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Some Remarks on Tall Cardinals and Failures of GCH

by

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Summary. We investigate two global GCH patterns which are consistent with the existence of a tall cardinal, and also present some related open questions.

1. Introduction and preliminaries. We begin with the following definition due to Hamkins [13]. Suppose κ is a cardinal and $\lambda \geq \kappa$ is an arbitrary ordinal. κ is λ *tall* if there is an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and $M^\kappa \subseteq M$; and κ is *tall* if κ is λ tall for every ordinal λ.

In [13], Hamkins made a systematic study of tall cardinals and established many of their basic properties. He also made the interesting observation [13, p. 84] that "strongness is to tallness as supercompactness is to strong compactness" and established in [13] many results that either support this thesis directly or are analogues of conjectures believed true about strongly compact and supercompact cardinals. In particular, [13, Corollary 3.2] shows the consistency relative to a strong cardinal of a tall cardinal κ with GCH holding at and below κ yet failing above κ. This provides a negative solution to an analogue of a question about strongly compact cardinals attributed to Woodin [16, Question 22.22, p. 310], which asks: if κ is strongly compact and GCH holds everywhere below κ, then does GCH hold everywhere? Note that the answer remains unknown in the context of ZFC (although as shown in [3], a negative solution may be obtained when the Axiom of Choice is false). In addition, it is possible to invert Woodin's question and ask: if κ is strongly compact and GCH fails everywhere below κ, then must GCH fail somewhere at or above κ (or is this even consistent)? Once again, an

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answer remains unknown in the context of ZFC (although as shown in [4], a negative solution to a weaker version of this question may be obtained when the Axiom of Choice is false). Of course, if κ is either supercompact or strong, then an easy reflection argument shows that the answer to the appropriate analogue of the first of the above questions must be yes. If κ is strong, then once again, an easy reflection argument shows that the answer to the appropriate analogue of the second of the above questions must also be yes (and the fact that it is relatively consistent for κ to be strong and for GCH to fail everywhere below κ will be addressed in the proof of Theorem 1). If κ is supercompact, then κ is also strongly compact, so by Solovay's theorem [21], GCH must hold at any singular strong limit cardinal above κ. Another easy reflection argument then shows that there must be unboundedly many in κ singular strong limit cardinals at which GCH holds. This establishes that the theory "ZFC + κ is supercompact + GCH fails everywhere below κ" is inconsistent.

The purpose of this paper is to show that as with Woodin's original question, it is possible to obtain negative answers to versions of this second question for tall cardinals. In particular, we have the following theorem.

THEOREM 1. Con(*ZFC + There is a supercompact cardinal with infinitely many inaccessible cardinals above it*) $\Rightarrow$ Con(*ZFC + There is a tall cardinal δ such that GCH fails everywhere below δ yet holds for every cardinal $\gamma \geq \delta$*).

If we weaken our requirements to a tall cardinal κ in which GCH fails only at every regular cardinal below κ yet holds for every cardinal $\delta \geq \kappa$, then it is possible to obtain this cardinal pattern from only a strong cardinal. Specifically, we have the following theorem.

THEOREM 2. Con(*ZFC + There is a strong cardinal*) $\Rightarrow$ Con(*ZFC + There is a tall cardinal κ such that GCH fails at every regular cardinal below κ yet holds for every cardinal $\delta \geq \kappa$*).

As corollaries to the proofs of Theorems 1 and 2, we will be able to force and obtain analogous cardinal patterns in which our tall cardinal κ is also the least measurable cardinal. It will also be possible to show that relative to the appropriate assumptions, it is the case that our witnessing models contain a proper class of strong cardinals.

We very briefly mention that we are assuming a basic knowledge of large cardinals and forcing, for which we refer readers to [15, 16]. A basic knowledge of Hamkins' paper [13] is also helpful. In particular, by [13, Theorem 2.10], any strong cardinal is also a tall cardinal.

For any regular cardinal δ and any ordinal α, $\mathrm{Add}(\delta, \alpha)$ is the usual partial ordering for adding α Cohen subsets of δ. The partial ordering $\mathbb{P}$ is

δ-*directed closed* if for any directed $D \subseteq \mathbb{P}$ such that $|D| < \delta$, there is some $p \in \mathbb{P}$ extending each member of D. Furthermore, $\mathbb{P}$ is (δ, ∞)-*distributive* if for any δ sequence $\langle D_\alpha \mid \alpha < \delta \rangle$ of dense open subsets of $\mathbb{P}$, $\bigcap_{\alpha<\delta} D_\alpha$ is dense open as well. Any partial ordering $\mathbb{P}$ which is δ^+-directed closed is automatically (δ, ∞)-distributive. For $\lambda > \delta$, δ *is strong up to* λ if δ is α strong for every $\alpha < \lambda$. If $G \subseteq \mathbb{P}$ is V-generic, we will abuse notation somewhat by using both $V^{\mathbb{P}}$ and $V[G]$ interchangeably.

The following fact is basic and will be used in several of our proofs.

FACT 1.1. *For every cardinal δ, there is a (possibly proper class) δ^+-directed closed reverse Easton iteration $\mathbb{P}(\delta)$ such that after forcing with $\mathbb{P}(\delta)$, GCH holds for all cardinals at and above δ.*

Sketch of proof. Define the (possibly proper class) reverse Easton iteration $\mathbb{P}(\delta) = \langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle \mid \alpha \in \mathrm{Ord}\rangle$, where $\mathbb{P}_0 = \mathrm{Add}(\delta^+, 1)$. For each ordinal α, if $\Vdash_{\mathbb{P}_\alpha}$ "There is a cardinal greater than δ violating GCH", then $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha = \dot{\mathrm{Add}}(\gamma^+, 1)$ where γ is the least cardinal greater than δ violating GCH". If this is not the case, i.e., if there is some $p \in \mathbb{P}_\alpha$ such that $p \Vdash_{\mathbb{P}_\alpha}$ "All cardinals greater than δ satisfy GCH", then we stop our construction and define $\mathbb{P}(\delta) = \mathbb{P}_\alpha/p$. Since by its definition, for any cardinal γ, $\mathrm{Add}(\gamma^+, 1)$ is γ^+-directed closed, $\mathbb{P}(\delta)$ is δ^+-directed closed. The arguments found in the proof of [1, Theorem 2.1] then show that $\mathbb{P}(\delta)$ is as desired. In particular, after forcing with $\mathrm{Add}(\gamma^+, 1)$ for some γ, all cardinals less than or equal to γ^+ are preserved, $2^\gamma = \gamma^+$, 2^γ of the ground model is collapsed to γ^+, and all cardinals greater than or equal to $(2^\gamma)^+$ of the ground model are preserved. ■

2. The proofs of Theorems 1 and 2 and related results. We turn now to the proofs of our theorems, beginning with the proof of Theorem 1.

Proof of Theorem 1. Let $\overline{V} \vDash$ "ZFC + κ is supercompact + There are infinitely many inaccessible cardinals greater than κ". Without loss of generality, we assume that $\overline{V} \vDash$ GCH as well. By work of Foreman and Woodin [8], for any fixed integer $n \geq 1$, we may assume that $\overline{V}$ has been generically extended to a model V' of ZFC in which the following hold:

(1) κ is $\beth_n(\kappa)$ supercompact.
(2) GCH fails everywhere below κ.
(3) $2^\kappa = \lambda$ where λ is weakly inaccessible.

Then, by forcing over V' with $\mathbb{P}(\lambda) = \mathbb{P}(2^\kappa)$, we may further assume that V' has been generically extended to a model V in which κ is 2^κ supercompact, properties (2) and (3) of V' remain true, and GCH holds for all cardinals greater than or equal to λ [1]. This follows by Fact 1.1 (and uses in particular that $V' \models$ "$\mathbb{P}(\lambda)$ is λ^+-directed closed"). We henceforth work over V.

By the proof of [5, Lemma 2.1] (see also the proof of [16, Proposition 26.11]), since κ is 2^κ supercompact, $\{\delta < \kappa \mid \delta$ is strong up to $\kappa\}$ is unbounded in κ. Thus, $V_\kappa \models$ "There is a proper class of strong cardinals". Consequently, we may let $\delta < \kappa$ be such that $V_\kappa \models$ "δ is a strong cardinal". Consider $(\mathbb{P}(\delta))^{V_\kappa}$, which we henceforth write as $\mathbb{P}(\delta)$. Since $V_\kappa \models$ "$\mathbb{P}(\delta)$ is δ^+-directed closed and δ is a tall cardinal", by [13, Theorem 3.1] (which says that any tall cardinal δ automatically has its tallness indestructible under (δ, ∞)-distributive forcing), $(V_\kappa)^{\mathbb{P}(\delta)} = V^* \models$ "δ is a tall cardinal". By Fact 1.1 and the fact that $V \models$ "GCH fails everywhere below κ", $V^* \models$ "ZFC + GCH fails everywhere below δ yet holds for every cardinal $\gamma \geq \delta$". This completes the proof of Theorem 1. ∎

Proof of Theorem 2. The reasoning is similar. Suppose $\overline{V} \models$ "ZFC$+\kappa$ is a strong cardinal". By passing to the appropriate inner model (see, e.g., [22]), we may assume that $\overline{V} \models$ GCH as well. Consequently, by work of Friedman and Honzik [11, Theorem 3.17], we may assume that $\overline{V}$ has been generically extended to a model V of ZFC such that $V \models$ "κ is a strong cardinal + For every regular cardinal δ, $2^\delta = \delta^{++}$". Consider once again $(\mathbb{P}(\kappa))^V$ (or as above, just $\mathbb{P}(\kappa)$). Then as in the proof of Theorem 1, $V^{\mathbb{P}(\kappa)} \models$ "κ is a tall cardinal such that GCH fails at every regular cardinal below κ yet holds for every cardinal $\delta \geq \kappa$". This completes the proof of Theorem 2. ∎

In Theorems 1 and 2, our tall cardinals δ and κ are strong in their respective universes over which we force with $\mathbb{P}(\delta)$ and $\mathbb{P}(\kappa)$. Therefore, in the models witnessing the conclusions of Theorems 1 and 2, δ and κ are both quite large in size (e.g., each is a measurable limit of measurable cardinals). Consider what happens if we first force with the Magidor iteration of Prikry forcing [18] which destroys every measurable cardinal below either δ or κ. The work of [18] shows that this partial ordering has size 2^δ or 2^κ. A theorem of Gitik [6, Lemma 2.1] shows that since δ and κ are initially strong cardinals, forcing with this partial ordering preserves the tallness of either δ or κ. If we then force with either $\mathbb{P}(\delta)$ or $\mathbb{P}(\kappa)$, since the Magidor iteration of Prikry forcing preserves both cardinals and the sizes of power sets (see [2] for a discussion of these facts), we have the following two corollaries to Theorems 1 and 2.

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(1) Strictly speaking, it is not necessary to force over V' to obtain V where GCH holds at and above $\lambda = 2^\kappa$ in order to prove Theorem 1 as stated. This is only done to show that in the model V^* witnessing the conclusions of Theorem 1, there is a proper class of strong cardinals. This issue will be discussed further in the paragraph immediately following the proof of Proposition 2.3.

Corollary 2.1. Con(*ZFC + There is a supercompact cardinal with infinitely many inaccessible cardinals above it*) $\Rightarrow$ Con(*ZFC + There is a tall cardinal δ such that GCH fails everywhere below δ yet holds for every cardinal $\gamma \geq \delta$ + δ is the least measurable cardinal*).

Corollary 2.2. Con(*ZFC + There is a strong cardinal*) $\Rightarrow$ Con(*ZFC + There is a tall cardinal κ such that GCH fails at every regular cardinal below κ yet holds for every cardinal $\delta \geq \kappa$ + κ is the least measurable cardinal*).

It is in fact the case that the model V^* witnessing the conclusions of Theorem 1 (and hence, by the Lévy–Solovay results [17], the model witnessing the conclusions of Corollary 2.1 as well) contains a proper class of strong cardinals. In particular, we have the following proposition.

Proposition 2.3. $V^* \models$ *"There is a proper class of strong cardinals".*

Proof. With a slight abuse of notation, write $\mathbb{P}(\delta)$ for $(\mathbb{P}(\delta))^{V_\kappa}$. It is then the case that $\mathbb{P}(\delta) \in V$. We will show that $V^{\mathbb{P}(\delta) * \dot{\mathrm{Add}}(\kappa^+,1)} \models$ "κ is $2^\kappa = \kappa^+$ supercompact". This suffices, since as in the proof of Theorem 1, it is then true that $\{\delta < \kappa \mid \delta$ is strong up to $\kappa\}$ is unbounded in κ in $V^{\mathbb{P}(\delta) * \dot{\mathrm{Add}}(\kappa^+,1)}$. Because $\Vdash_{\mathbb{P}(\delta)}$ "$\dot{\mathrm{Add}}(\kappa^+, 1)$ is κ^+-directed closed", $\{\delta < \kappa \mid \delta$ is strong up to $\kappa\}$ is unbounded in κ in $V^{\mathbb{P}(\delta)}$ as well. From this, it immediately follows that in $V^* = (V_\kappa)^{\mathbb{P}(\delta)}$, there is a proper class of strong cardinals.

To see this, we use an argument found in the proof of [1, Theorem 2.1], quoting verbatim when appropriate. Let $j : V \to M$ be an elementary embedding witnessing the λ supercompactness of κ in V generated by a supercompact ultrafilter over $P_\kappa(\lambda)$. In particular, $M^\lambda \subseteq M$. We use a standard lifting argument to show that j lifts in $V^{\mathbb{P}(\delta) * \dot{\mathrm{Add}}(\kappa^+,1)}$ to $j : V^{\mathbb{P}(\delta) * \dot{\mathrm{Add}}(\kappa^+,1)} \to M^{j(\mathbb{P}(\delta) * \dot{\mathrm{Add}}(\kappa^+,1))}$. Specifically, let G_0 be V-generic over $\mathbb{P}(\delta)$, and let G_1 be $V[G_0]$-generic over $\mathrm{Add}(\kappa^+, 1)$. Observe that $j(\mathbb{P}(\delta) * \dot{\mathrm{Add}}(\kappa^+, 1)) = \mathbb{P}(\delta) * \dot{\mathrm{Add}}(\kappa^+, 1) * \dot{\mathbb{Q}} * \dot{\mathrm{Add}}(j(\kappa^+), 1)$. Working in $V[G_0][G_1]$, we first note that since $\mathbb{P}(\delta) * \dot{\mathrm{Add}}(\kappa^+, 1)$ is $(2^\kappa)^+ = \lambda^+$-c.c., $M[G_0][G_1]$ remains λ closed with respect to $V[G_0][G_1]$. This means that $\mathbb{Q}$ is λ^+-directed closed in both $M[G_0][G_1]$ and $V[G_0][G_1]$.

Since $M[G_0][G_1] \models$ "$|\mathbb{Q}| = j(\kappa)$", the number of dense open subsets of $\mathbb{Q}$ present in $M[G_0][G_1]$ is $(2^{j(\kappa)})^M$. In V, since M is given via an ultrapower by a supercompact ultrafilter over $P_\kappa(2^\kappa)$, this is calculated as $|\{f \mid f : [2^\kappa]^{<\kappa} \to 2^\kappa\}| = |\{f \mid f : 2^\kappa \to 2^\kappa\}| = 2^{2^\kappa} = 2^\lambda$. Since $V \models$ "$2^\lambda = \lambda^+$" and λ^+ is preserved from V to $V[G_0][G_1]$, we may let $\langle D_\alpha \mid \alpha < \lambda^+\rangle \in V[G_0][G_1]$ enumerate the dense open subsets of $\mathbb{Q}$ present in $M[G_0][G_1]$. We may now use the fact that $\mathbb{Q}$ is λ^+-directed closed in $V[G_0][G_1]$ to meet each D_α and thereby construct in $V[G_0][G_1]$ an $M[G_0][G_1]$-generic object H_0 over $\mathbb{Q}$. Our construction guarantees that $j''G_0 \subseteq G_0 * G_1 * H_0$, so j lifts in $V[G_0][G_1]$ to $j : V[G_0] \to M[G_0][G_1][H_0]$.

It remains to lift j in $V[G_0][G_1]$ through $\mathrm{Add}(\kappa^+,1)$. Because $V[G_0] \vDash$ "$|\mathrm{Add}(\kappa^+,1)| = 2^\kappa = (2^\kappa)^V = \lambda$", $M[G_0][G_1][H_0] \vDash$ "$|\mathrm{Add}(j(\kappa^+),1)| = 2^{j(\kappa)} = (2^{j(\kappa)})^M$". Therefore, since $M[G_0][G_1][H_0]$ remains λ closed with respect to $V[G_0][G_1]$, $M[G_0][G_1][H_0] \vDash$ "$\mathrm{Add}(j(\kappa^+),1)$ is $j(\kappa^+)$-directed closed", and $j(\kappa^+) > j(\kappa) > \lambda$, there is a master condition $q \in \mathrm{Add}(j(\kappa^+),1)$ for $j''\{p \mid p \in G_1\}$. Further, the number of dense open subsets of $\mathrm{Add}(j(\kappa^+),1)$ present in $M[G_0][G_1][H_0]$ is $(2^{2^{j(\kappa)}})^M$. This is calculated in V as $|\{f \mid f : [2^\kappa]^{<\kappa} \to 2^{2^\kappa}\}| = |\{f \mid f : 2^\kappa \to (2^\kappa)^+\}| = |\{f \mid f : \lambda \to \lambda^+\}| = 2^\lambda = (2^\kappa)^+ = \lambda^+$. Working in $V[G_0][G_1]$, since $\mathrm{Add}(j(\kappa^+),1)$ is λ^+-directed closed in both $M[G_0][G_1][H_0]$ and $V[G_0][G_1]$, we may consequently use the arguments of the preceding paragraph to construct an $M[G_0][G_1][H_0]$-generic object H_1 over $\mathrm{Add}(j(\kappa^+),1)$ containing q. Since by the definition of H_1, $j''(G_0 * G_1) \subseteq G_0 * G_1 * H_0 * H_1$, j lifts in $V[G_0][G_1]$ to $j : V[G_0][G_1] \to M[G_0][G_1][H_0][H_1]$. As $V[G_0][G_1] \vDash$ "$|\lambda| = \kappa^+$", this means that $V^{\mathbb{P}(\delta) * \mathrm{Add}(\kappa^+,1)} \vDash$ "κ is $2^\kappa = \kappa^+$ supercompact". This completes the proof of Proposition 2.3. ■

We take this opportunity to observe that if we did not wish to show that $V^* \vDash$ "There is a proper class of strong cardinals", it would be unnecessary to force over V' with $\mathbb{P}(\lambda)$. The proof of Proposition 2.3 requires a sufficient amount of GCH above λ, which is why we needed to generically extend V' to V. (We could have, of course, only forced exactly the amount of GCH required to allow the arguments of Proposition 2.3 to go through, as opposed to forcing GCH to hold for all cardinals greater than or equal to λ.)

We turn our attention now to proving a version of Theorem 2 in which our witnessing model contains a proper class of strong cardinals. One might expect to proceed by starting with a model containing a proper class of strong cardinals in which GCH holds, then use [11, Theorem 3.17] to obtain a model containing a proper class of strong cardinals in which $2^\delta = \delta^{++}$ for every regular cardinal δ, and then force GCH to hold on a proper class of cardinals above a fixed strong cardinal κ. The problem is that the usual argument for the preservation of a strong cardinal λ after a reverse Easton iteration (as found, e.g., in [14, Theorem 4.10]) requires that $2^\lambda = \lambda^+$ in the model over which the forcing has been done. This will, of course, not be the case in the approach just suggested, and is the reason Proposition 2.3 is used to show that $V^* \vDash$ "There is a proper class of strong cardinals". It is, however, possible to proceed in a different fashion, by using stronger assumptions. Specifically, we have the following result.

Theorem 3. Con(*ZFC + There is a cardinal λ such that λ is 2^λ supercompact and $2^\delta = \delta^{++}$ for every regular cardinal $\delta \le \lambda$*) $\Rightarrow$ Con(*ZFC +*

There is a tall cardinal κ such that GCH fails at every regular cardinal below κ yet holds for every cardinal $\delta \geq \kappa$ + There is a proper class of strong cardinals).

Note that a model witnessing the hypotheses of Theorem 3 may be obtained starting with a model for "ZFC + There exists a supercompact cardinal" (or even weaker assumptions—for the optimal hypotheses, see [7]) by using Menas' techniques from [19, Theorem 18].

Proof of Theorem 3. Let $\overline{V} \models$ "ZFC + There is a cardinal λ such that λ is 2^λ supercompact and $2^\delta = \delta^{++}$ for every regular cardinal $\delta \leq \lambda$". As in the proof of Theorem 1, we may begin by forcing with $\mathbb{P}(\lambda^{++}) = \mathbb{P}((2^\lambda))$ to generically extend $\overline{V}$ to a model V such that $V \models$ "ZFC + λ is $2^\lambda = \lambda^{++}$ supercompact + GCH fails for every regular cardinal $\delta \leq \lambda$ + GCH holds for every regular cardinal $\delta \geq \lambda^{++}$". Let $\kappa < \lambda$ be such that $V \models$ "κ is strong up to λ". As in the proof of Proposition 2.3, we slightly abuse notation and write $\mathbb{P}(\kappa)$ for $(\mathbb{P}(\kappa))^{V_\lambda}$. If we now force over V with $\mathbb{P}(\kappa) * \dot{\mathrm{A}}\mathrm{dd}(\lambda^+, 1)$, the arguments used in the proofs of Theorems 1 and 2 and Proposition 2.3 show that $V^* = (V_\lambda)^{\mathbb{P}(\kappa)} \models$ "ZFC+κ is a tall cardinal + GCH fails at every regular cardinal below κ yet holds for every cardinal $\delta \geq \kappa$ + There is a proper class of strong cardinals". This completes the proof of Theorem 3. ∎

In analogy to Corollary 2.2, we have the following corollary to Theorem 3.

Corollary 2.4. Con(*ZFC + There is a cardinal λ such that λ is 2^λ supercompact and $2^\delta = \delta^{++}$ for every regular cardinal $\delta \leq \lambda$*) $\Rightarrow$ Con(*ZFC + There is a tall cardinal κ such that GCH fails at every regular cardinal below κ yet holds for every cardinal $\delta \geq \kappa$ + κ is the least measurable cardinal + There is a proper class of strong cardinals*).

3. Concluding remarks. We conclude with some open questions and related remarks raised by the results and proofs of this paper. In particular:

1. Are the theories "ZFC+κ is strongly compact + GCH holds everywhere below κ yet fails for some regular cardinal $\delta > \kappa$", "ZFC + κ is strongly compact + GCH fails everywhere below κ yet holds for all regular cardinals $\delta \geq \kappa$", and "ZFC + κ is strongly compact + GCH fails for all regular cardinals below κ yet holds for all regular cardinals $\delta \geq \kappa$" consistent? As we have already noted, by Solovay's theorem [21], if κ is strongly compact, then GCH must hold at any singular strong limit cardinal above κ. Consequently, if GCH holds at every regular cardinal above κ, then GCH must hold at *every* cardinal above κ (since all singular cardinals above κ are then strong limit cardinals as well).

2. Is the theory "ZFC + κ is strongly compact + GCH fails everywhere below κ" consistent? Note that in this question, we are not imposing any

constraints on the size of 2^δ for cardinals $\delta \geq \kappa$. In addition, observe that by [19, Theorem 18], the theory "ZFC + κ is supercompact + GCH fails for every regular cardinal" is consistent relative to the theory "ZFC + κ is supercompact".

3. What is the consistency strength of the theories "ZFC + There is a tall cardinal δ such that GCH fails everywhere below δ yet holds for every cardinal $\gamma \geq \delta$", "ZFC + GCH fails everywhere + There is a strong cardinal", and "ZFC + GCH fails everywhere + There is a proper class of strong cardinals"? In [8, p. 35], it is stated that Woodin can obtain a model for the theory "ZFC + GCH fails everywhere" (in fact, for the theory "ZFC + $2^\delta = \delta^{++}$ for every cardinal δ") starting from a $\wp^2(\kappa)$ hypermeasurable cardinal κ (also known as a $\kappa + 2$ strong cardinal κ) (2). We conjecture that the consistency of the first two of the above theories can be established relative to the existence of a cardinal λ which is strong up to a $\wp^2(\kappa)$ hypermeasurable cardinal κ, and the consistency of the last theory can be established relative to a $\wp^2(\kappa)$ hypermeasurable cardinal κ which is a limit of cardinals λ which are strong up to κ. However, it is unclear if these assumptions will provide equiconsistencies in each case.

4. What is the consistency strength of the theories "ZFC + There is a tall cardinal δ such that GCH fails everywhere below δ yet holds for every cardinal $\gamma \geq \delta$ + There is a proper class of strong cardinals" and "ZFC + There is a tall cardinal κ such that GCH fails at every regular cardinal below κ yet holds for every cardinal $\delta \geq \kappa$ + There is a proper class of strong cardinals"? We conjecture that these lie somewhere below the consistency strength of a cardinal λ which is 2^λ supercompact. Note that [13, Corollary 3.14] (which Hamkins credits originally to Gitik) tells us that the theories "ZFC + There is a strong cardinal" and "ZFC + There is a tall cardinal" are equiconsistent. This indicates that the hypotheses and conclusion of Theorem 2, namely "ZFC + There is a strong cardinal" and "ZFC + There is a tall cardinal κ such that GCH fails at every regular cardinal below κ yet holds for every cardinal $\delta \geq \kappa$", are equiconsistent as well.

5. Is the theory "ZFC + There is a tall cardinal κ such that GCH holds everywhere below κ yet fails at a singular strong limit cardinal above κ" consistent? Gitik has pointed out [12] that it is impossible to do Prikry forcing above a tall cardinal κ while preserving κ's tallness without first doing some sort of preparation forcing below κ. Thus, an analogue of [13, Theorem 3.1], which tells us that any tall cardinal δ is automatically indestructible under

(2) Models of ZFC in which GCH fails everywhere constructed using strongness hypotheses may also be found in [9], [10], and [20].

(δ, ∞)-distributive forcing, does not seem to be valid. This suggests that obtaining a model for this theory seems to be quite a difficult task.

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