0$. Hence, by Remark 1.1(i), we get $M(\alpha)\text{-}\operatorname{trdef} X = \omega_0$. Furthermore, since $Q_\alpha \in A(\alpha)$, by Propositions 3.3 and 3.4(ii) it follows that $E_n = \pi_n^{-1}(Q_\alpha) \in A(\alpha)$. Then Proposition 3.1(i) yields $X \in A(\alpha)$. Hence $A(\alpha)\text{-}\operatorname{trind} X = A(\alpha)\text{-}\operatorname{trsur} X = -1$.

(ii) Use Propositions 4.5(iii), 4.3 and Remark 1.1(ii) to get $A(\alpha)$-trind $X = \infty$ and hence $A(\alpha)$-trsur $X = \omega_0$ by a similar argument to the one above. To prove that $M(\alpha)$-trind $X = M(\alpha)$-trdef $X = -1$, it suffices to show that $X \in M(\alpha)$. Since $P_\alpha \in M(\alpha)$, Propositions 3.3 and 3.4(ii) show that $E_n = \pi_n^{-1}(P_\alpha) \in M(\alpha)$. Hence $\mathbb{I}^n \setminus E_n \in A(\alpha)$ and $\bigcup_{n=2}^{\infty}(\mathbb{I}^n \setminus E_n) \in A(\alpha)$. Therefore, $X = Y \setminus \bigcup_{n=2}^{\infty}(\mathbb{I}^n \setminus E_n) \in M(\alpha)$.

(iii) Use Proposition 4.5(iv) to get $M(\alpha)$-trind $X = A(\alpha)$-trind $X = \infty$ and $M(\alpha)$-trdef $X = A(\alpha)$-trsur $X = \omega_0$ as above. Since $D_\alpha \in A(\alpha+1) \cap M(\alpha+1)$, an argument similar to (ii) shows that $X \in A(\alpha+1) \cap M(\alpha+1)$ and hence $A(\alpha) \cap M(\alpha)$-trind $X = -1$.

(iv) By Proposition 4.5(v), $E_n \cap \bigcap_{i=1}^{m} L_i^n \notin \mathcal{AB}$. Hence Proposition 4.3 yields $\mathcal{AB}$-trind $X = \infty$. The proposition is proved.

Proof of Theorem 1.1. Let $\pi_n : \mathbb{I}^n \to \mathbb{I}$ be the projection onto the nth factor. For each ordinal α with $1 \le \alpha < \omega_1$ we define

$$X_\alpha = (\{0\} \times \mathbb{I}^\infty) \cup \Big(\bigcup_{n=2}^{\infty} \{1/n\} \times \pi_n^{-1}(P_\alpha)\Big),$$

$$Y_\alpha = (\{0\} \times \mathbb{I}^\infty) \cup \Big(\bigcup_{n=2}^{\infty} \{1/n\} \times \pi_n^{-1}(Q_\alpha)\Big),$$

$$Z_\alpha = (\{0\} \times \mathbb{I}^\infty) \cup \Big(\bigcup_{n=2}^{\infty} \{1/n\} \times \pi_n^{-1}(D_\alpha)\Big).$$

It follows from Proposition 4.6 that X_α, Y_α and Z_α satisfy conditions (i)–(iv). The second part of Theorem 1.1 is a direct consequence of the following facts.

LEMMA 4.2. *Let X be a space with either $A(\alpha)$-trind $X = -1$ and $M(\alpha)$-trind $X \le \mu$, or $A(\alpha)$-trind $X = \mu$ and $M(\alpha)$-trind $X = -1$, where μ is an ordinal or the integer -1. Then $A(\alpha) \cap M(\alpha)$-trind $X \le \mu$.*

Proof. We consider only the case $A(\alpha)$-trind $X = -1$ and $M(\alpha)$-trind $X \le \mu$. We apply induction on $\mu \ge -1$. If $\mu = -1$ then $X \in A(\alpha) \cap M(\alpha)$. Hence $A(\alpha) \cap M(\alpha)$-trind $X = -1$. Thus the assertion is valid for $\mu = -1$. Assume that it holds for $\mu < \gamma \ge 0$. Let now $\mu = \gamma$. For each $x \in X$ and each neighborhood U of x there is a neighborhood $V \subset U$ of x such that $M(\alpha)$-trind $\operatorname{Bd} V < \gamma$. Note that $A(\alpha)$-trind $\operatorname{Bd} V = -1$. Hence by the inductive assumption, we have $A(\alpha) \cap M(\alpha)$-trind $\operatorname{Bd} V < \gamma$. Therefore $A(\alpha) \cap M(\alpha)$-trind $X \le \gamma$. The lemma is proved.

PROPOSITION 4.7. *Let X be a space such that $A(\alpha)$-trind $X \le \mu_1$ and $M(\alpha)$-trind $X \le \mu_2$, where μ_1 and μ_2 are ordinals. Then*

$$A(\alpha) \cap M(\alpha)\text{-trind}\, X \leq \begin{cases} \mu_1 + n(\mu_2) + 1 = \mu_2 + n(\mu_1) + 1 & \textit{if } \lambda(\mu_1) = \lambda(\mu_2), \\ \mu_1 & \textit{if } \lambda(\mu_1) > \lambda(\mu_2), \\ \mu_2 & \textit{if } \lambda(\mu_2) > \lambda(\mu_1). \end{cases}$$

Proof. We apply induction on $\nu = \max\{\mu_1, \mu_2\} \geq 0$. If $\nu = 0$ then $\mu_1 = \mu_2 = 0$. For each $x \in X$ and each neighborhood U of x there is a neighborhood $V \subset U$ of x such that $A(\alpha)$-trind Bd $V = -1$. Observe that $M(\alpha)$-trind Bd $V \leq 0$. Lemma 4.2 implies that $A(\alpha) \cap M(\alpha)$-trind Bd $V \leq 0$. Hence $A(\alpha) \cap M(\alpha)$-trind $X \leq 1$, and the assertion is valid for $\nu = 0$.

Suppose $\nu > 0$ and the assertion holds for every $\gamma < \nu$.

Case 1: $\lambda(\mu_1) > \lambda(\mu_2)$. Then $\nu = \mu_1$. For each $x \in X$ and each neighborhood U of x there is a neighborhood $V \subset U$ of x such that $A(\alpha)$-trind Bd $V < \mu_1 = \nu$. Since $M(\alpha)$-trind Bd $V \leq \mu_2 < \lambda(\mu_1) \leq \mu_1 = \nu$, by the inductive assumption we have $A(\alpha) \cap M(\alpha)$-trind Bd $V < \mu_1$. Hence $A(\alpha) \cap M(\alpha)$-trind $X \leq \mu_1$.

Similarly the assertion is valid for $\lambda(\mu_1) > \lambda(\mu_2)$.

Case 2: $\lambda(\mu_1) = \lambda(\mu_2)$ and $\mu_1 > \mu_2$. Then $\nu = \mu_1$. For each $x \in X$ and each neighborhood U of x there is a neighborhood $V \subset U$ of x such that $A(\alpha)$-trind Bd $V < \mu_1$. The inductive assumption yields

$$\begin{aligned} A(\alpha) \cap M(\alpha)\text{-trind Bd}\, V &\leq \mu_1 - 1 + n(\mu_2) + 1 = \mu_1 + n(\mu_2) \\ &< \mu_1 + n(\mu_2) + 1. \end{aligned}$$

Hence $A(\alpha) \cap M(\alpha)$-trind $X \leq \mu_1 + n(\mu_2) + 1$. Analogously the assertion is valid for $\mu_2 > \mu_1$.

Case 3: $\mu_1 = \mu_2 = \mu$. If μ is not a limit ordinal, then for each $x \in X$ and each neighborhood U of x there is a neighborhood $V \subset U$ of x such that $M(\alpha)$-trind Bd $V \leq \mu - 1$. By Case 2, we have

$$A(\alpha) \cap M(\alpha)\text{-trindBd}\, V \leq \mu + n(\mu - 1) + 1 = \mu + n(\mu).$$

So $A(\alpha) \cap M(\alpha)$-trind $X \leq \mu_1 + n(\mu) + 1$. If μ is a limit ordinal, then for each $x \in X$ and each neighborhood U of x there is a neighborhood $V \subset U$ of x such that $M(\alpha)$-trind Bd $V < \mu$. By Case 1, we have $A(\alpha) \cap M(\alpha)$-trind Bd $V \leq \mu$. Hence $A(\alpha) \cap M(\alpha)$-trind $X \leq \mu + 1$. This completes the proof.

We do not know of any example of a space X such that $\mathcal{C}_0$-ind $X = \mathcal{S}_0$-ind $X = 0$ and $\mathcal{C}_0 \cap \mathcal{S}_0$-ind $X = 1$.

Proof of Theorem 1.2. Let F be a Bernstein set in $\mathbb{I}$. Set $X = (\{0\} \times \mathbb{I}^\infty) \cup (\bigcup_{n=2}^\infty \{1/n\} \times \pi_n^{-1}(F))$. Then Proposition 4.6 and the hierarchy of absolute Borel classes show that X is as desired.

REMARK 4.2. By Proposition 4.6 the space X from (4.1), where $E_n = \{(x_i) \in \mathbb{I}^n : 0 < x_n \leq 1\}$ for each $n \geq 2$, has $\operatorname{trcmp} X = \infty$ and $\operatorname{trdef} X = \mathcal{K}_0\text{-}\operatorname{trsur} X = \omega_0$. Evidently, $A(1) \cap M(1)\text{-}\operatorname{trind} X = -1$.

Recall ([3, Theorem 7.1.6]) that for any space X, $\operatorname{trind} X < \omega_1$ or $\operatorname{trind} X = \infty$. In [6] R. Pol showed that for each ordinal $\beta < \omega_1$ there exists a separable completely metrizable σ-compact space R_β with $\operatorname{trcmp} R_\beta = \beta$. Notice that $\mathcal{C}_0\text{-}\operatorname{trind} R_\beta = \mathcal{S}_0\text{-}\operatorname{trind} R_\beta = -1$. So it is natural to pose

PROBLEM 4.1. Do there exist for each ordinal $1 \leq \alpha < \omega_1$ and each ordinal $0 \leq \beta < \omega_1$ spaces $X_{\alpha,\beta}, Y_{\alpha,\beta}$ such that $M(\alpha)\text{-}\operatorname{trind} X_{\alpha,\beta} = A(\alpha)\text{-}\operatorname{trind} Y_{\alpha,\beta} = \beta$ and $A(\alpha)\text{-}\operatorname{trind} X_{\alpha,\beta} = M(\alpha)\text{-}\operatorname{trind} Y_{\alpha,\beta} = -1$?

REMARK 4.3. Let $\mathbb{Q}$ denote the space of rationals. In [6] R. Pol observed that using Aarts' argument in the proof of the equality $\operatorname{cmp}(\mathbb{I}^n \times \mathbb{Q}) = n$ ([1]) one can show that $\operatorname{trcmp}(X \times \mathbb{Q}) = \alpha$ for any $X \in \mathcal{K}_0$ with $\operatorname{trind} X = \alpha$. We can add that even $M(1)\text{-}\operatorname{trind}(X \times \mathbb{Q}) = M(1)\text{-}\operatorname{trdef}(X \times \mathbb{Q}) = \alpha$. (Evidently $A(1)\text{-}\operatorname{trind}(X \times \mathbb{Q}) = -1$.) Recall ([3]) that for any $\alpha < \omega_1$ there is a space $X_\alpha \in \mathcal{K}_0$ such that $\operatorname{trind} X_\alpha = \alpha$.

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